

Pension Plan Embedded Option Valuation

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Executive Summary

Embedded options in pension plans play an increasing role in estimating pension liability values. With the credit interest rate floor of a cash balance plan—a simple type of option identified by the Project Oversight Group—as a model, this research uses three valuation and risk analysis approaches to explore the existing techniques to value embedded options on an economic basis. The three approaches are:

- (1) Closed-form valuation. Analytic option pricing solutions often do not exist for common embedded options in pension plans. Most embedded option payouts are linked to a cumulative index and those options are exercisable at multiple time points. In certain contexts, a closed-form valuation may provide a reasonable estimate in a computationally inexpensive manner. However, the complexity of the parameter calibration and high sensitivity of the results limit such applications.
- (2) Stochastic valuation. Based on stochastic cash flow projection, the fair value of embedded options can be determined as the average present value of projected cash flows under a set of stochastic risk-neutral scenarios that are market consistent. The stochastic approach also allows an in-depth analysis of the risk distribution and supports various capital management needs based on stochastic real-world scenario analysis. This approach is preferred to value and analyze embedded options in pension plans due to its flexibility, versatility and the wide range of information it can provide.
- (3) Replicating portfolio. A replicating portfolio approach uses market instruments to replicate the value and sensitivities of embedded options. It is not used for accurate valuation purposes but can help estimate the change in embedded option values given a real or “what if” market scenario. It is often used in asset liability management or economic modeling to represent the embedded optionality without a full-blown stochastic analysis. It helps us identify risks, design hedging programs and effectively communicate the features of embedded options on the liability side using the language with which investment communities are familiar.

A case study that applies different valuation approaches to a credit interest rate floor option illustrates the process of assumption setting, fair valuation and risk analysis. Accompanying this case study, several tools were built with comprehensive structures and documented implementation processes. The tools provide functions like economic assumption calibration, economic scenario generation (ESG), scenario validation and option value calculations using the three approaches listed above. To make this paper as self-contained as possible, some well-known models that have been built into the tools and used in the case study are described as well. However, models and assumptions used in the case study or the tools may not be suitable for actual valuation since many technical details are omitted to make the tools as widely accessible as possible. It is hoped that the report and the tools introduce pension plan embedded option valuation in an easy and practical way.

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1. Introduction

Since the 2008 financial crisis, the value of retirement accounts has decreased and the funding level and capital position of pension plans have deteriorated. In addition to the drop in asset value, various embedded options in pension plans became in the money, which led to a bigger funding shortfall. However, the true economic cost of the embedded options, as well as potential accounting and capital implications, was not fully understood and quantified.

Pension plans have been providing guarantees to plan participants in a variety of ways, such as guaranteed cost-of-living adjustments, a credit interest rate floor on a cash balance plan and lump-sum subsidy at retirement. To focus on the structure and process of embedded option valuation, a very straightforward type—a credit interest rate floor on a cash balance plan—was chosen for this report.

Cash balance plans often have embedded options in the form of a floor to the interest rate credit. The account is credited annually with a fixed percentage of salary and also earns interest credits at the maximum of n-year Treasury bond yield and a fixed rate. For example, an account is credited annually with 4 percent of salary and interest credits determined by the maximum of 10-year Treasury bond yield and 4 percent. The salary is the actual salary the plan participant earns when the account is credited.

Modern finance theory has developed sophisticated techniques for financial option valuation. It is beneficial to understand whether and how those valuation methods can be applied to fair valuation and risk assessment of embedded options in pension plans. This research explores the application of modern valuation techniques for embedded options offered in pension plans. Three approaches are discussed regarding the underlying theory and its application.

- (1) Closed-form valuation. By taking the assumptions of the Black-Scholes-Merton¹ model, some embedded options in pension plans can be evaluated using an option pricing formula. Those guarantees can be translated to a set of standard financial options with adjustment to reflect the features of the pension plan. The Black-Scholes model was developed to value European options, which can only be exercised at the expiration date. The differences between European options and pension plan options are as follows.
 - (a) The payoff condition of the embedded option in a pension plan could be more complicated than a European option. It normally has multiple discrete exercise dates with the underlying asset/rate based on cumulative measures such as cumulative investment return or the consumer price index (CPI).
 - (b) The payoffs of an embedded option often are contingent on the policyholder's death or survival.

¹ The Black-Scholes-Merton model, aka Black-Scholes model, is a mathematical model of option pricing/valuation. It includes a pricing formula for European options. Given some key assumptions, "it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock." Black and Scholes, "Pricing of Options," 641.

- (c) The underlying fund value needs to reflect the deduction of management fees.
- (2) Stochastic approach. If closed-form valuation cannot provide a reasonable estimation, the stochastic approach may be the only method to quantify the option's value. It can be used for two purposes.
- (a) Fair valuation using risk-neutral scenarios. Risk-neutral scenarios adjust the probability assigned to each scenario to get a market-consistent model price. The expected value under risk-neutral scenarios is expected to be the same as the market value in a nonarbitrage world.
- (b) Risk analysis using real-world scenarios. Real-world scenarios can be used for assessing the risks of embedded options. It gives a comprehensive and realistic picture of the possible outcomes. This is very helpful for economic capital management.
- (3) Replicating portfolio approach. It is more suitable for risk management than for valuation and is often used in asset liability management, economic capital modeling and other "what if" analysis. Based on the liquid assets available in the market, the replicating portfolio approach selects appropriate securities that mimic the characteristics of embedded options in pension plans. Available investment instruments include, but are not limited to, fixed income bonds, stocks, swaptions, caps, floors, stock index options, Treasury Inflation Protection Securities (TIPS)² and catastrophe (CAT) bonds.³ Using the replicating portfolio approach, the sensitivities of the fair value of embedded options with respect to financial market movements can be better understood, including interest rate, equity price and market volatility. During market turmoil, the replicating portfolio can give a quick estimation of the change in the fair value of embedded options. In addition, it also provides good insight into the types of hedging instruments that can help mitigate the risks arising from the embedded options.

A major part of this research is building a set of embedded option valuation tools to illustrate several valuation approaches for educational purposes. The tools are as follows.

- (1) Economic assumption calibration tool. This spreadsheet calibrates interest rate, equity and inflation models to asset instrument prices. This tool can be used to fit a one-factor Hull-White interest rate model, lognormal equity return model and lognormal inflation model to the market price of asset instruments including zero coupon bonds, interest rate caps and floors, equity index options and caplets on inflation rates.
- (2) Economic scenario generation (ESG) tool. This spreadsheet model generates risk-neutral and real-world scenarios. This tool can be used to generate stochastic interest rates (one-factor Hull-White model), bond yields, credit spreads (one-factor Hull-White intensity model), equity returns (lognormal model) and inflation rates (lognormal model).

² TIPS are inflation indexed bonds that have principal adjusted to CPI realization.

³ A CAT bond is a market instrument that transfers catastrophe risk from the issuer to the investor. The issuer will pay coupon if no specified catastrophe happens while the investor will lose the principal if some catastrophic event occurs. See U.S. General Accounting Office, "Catastrophe Insurance Risks."

They can be generated individually or together with correlation built among economic variables.

- (3) Economic scenario validation tool. This spreadsheet model validates the market consistency of stochastic scenarios. Two types of tests are included: martingale and equity implied volatility.
- (4) Replicating portfolio tool. This spreadsheet model replicates embedded options with assets including bonds, equity, equity index options, forward-starting swaps and swaptions. It uses the singular value decomposition method⁴ to find an asset portfolio with good fit.
- (5) Pension plan valuation model. This spreadsheet model can project scenario-based cash flows and calculate the fair value of embedded options. In practice, an advanced actuarial valuation system needs to be used for stochastic valuation of large pension plans.

The following three sections discuss three valuation approaches to embedded options. In Section 5, a case study of valuing the credited interest rate floor in a cash balance plan illustrates how the principles developed in sections 2 through 4 can be applied. Supplementary information of the valuation tools is included in the appendixes.

⁴ Details about singular value decomposition are included in [Section 4.1](#).

2. Closed-Form Valuation

2.1 Black-Scholes-Merton Framework

In this report, closed-form valuation means using a mathematic formula to directly calculate the value of embedded options in pension plans. To make use of the simplicity of the closed-form valuation method, some assumptions are made. In this section, classical Merton (1971)⁵ and Black and Scholes continuous time framework is assumed, which includes $n + 1$ securities (funds) that are traded continuously and one of the assets is risk free. The price of the risk-free asset follows the differential equation below.

$$\frac{dP^{rf}(t)}{P^{rf}(t)} = rdt \quad r > 0$$

The n risky assets have the following stochastic price-evolving process.

$$\frac{dP^i(t)}{P^i(t)} = \mu_i dt + \sigma_i dZ^i(t) \quad P^i(t) = P_{initial}^i \cdot \exp\left[\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i Z^i(t)\right],$$

in which $Z^i(t)$ is standard Brownian motion correlated with each other and $P^i(t) / P_{initial}^i$

follows lognormal distribution with parameters $\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t, \sigma_i Z^i(t)\right)$.

The value of European call options and put options can be calculated using the closed-form formula. A call option gives the option holder the right to buy the asset at a certain price (exercise price K) at prescribed date(s). A European call option allows the holder to exercise the option only at the option maturity date T . If the asset price P_T at maturity is higher than the exercise price, the option will be exercised to gain a payoff of $P_T - K$. The value of the European call option can be calculated using the famous Black-Scholes formula below.

$$V_c = P_t \Phi(d_1) - Ke^{-r_{T-t}T-t} \Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{P_t}{K}\right) + \left(r_{T-t} + \frac{\sigma_{T-t}^2}{2}\right)(T-t)}{\sigma_{T-t} \sqrt{T-t}} \quad d_2 = d_1 - \sigma_{T-t} \sqrt{T-t} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

where σ_t is the implied volatility of a call option maturing at time t and r_t is the continuous risk-free rate with a term of t .

The put option gives the option holder the right to sell the asset at a certain price (exercise price K) at prescribed date(s). The European put option value can be calculated using the formula below with the same notations as those used in European call option price formula.

⁵ Merton, "Theory of Rational Option."

$$V_p = -P_t \Phi(-d_1) + Ke^{-r_{T-t} \cdot (T-t)} \Phi(-d_2)$$

2.2 Cash Balance Plan with a Credit Interest Rate Floor

A cash balance plan normally has the account credited annually with a fixed percentage of current salary and investment income. With a floor, the credit interest rate is determined as the greater of Treasury bond yield and a fixed rate X percent. It can be broken down into two parts:

(1) Treasury bond yield

$$\frac{dS(t)}{S(t)} = \mu_t dt + \sigma dZ(t)$$

S : asset index with return equal to Treasury bond yield

and,

(2) A compound annual ratchet⁶ $(1 + X\% - \frac{S(t)}{S(t-1)})^+$

The account value is determined as $F_t \prod_{i=t}^{t_{ra}} \left[1 + X\% + \left(\frac{S(t)}{S(t-1)} - (1 + X\%) \right)^+ \right]$.

If only accumulated benefits are considered, account value at time t $F_t = F_{t-1} \cdot e^{rt}$ ⁷

r: continuous risk-free rate

The cost of the embedded option can be valued as below. Here, a flat volatility term structure has to be assumed in order to have an analytic solution.

$$V_{abo,t} = \sum_{i=t+1}^{t_{ra}-1} \left[D_i \cdot \frac{IF_i}{IF_t} \cdot \left\{ \frac{DF_i}{DF_t} \cdot F_t \prod_{j=t}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - F_t \right\} \right] \quad (I)$$

$$+ \frac{IF_{t_{ra}}}{IF_t} \cdot \left\{ \frac{DF_{t_{ra}}}{DF_t} \cdot F_t \prod_{j=t}^{t_{ra}} \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - F_t \right\} \quad (II)$$

Part I is the value of the embedded option for plan participants who exit the plan due to death or termination. Part II is the value of the embedded option for plan participants who remain in the plan until retirement.

$$IF_t = IF_{t-1} (1 - l_t - q_{start\ age+t})$$

⁶ Hardy, *Investment Guarantees*, 247–57.

⁷ There may be a fund management fee collected in other types of pension plans and it needs to be reflected in the fund value projection.

IF_t : survival factor at time t

q_x : mortality rate for the plan participant at age x

l_t : termination rate for the plan participant at time t

D_x : $q_{start\ age+t} + l_t$, decrement rate for the plan participant at time t

$$DF_t = DF_{t-1} \cdot e^{-r_t}$$

r_t : continuous risk-free rate at time $t - 1$

The value of CAR option with payoff $\prod_{j=t}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right]$:

$$V_{CAR,t}^i = \left[(1 + X\%) \cdot e^{-r} + (1 \cdot N(d_1) - (1 + X\%)e^{-r}N(d_2)) \right]^{i-t}$$

$$d_1 = \frac{\log(1/(1 + X\%)) + \frac{1}{2}(r + \sigma^2)(1)}{\sigma\sqrt{1}} \quad d_2 = d_1 - \sigma$$

CAR : Compound Annual Ratchet

However, the validity of the valuation formula above depends on an underlying assumption that the portfolio return or the credit interest rate is attainable. In other words, if there is not any trading strategy that can achieve the specified return, a standard closed-form valuation method based on no arbitrage pricing theory will not generate a fair price. For example, if the annual credit interest rate is the then-current 10-year Treasury bond yield, unless the bond yield term structure is flat, there is no trading strategy that can generate the specified return. For example, if one-year bond yield is 3 percent and 10-year bond yield is 5 percent, a locked-in investment return of 5 percent in the next year is not achievable. Otherwise, there will be an arbitrage opportunity. Investors can borrow money for one year at a low rate and invest it to get a locked-in return of 5 percent in one year. Therefore, the value of the embedded option needs to be adjusted, like below.

$$V_{abo,t} = \sum_{i=t+1}^{t_{ra}-1} \left[D_i \cdot \frac{IF_i}{IF_t} \cdot \frac{DF_i}{DF_t} \cdot \left\{ F_t \prod_{j=t}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - F_t \cdot \frac{AF_i}{AF_t} \right\} \right] \\ + \frac{IF_{t_{ra}}}{IF_t} \cdot \frac{DF_{t_{ra}}}{DF_t} \cdot \left\{ F_t \prod_{j=t}^{t_{ra}} \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - F_t \cdot \frac{AF_{t_{ra}}}{AF_t} \right\}$$

$$AF_t = AF_{t-1} \cdot e^{\mu_t}$$

μ_t : continuous Treasury bond yield at time $t - 1$

The value of option with payoff $\prod_{j=t}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right]$

$$V_{CAR,t}^i = \left[(1 + X\%) \cdot e^{-r} + (1 \cdot N(d_1) - (1 + Y\%)e^{-r}N(d_2)) \right]^{i-t}$$

$$d_1 = \frac{\log(1/(1 + Y\%)) + \frac{1}{2}(r + \sigma^2)(1)}{\sigma\sqrt{1}} \quad d_2 = d_1 - \sigma$$

$$Y\% = X\% - \left[\left(\frac{AF_i}{AF_t} \cdot \frac{DF_i}{DF_t} \right)^{\frac{1}{i-t}} - 1 \right]$$

The adjustment of the floor from X percent to Y percent in the valuation formula enables us to

apply the standard Black-Scholes model. The value of a floor on 10-year Treasury bond yield is estimated as the value of a floor on the risk-free rate after excluding the term premium from the floor.

If projected benefits are considered, projected account value at time t

$$F_t = F_{t-1} \cdot e^{rt} + C_t$$

C_t : Contribution at time t

The cost of embedded option can be valued as below.

$$V_{pbo,t} = V_{abo,t} + \sum_{k=t}^{t_{ra}-1} (A + B)$$

$$A = \sum_{i=k+1}^{t_{ra}-1} \left[D_i \cdot \frac{IF_i}{IF_t} \cdot \frac{DF_i}{DF_t} \cdot \left\{ C_k \prod_{j=k}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - C_k \cdot \frac{AF_i}{AF_k} \right\} \right]$$

$$B = \frac{IF_{t_{ra}}}{IF_t} \cdot \frac{DF_{t_{ra}}}{DF_t} \cdot \left\{ C_k \prod_{j=k}^{t_{ra}} \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right] - C_k \cdot \frac{AF_{t_{ra}}}{AF_k} \right\}$$

The value of option with pay off $\prod_{j=k}^i \left[1 + X\% + \left(\frac{S(j)}{S(j-1)} - (1 + X\%) \right)^+ \right]$:

$$V_{CAR,k}^i = \left[(1 + X\%) \cdot e^{-r} + (1 \cdot N(d_1) - (1 + Y\%) e^{-r} N(d_2)) \right]^{i-k}$$

$$d_1 = \frac{\log(1/(1 + Y\%)) + \frac{1}{2}(r + \sigma^2)(1)}{\sigma \sqrt{1}} \quad d_2 = d_1 - \sigma$$

$$Y\% = X\% - \left[\left(\frac{AF_i}{AF_k} \cdot \frac{DF_i}{DF_k} \right)^{\frac{1}{i-t}} - 1 \right]$$

3 Stochastic Valuation

The stochastic valuation method uses stochastic scenarios to project stochastic cash flows; those cash flows are then discounted back using scenario-dependent discount rates. The average present value of the liability values discounted using scenario-dependent interest rates is the estimated liability value. With embedded options considered in cash flow projections, the value of the options can be estimated by discounting the extra cash flow caused by the option.

There are two types of stochastic scenarios: risk-neutral scenarios and real-world scenarios. Market-consistent risk-neutral valuation uses risk-neutral scenarios for discounting while adjusting the probability of scenarios to match the average discount value with market value. On the other hand, market-consistent real-world scenarios use the sum of the risk-free rate and implied risk premium as the discount rate. Theoretically, both risk-neutral scenarios and real-world scenarios can be used for market-consistent valuation; however, risk-neutral scenarios are preferred as it is not practical to determine the implied risk premium required for real-world market-consistent valuation. Using risk-neutral scenarios, only the average discount value is useful as the distributions implied by risk-neutral scenario probability are not realistic. Real-world scenarios are used for other purposes and real-world analysis can tell us the possible outcomes. The distribution of the outcomes, and risk measures such as value at risk (VaR) and conditional tail expectation (CTE) available in real-world analysis, are very useful in pension plan funding management and risk-return analysis.

3.1 Economic Assumption Setting

The stochastic valuation process is vulnerable to “garbage-in, garbage-out” if the input assumptions and data are treated lightly. One of the key considerations in valuing financial options is how to determine and calibrate volatility, its term structure and the interdependency of economic variables. The embedded option value is very sensitive to volatility in most cases. In this section, we will go through the assumption-setting process with a focus on calibration to market price of financial instruments.

3.1.1 Yield Curve Extension

3.1.1.1 Initial Yield Curve

There are two choices of risk-free curve commonly accepted: swap curve and government bond yield curve. The longest liquid term of the government bond market is normally longer than that for the swap curve. Therefore, the government bond yield curve might be more appropriate for long-term liability valuation. Sometimes, swap curves are used but extended based on an assumed relationship between swap curve and government bond yield. For example, when there is a lack of liquidity for the long-term swap rate, the long-term

government bond yield plus a spread is used to represent the risk-free rate. The spread is determined as the difference between the swap rate and government bond yield at the longest liquid term for the swap market. The risk-free curve is the baseline for generating risk-neutral scenarios. In some cases, an illiquidity premium is added to the risk-free curve for discounting purposes. The illiquidity premium is an important factor in determining fair value of embedded options but it is not covered in this paper.⁸

Due to the long-term nature of pension plan liability, the market available yield curve needs to be interpolated and extrapolated for discounting liability cash flow. There are some desirable properties of the extended yield curve.⁹

- (1) Accuracy. The yield curve should accurately replicate the price of liquid financial instruments.
- (2) Continuity. The curve should be continuous.
- (3) Smoothness. The curve should be smooth.
- (4) Volatility decay. Volatility should decrease with time (term).

Property 1 reflects the requirement that the interpolation should be accurate. Where there is a liquid market, the yield curve should accurately price the market instruments. Property 4 relates to extrapolation. The volatility of long-term rates is normally lower than short-term rates.¹⁰ Properties 2 and 3 relate to both interpolation and extrapolation. It requires both the curve and the first derivative of the curve to be continuous.

3.1.1.2 Interpolation

In this report and the accompanying tool, cubic spline method¹¹ is used for yield curve interpolation. The assumed form used in cubic polynomial curve fit for each segment (x_i, x_{i+1}) is

$$y = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

It allows a perfect fit to the market data and ensures the continuity and smoothness of the resulting yield curve. Assuming there are n data points derived from market data, a typical curve fit involves forming one equation through all n points. In contrast, a spline allows each segment to have a unique equation. However, because the method involves connecting individual segments, it may create undesired oscillation in the fitted curve. Overall, the cubic spline provides a good curve fitting for arbitrary data points.

The cubic spline method is also easy to implement. [Appendix D](#) describes the algorithm used

⁸ Committee of European Insurance and Occupational Pensions Supervisors' "Task Force Report on the Liquidity Premium" is a good reference material for liquidity premium and methods of estimation.

⁹ Sorensen, "Interest Rate Calibration," 5.

¹⁰ Karasinski, "What Drives Interest Rates Volatility?" provides some empirical evidence.

¹¹ Mathews and Fink, *Numerical Methods*, 281–87. The algorithm used in the economic scenario generation tool is based on this reference.

for the case study.

3.1.1.3 Extrapolation

In some cases, constant forward rate or constant spot rate assumption is used for yield curve extrapolation. They are simple to use but may not be appropriate for long-term interest rate assumption. The extrapolated long-term rates on the fitted curve can be very volatile across time and there is no economic theory to support these assumptions.

Nelson-Siegel Model

The idea of the Nelson-Siegel (N-S)¹² approach is to fit the empirical form of the yield curve with a prespecified functional form of the spot rates. It assumes that the spot rates are affected by three components. Short-term, medium-term and long-term components will affect the spot rates at different levels. The effects of the short-term and medium-term contributions will vanish as time passes.

The Nelson-Siegel model specifies the spot rate function by:

$$r_m = \beta_0 + \beta_1 \left(\frac{1 - e^{(-m/\tau_1)}}{m/\tau_1} \right) + \beta_2 \left(\frac{1 - e^{(-m/\tau_2)}}{m/\tau_2} - e^{(-m/\tau_2)} \right)$$

$\beta_0, \beta_1, \beta_2, \tau_1, \tau_2$ are parameters to be estimated. Here, β_0 corresponds to the long-term spot rate, β_1 corresponds to the short-term contribution component, β_2 corresponds to the medium-term contribution component, τ_1 corresponds to the volatility decay factor for short term, and τ_2 corresponds to the volatility decay factor for medium term. m is the term to maturity.

To estimate the parameters, an objective function needs to be chosen. Intuitively, the modeled price should be as close to the market price as possible. The goal is to have a minimum sum of squared errors between the modeled prices and the market prices.

Note that there are five parameters to be estimated, which will make the results unstable and largely depend on the initial guess of the value of the parameters in the optimization process.

One way to get rid of this problem is to predetermine some parameters according to economic theory and historical data. The long-term rate (β_0) and the volatility decay factors (τ_1, τ_2) can be determined exogenously. The number of parameters to be calibrated in the optimization process is reduced to only two (β_1 and β_2). It makes the estimation of

¹² Nelson and Siegel, "Parsimonious Modeling," 473–89.

parameters more stable with different initial values but without loss of accuracy.

Long-Term Interest Rate

Fixing the long-term target rate will generate less volatile long-term interest rates over time. Long-term rates at different times will not differ too much from each other as the rate is largely determined by the availability of economic resources. Other methods such as constant forward rate extrapolation may have a volatile ultimate forward rate implied from the market yield curves. In contrast, N-S extrapolation with a predetermined fixed long-term target will produce a less fluid long-term part of the fitted yield curve. This method would generate more stable and reasonable results for the valuation of long-term cash flows.

The long-term forward interest rate consists of five components:

- (1) Long-term expected real return
- (2) Long-term expected inflation
- (3) Currency risk premium
- (4) Long-term premium
- (5) Convexity adjustment

The first two components are the real return and inflation rate in equilibrium. The currency risk premium is the risk premium that global investors will demand for holding certain currencies. The term premium is the difference between the expected return on risk-free long-dated bonds and the risk-free short-term interest rate. Liquidity preference theory assumes positive term premium or liquidity premium for holding long-term bonds. On the other hand, under market segmentation theory and preferred habitat theory, the term premium can be positive or negative as the supply and demand of short- and long-term bond markets are independent. For the long-term interest rates, liquidity preference theorem may be more appropriate because in equilibrium, supply and demand breaks even and liquidity plays a key role in term premium. Therefore, a positive long-term premium is preferred. The fifth component, convexity adjustment, is used to reflect the difference between the long-term forward rate and expected long-term future interest rate, caused by the nonlinear relationship between the bond price and interest rates.

Volatility Decay Factors

The two parameters (τ_1, τ_2) in the N-S function represent the short-term and medium-term volatility decay factors respectively. The higher value of these two parameters means a lower decay speed of volatility. To estimate the values, let's revisit the N-S function.

$$r_m = \beta_0 + \beta_1 \left(\frac{1 - e^{\left(\frac{-m}{\tau_1}\right)}}{m/\tau_1} \right) + \beta_2 \left(\frac{1 - e^{\left(\frac{-m}{\tau_2}\right)}}{m/\tau_2} - e^{\left(\frac{-m}{\tau_2}\right)} \right)$$

Assume the β_0 , β_1 and β_2 are independent variables and we get the following equation.

$$Var(r_m) = Var(\beta_0) + Var(\beta_1) * \left(\frac{1 - e^{(-m/\tau_1)}}{m/\tau_1} \right)^2 + Var(\beta_2) * \left(\frac{1 - e^{(-m/\tau_2)}}{m/\tau_2} - e^{(-m/\tau_2)} \right)^2$$

If the 0.5-year interest rate and the one-year interest rate are treated as short-term rates, and the 10-year interest rate and the 15-year interest rate are treated as medium-term rates, the value of τ_1 and τ_2 can be estimated by the following process.

τ_2 , which is related to the medium-term volatility decay factor, can be calculated using the following equation.

$$\frac{\frac{1 - e^{(-10/\tau_2)}}{10/\tau_2} - e^{(-10/\tau_2)}}{\frac{1 - e^{(-15/\tau_2)}}{15/\tau_2} - e^{(-15/\tau_2)}} = \frac{\sigma_{10}}{\sigma_{15}}$$

σ_{10} and σ_{15} could be estimated as the implied volatility of the 10-year and 15-year at-the-money (ATM) swaptions, respectively. Here, it is assumed there is no short-term volatility effect in the volatility of interest rates with terms more than 10 years.

τ_1 , which is related to the short-term volatility decay factor, can be calculated by the following process.

- (1) Determine the medium-term effect that should be removed from the total volatility of the 0.5-year and one-year interest rates.

$$Vol_{0.5,remove} = \sigma_{10} \cdot \frac{\frac{1 - e^{(-0.5/\tau_2)}}{0.5/\tau_2} - e^{(-0.5/\tau_2)}}{\frac{1 - e^{(-10/\tau_2)}}{10/\tau_2} - e^{(-10/\tau_2)}} \quad Vol_{1,remove} = \sigma_{10} \cdot \frac{\frac{1 - e^{(-1/\tau_2)}}{1/\tau_2} - e^{(-1/\tau_2)}}{\frac{1 - e^{(-10/\tau_2)}}{10/\tau_2} - e^{(-10/\tau_2)}}$$

$$(2) \text{ Estimate } \tau_1 \text{ by solving function } \frac{\left(\frac{1 - e^{(-0.5/\tau_1)}}{0.5/\tau_1} \right)}{\left(\frac{1 - e^{(-1/\tau_1)}}{1/\tau_1} \right)} = \frac{\sigma_{0.5} - Vol_{0.5,remove}}{\sigma_1 - Vol_{1,remove}}$$

3.1.2 Interest Rate Model (One-Factor Hull-White Model)

Now that we have a method to determine the basic yield curve, we need to model interest rate volatility.

There are many interest rate models that can be used to represent the dynamics of interest rate movements. In this paper, the one-factor Hull-White model is used because of its simplicity and appropriateness for illustration. Other more complicated interest rate models exist that are more flexible and can better fit to market prices and market volatility.

Hull and White¹³ developed a no-arbitrage term structure model that allows mean reversion. Under this model, the analytic formula is used to value several market instruments. The interest rate evolves according to the diffusion process given below.

$$dr = (\theta(t) - ar)dt + \sigma dz$$

Cap and Floor

Market instruments such as cap/floor and swaption can be used to calibrate the volatility parameters. Given the analytical tractability of the one-factor Hull-White model, it is relatively easy to calibrate the price parameters of caps or floors.

Cap/floor price can be calculated as a portfolio of put/call options on bonds where closed-form valuation exists for the one-factor Hull-White model.

Analytical price of a call option on a bond:

$$c = LP(0,t)N(h) - XP(0,T)N(h - \sigma_p)$$

where

$$h = \frac{1}{\sigma_p} \ln \frac{LP(0,t)}{P(0,T)X} + \frac{\sigma_p}{2} \quad \sigma_p = \frac{\sigma}{a} \left[1 - e^{-a(t-T)} \right] \sqrt{\frac{1 - e^{-2aT}}{2a}}$$

t: maturity of the bond

T: maturity of the option

L: notional amount

X: exercise price of the option on bonds calculated as L growing at exercise rate till the exercise dates of the caplets/floorlets

Swaption

Swaption is another choice for interest rate volatility calibration. The trinomial tree version of the one-factor Hull-White model needs to be used for calibration purposes as there is no closed-form solution for swaption price. This trinomial tree model was presented by Hull and

¹³ Hull and White, "One-Factor Interest Rate," 235–54.

White in 1996.¹⁴ With the interest rate trinomial tree, swaption price can be valued using a cash flow discounting method along each path with corresponding probabilities.

3.1.3 Holistic ESG: Dependency

In a global scenario, the interaction of different components and the underlying drivers need to be considered. This includes both the correlation between different economies and the correlation between different economic variables such as interest rate, credit spread, equity return and inflation rate. The correlation among economic variables exerts another constraint we must reflect when generating economic scenarios. The case study in Section 5 uses the correlation matrix approach. Although methods like copulas can better capture tail risk, a correlation matrix can give us reasonable estimation of the mean value. This is essential for fair valuation.

The Cholesky decomposition¹⁵ method is used to decompose the correlation matrix given that it is semi-positive definite. It is instrumental in generating correlated diffusion processes that have a specified correlation matrix.

Cholesky Decomposition

A symmetric and positive-definite matrix can be decomposed as $A=LL^T$.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix} \times \begin{pmatrix} l_{11} & l_{21} & \dots & l_{n1} \\ 0 & l_{22} & \dots & l_{n2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & l_{nn} \end{pmatrix} \quad \begin{aligned} l_{ii} &= \sqrt{(a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2)} \\ l_{ji} &= (a_{ji} - \sum_{k=1}^{i-1} l_{jk}l_{ik})/l_{ii} \end{aligned}$$

For example, the variance-covariance matrix has dimension (3×3) ; then we can get

$$\begin{aligned} l_{11} &= \sqrt{a_{11}} & l_{21} &= a_{21} / l_{11} & l_{22} &= \sqrt{(a_{22} - l_{21}^2)} \\ l_{32} &= (a_{32} - l_{31}l_{21})/l_{22} & l_{33} &= \sqrt{(a_{33} - l_{31}^2 - l_{32}^2)} \end{aligned}$$

For $\epsilon_i \sim N(0,1)$ and i.i.d., $L \epsilon$ has variance-covariance matrix A; ϵ_i can be generated from $N^{-1}(U_i)$, where U_i is uniformly distributed; and $L \epsilon$ is used as correlated random numbers for global scenario generation. Independent normal random numbers multiplied by matrix L can generate correlated normal random numbers and then correlated diffusion processes. In this way, appropriate correlation is built in the generated stochastic scenarios.

¹⁴ Hull and White, "Using Hull-White," 26–36.

¹⁵ "Cholesky decomposition ... is a decomposition of a Hermitian, positive-definite matrix into the product of a lower triangular matrix and its conjugate transpose." *Wikipedia*, s.v. "Cholesky decomposition," http://en.wikipedia.org/wiki/Cholesky_decomposition.

3.2 Valuation and Risk Assessment

Stochastic analysis can help both fair valuation and quantification of risks for embedded options in pension plans.

The fair value of embedded options in pension plans is calculated by projecting pension plan cash flows along each scenario and discounting back the cash flows attributable to the embedded options using scenario-dependent rates. Risk-neutral scenarios are used for valuation purposes. They can produce market-consistent model prices for financial instruments traded in the market and provide reasonable estimation of the value of future cash flows that are not traded in the real market.

On the other hand, with real-world scenarios, people can have a more realistic forecast of the future. It provides possible outcomes under different assumed market conditions. A common measure is value at risk (VaR), which is the value at a certain probability of the distribution. For example, 95 percent VaR is the 95th percentile of the loss distribution. Sometimes for skewed loss distribution with a long tail, VaR may underestimate the tail risk as it does not consider the magnitude of the loss in the tail. Conditional tail expectation (CTE), aka tail value at risk (TVaR), provides more comprehensive information about the tail risk. These risk measures help assess the risk profiles of pension plan asset holdings, liabilities and the cost of embedded options. Combined with further modeling and a practical accounting understanding, such risk analysis would also help plan sponsors prepare appropriate capital funding plans, especially for stress scenarios.

4. Replicating Portfolio

The replicating portfolio uses a set of asset instruments to replicate the characteristics of the liability. The set of instruments needs to replicate not only the liability value but also its distribution. By doing this, both the value and the market risk sensitivities can be replicated. In the short term, the market value of the asset portfolio could also be a reasonable and quick estimate for the liability value.

4.1 Methodology

Stochastic liability cash flows and stochastic scenarios such as interest rate, par yield and equity index are necessary inputs for replication of the liability value. In addition to replicating the baseline value, the replicating portfolio tries to match the value under stressed scenarios. Stress scenarios have shocked economic variables such as yield curve and equity volatility.

Many optimization methods can be used to find the best match between the liability profile and available liquid assets. In this paper, singular value decomposition is applied to decompose the matrix that contains information about stochastic cash flows and/or values under shocked scenarios. Instead of applying optimization techniques to find the global optimal portfolio that matches the value and cash flow under each stochastic scenario, the scenario average value of target variables is replicated under both baseline and shocked situations. This approach prevents the situation when optimization techniques fail to find a global optimal solution. However, to achieve reasonable fit under most scenarios, trial and error needs to be used to change the financial instruments selected in the replicating portfolio. This method works well if the basic features of the liability cash flows and embedded options are well understood. Appropriate asset types can be chosen based on the features.

Replication Model Setting

The following linear equation is set up to solve the weight for each asset instrument.

$$A \cdot x = b$$

$x_{(ni+m) \times 1}$: weight for each selected investment instrument ($x_1, x_2, \dots, x_{ni}, 0, \dots, 0$)

$$A_{1(ni \times ni)} = \left[\sum_{n=1}^{ns} z_{i,n} \cdot z_{j,n} \cdot \right] \quad b_{1(ns \times 1)} = \left[\sum_{n=1}^{ns} z_{i,n} \cdot L_n \right]$$

$z_{i,n}$: value of selected instrument i under scenario n

L_n : value of discounted liability cash flows under scenario n

ni : number of investment instruments

ns : number of stochastic scenarios

$$A = \begin{bmatrix} A_1 & p_1 & \dots & \dots & p_m \\ p_1 & & & & \\ \dots & & & & \\ \dots & & & 0 & \\ p_m & & & & \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ L_1 \\ \dots \\ L_m \end{bmatrix}$$

$p_{i,j}$: value of instrument i under sensitivity j

$P_j = (P_{1,j}, P_{2,j}, \dots, P_{ni,j})$

L_j : value of discounted liability cash flow under sensitivity j

m : number of sensitivities. The sensitivities include the deterministic, stochastic baseline and stochastic stressed scenarios.

Singular Vector Decomposition (SVD)¹⁶—Technical Notes

Matrix A is $(ni + m) \times (ni + m)$ dimension. The rank of $A^T A$ is r .

$V = \{V_1, V_2, \dots, V_r\}$ is r eigenvector of $A^T A$. V_i is $(ni + m) \times 1$ matrix.

$$(A^T A)V_i = \lambda_i V_i \quad w_i = \sqrt{\lambda_i}$$

$U = \{U_1, U_2, \dots, U_r\}$ is r eigenvector of AA^T . U_i is $(ni + m) \times 1$ matrix.

$$(AA^T)U_i = \lambda_i U_i \quad U_i = \frac{1}{w_i} A \cdot V_i$$

$$W = \begin{bmatrix} w_i & & & \\ & \dots & & 0 \\ & & w_r & \\ & 0 & & 0 \\ & & & & 0 \end{bmatrix} \quad A = U \cdot W \cdot V^T$$

Base on the SVD results, the weight for each selected asset instruments can be solved.

$$A \cdot x = b$$

$$\Rightarrow U \cdot W \cdot V^T \cdot x = b$$

$$As U \cdot U^T = I \text{ and } V \cdot V^T = I$$

$$x = V \cdot \text{diag}(1/w_i) \cdot U^T \cdot b$$

One possible type of interest rate stress scenario used in the replicating process is based on principal component analysis (PCA). It explains both the volatility of the yield curve and the correlation among different terms on the yield curve. PCA is explained in [Appendix C](#) and applied in the case study in Section 5.

4.2 Financial Instruments

The following financial instruments have been built in the replication tool for illustration.

- Zero coupon bond
- Forward-starting swap
- Equity index
- Equity index call/put

¹⁶ Shlens, "A Tutorial," 7–9.

- Payer/receiver swaption

Details about the features, pricing formula and sensitivities of these asset types can be found in [Appendix B](#).

5. Case Study

In this section, we will go through the valuation of the credit interest rate floor in a cash balance plan step by step. The credit interest rate is determined as the maximum of the floor and 10-year Treasury bond yield. First, the process of calibrating economic assumptions is discussed. Most of the data used in the calibration are based on December 2010 market data but some of them are not. It is necessary to keep in mind that all the market data used in the case study are for illustration purposes only. They do not represent any historical market situation or forward-looking market expectation. Those include, but are not limited to, the yield curve and the market price of bonds and financial derivatives.

5.1 Economic Assumption

Initial Yield Curve

In the case study, USD is assumed to be the only currency the pension plan will invest in. The U.S. Treasury bond yield curve is used as the risk-free curve. However, it is common that the market data do not cover all the term points and that the long-term bond market is not liquid or even does not exist. Since a pension plan usually has a projection period longer than the asset market, the yield curve needs to be extended.

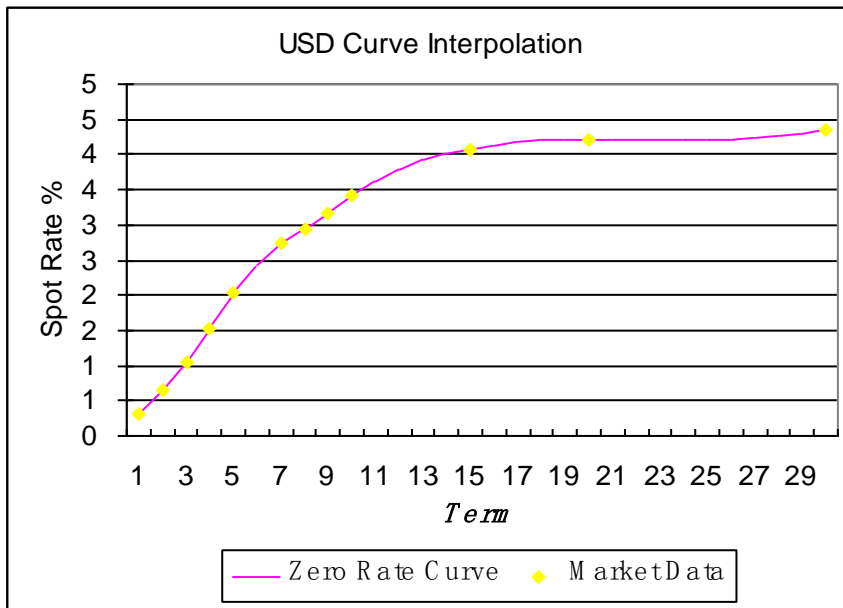
- (a) Input market data. Assume at valuation date, the USD Treasury bond market has the following yield curve.

Table 5.1.1

Term	1	2	3	4	5	7	8	9	10	15	20	30
USD Risk Free Rate (%)	0.30	0.64	1.05	1.54	2.03	2.74	2.94	3.17	3.42	4.08	4.22	4.35

- (b) Yield curve interpolation. The cubic spline method is used for yield curve interpolation. The term structure is interpolated till the longest market available term point (30 years). Figure 5.1.1 shows that the interpolated yield is a smooth yield curve that can accurately replicate all the market data.

Figure 5.1.1 USD Curve Interpolation



- (c) Yield curve extrapolation. Pension plan liability cash flows usually are longer term than liquid assets. For valuation purpose, we need to extrapolate the yield curve beyond the longest market available term (30 years) to 100 years. The Nelson-Siegel model is used. As discussed in [Section 3.1.1](#), three parameters are fixed based on economic forecast and market data: long-term interest rate and decay factors for short-term and medium-term components.

$$r_m = 0.0514 + \beta_1 \left(\frac{1 - e^{-m/0.32}}{m/0.32} \right) + \beta_2 \left(\frac{1 - e^{-m/5.40}}{m/5.40} - e^{-m/5.40} \right)$$

where: $\beta_0=5.14\%$
 $\tau_1=0.32$
 $\tau_2=5.40$

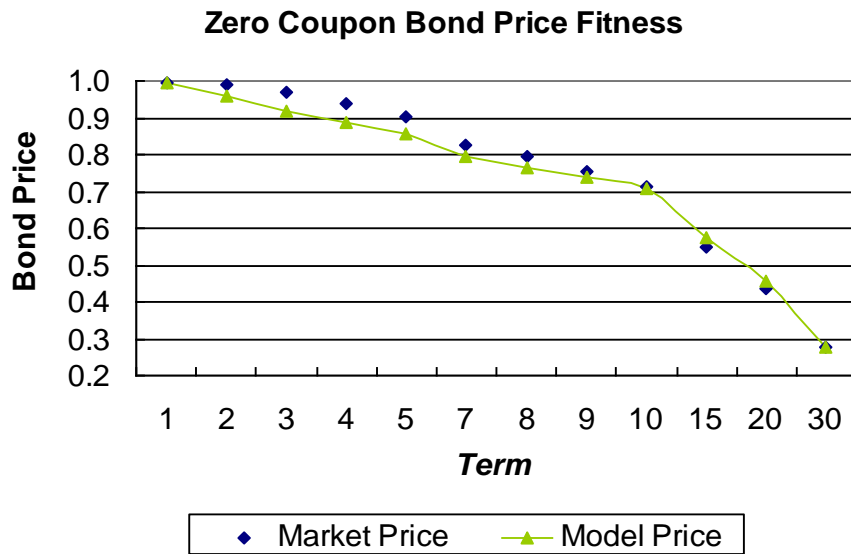
Undetermined parameters β_1 and β_2 are solved by minimizing the square of the difference between market price and model price of zero coupon bonds. The resulting parameters are shown below.

Table 5.1.2

β_0	Contribution of long term component	5.14%
β_1	Contribution of short term component	-14.85%
β_2	Contribution of median term component	-4.14%
τ_1	Decay factor for short term component	0.32
τ_2	Decay factor for medium term component	5.40

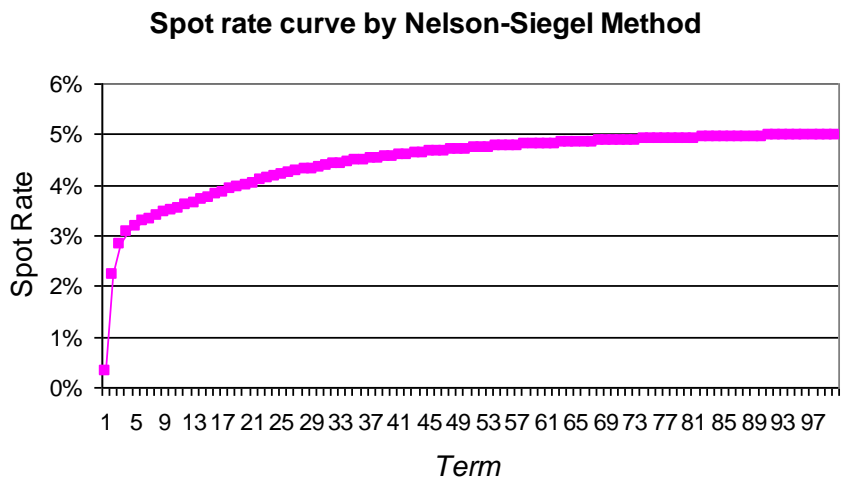
Below is a graph that shows the fit of the model prices to market prices. We can see that the model is capable of producing a fair fit to the market data in this case.

Figure 5.1.2 Zero Coupon Bond Price Fitness



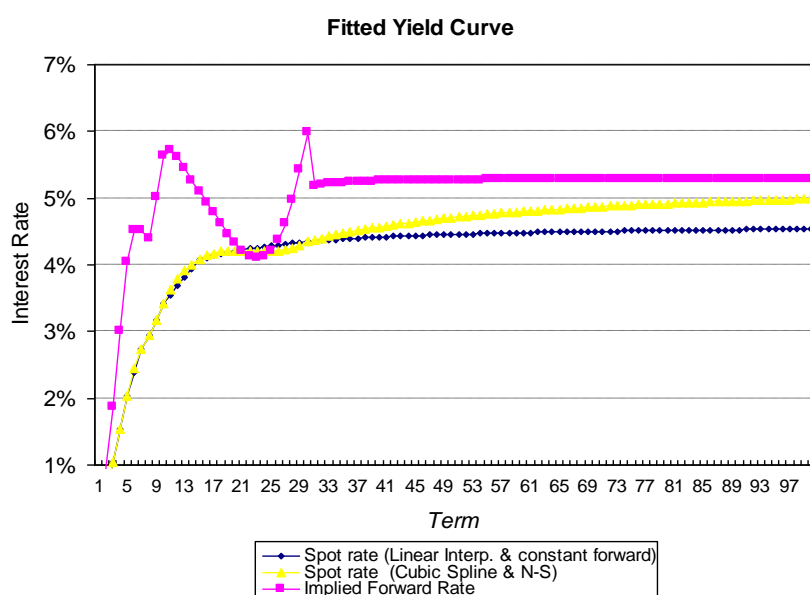
The yield curve is then extrapolated from 30 years to 100 years.

Figure 5.1.3 Spot Rate Curve by Nelson-Siegel Method



Combining the interpolated part (0 to 30 years) with the extrapolated part (30 to 100 years), the following figure shows the initial yield curve used in the case study and its implied forward rate curve. The up-and-down volatile pattern in the implied forward rate curve at short and medium terms is implied from market data. Cubic spline interpolation may cause oscillation due to the sparseness of the data. The extended yield curve is also compared with using linear interpolation and constant forward extrapolation methods. The difference is notable for long-term rates and therefore has a material impact on the valuation results if there are sizable long-term liability cash flows. As mentioned before, the Nelson-Siegel model with fixed long-term target interest rates will generate more stable valuation results for long-term cash flows from period to period. It is also supported by sound economic theory reasoning.

Figure 5.1.4 Fitted Yield Curve



Interest Rate Model

The one-factor Hull-White model is used for stochastic interest rate scenario generation.

$$dr = (\theta(t) - ar)dt + \sigma dz$$

The initial yield curve can be fitted exactly by determining a series of $\theta(t)$. The other two parameters (α and σ) can be calibrated using market instruments such as caps, floors and swaptions. In this case study, calibration to the prices of caps and floors is demonstrated.

- (a) Get market price of caps and floors with different maturities and exercise rates.
 Notional amount: \$10,000
 Payment frequency: Quarterly
 For example, for a floor with an exercise rate equal to 1.75 percent in terms of par yield with quarterly frequency, or equivalently, 1.74 percent in terms of continuous rate, the four-year at-the-money cap or floor costs \$273.4.
- (b) Calculate model price of caps and floors using the pricing formula as described in [Section 3.1.2](#).
- (c) Determine the error (the sum of the square of the difference between market price and model price) and the R-squared¹⁷ of market price and model price.

Table 5.1.3 Caps/Floors Market Price vs. Model Price

¹⁷ R-squared is the ratio of explained variance and total variance. A high value of R-squared indicate a goodness of fit of a model. Here R stands for the Pearson product moment correlation coefficient.

Floors Market Price				Market Price	Model Price	Squared error
3 Month Libor Rate	c	Exercise Rate	Continuous rate			
Term	Swap rate					
1	1%	0.999%	43.5	59.3	249.1	
1	2%	1.995%	118.2	127.7	89.7	
1	3%	2.989%	193.7	201.8	65.8	
1	4%	3.980%	269.6	276.6	49.1	
1	0.45%	0.453%	10.9	30.6	391.7	
2	0.80%	0.802%	59.7	94.1	1,187.0	
3	1.28%	1.274%	151.1	192.2	1,690.6	
4	1.75%	1.740%	273.4	307.8	1,188.2	
5	2.17%	2.162%	409.8	431.3	463.7	
7	2.82%	2.800%	673.4	685.3	141.0	
10	3.38%	3.349%	1,035.5	1,056.1	425.5	
20	4%	3.961%	1,902.0	2,379.2	227,699.6	
<i>R-Squared</i>					98.8%	

Caps Market Price				Market Price	Model Price	Squared error
6 Month Libor Rate	P	Exercise Rate	Continuous rate			
Term	Swap Rate					
1	1%	0.999%	4.3	7.2	8.2	
1	2%	1.995%	1.2	0.7	0.2	
1	3%	2.989%	0.5	0.0	0.2	
1	4%	3.980%	0.2	0.0	0.0	
1	0.45%	0.453%	10.9	19.5	74.2	
2	0.80%	0.802%	59.7	74.1	207.6	
3	1.28%	1.274%	151.1	146.0	25.6	
4	1.75%	1.740%	273.4	252.0	455.2	
5	2.17%	2.162%	409.8	386.6	540.0	
7	2.82%	2.800%	673.4	626.8	2,172.7	
10	3.38%	3.349%	1,035.5	1,005.9	875.3	
20	4%	3.961%	1,902.0	1,429.9	222,841.4	
<i>R-Squared</i>					98.1%	

(d) Solve the parameter α and σ by either maximizing R-squared or minimizing the error. The corresponding parameters b and τ used in the discrete version of one-factor Hull-White model are also calculated.

Table 5.1.4

sigma	0.01
a	0.05
b	0.952
Tau	0.0116

Economic Scenario Generation

In the case study, 500 scenarios with 100-year projection periods are generated for illustration purpose. In practice, 1,000 or more scenarios are normally used for fair valuation calculation and 10,000 or more scenarios are used for capital management and risk analysis.

5.2 Pension Plan Assumption Setting

Actuarial Assumption

To project pension plan liability cash flows, assumptions of mortality and termination rates are required. Ultimate rates are assumed for mortality rates with differentiation between male and female. The termination rate is assumed to be a general rate of 3 percent before reaching the earliest possible retirement age without any difference regarding gender, age or salary

level. After that, an annual termination rate of 10 percent is assumed before the maximum possible retirement age when the entire account value is paid to the plan participant. Those are for illustration and can be varied by the user.

Demographic Information

The demographic information of plan participants is another input item. Inputs include age at hire, age at valuation, gender, salary, current pension account value and years until retirement. The table below lists employee information used in the case study.

Table 5.2.1

Employee ID	Age at Hire	Age at Valuation	Gender	Current Salary	Retirement Account Value	Years till Retirement
1	25	45	M	100000	120000	20
2	30	40	M	80000	50000	25
3	30	30	F	65000	0	30
4	25	50	F	150000	200000	15
5	25	60	M	100000	20000	5

Pension Plan Scheme

Plan assumption: A salary growth rate of 4 percent and a contribution rate of 5 percent are assumed. Vesting period is assumed to be three years.

Embedded option parameters: Set the floor on the credit interest rate for the cash balance plan. The credit interest rate is the maximum of the floor and 10-year Treasury bond yield.

Liability Cash Flow Projection

The model projects account value, survival factor, discount factor and payment. It also calculates the extra cash flow/cost due to embedded options under each scenario for stochastic valuation. It presents the cost of the option as a percentage of the present value of future benefits assuming no such option. This might be used to approximately true up the valuation result for plans with the same embedded option that have not reserved for it.

5.3 Comparison of Closed-Form and Stochastic Valuation Results

In the examples of a cash balance plan with a floor on the credit interest rate, two fair valuation approaches are illustrated, including closed-form valuation based on the deterministic risk-neutral scenario and stochastic valuation with 500 risk-neutral scenarios. For the stochastic approach, in practice 1,000 risk-neutral scenarios or more are normally used for pricing purpose and 10,000 real-world scenarios or more are used for capital management and risk analysis. Real-world analysis often focuses on a high percentile like the 99.95th. Even if 10,000 scenarios are used, only five scenarios will be equal or above the 99.95th percentile, which might still not be enough to get a robust estimate.

Scenario 1: Credit rate equal to the maximum of 10-year Treasury bond yield and 5 percent.

A flat volatility of 1.9 percent is used for closed-form valuation representing the weighted average volatility until retirement for the sample pension plan. It is calculated as the average volatility of 10-year bond yield¹⁸ inferred from the calibrated one-factor Hull-White interest rate model.

Table 5.3.1 5 Percent Floor

5% floor

Cash Balance Plan - Credit Rate Floor	Option Cost	PV (Benefit w/o Option)	% of PV(Benefit)
Closed Form Valuation	75,957	721,327	10.53%
Stochastic Valuation	75,441	733,525	10.28%
95% Confidence Interval of Cost	68,166~82,716		

Scenario 2: Credit rate equal to the maximum of 10-year Treasury bond yield and 4 percent, ceteris paribus.

Table 5.3.2 4 Percent Floor

4% floor

Cash Balance Plan - Credit Rate Floor	Option Cost	PV (Benefit w/o Option)	% of PV(Benefit)
Closed Form Valuation	37,069	721,327	5.14%
Stochastic Valuation	36,457	733,525	4.97%
95% Confidence Interval of Cost	31,949 ~ 40,964		

Scenario 3: Credit rate equal to the maximum of 10-year Treasury bond yield and 3 percent, ceteris paribus.

Table 5.3.3 3 Percent Floor

3% floor

Cash Balance Plan - Credit Rate Floor	Option Cost	PV (Benefit w/o Option)	% of PV(Benefit)
Closed Form Valuation	14,959	721,327	2.07%
Stochastic Valuation	14,053	733,525	1.92%
95% Confidence Interval of Cost	11,691~16,414		

Closed-form valuation results are different from stochastic valuation results but they are

¹⁸ Average volatility of 15 years is used where 15 years is the estimated average duration before retirement for the sample pension plan.

within the 95 percent confidence interval. There are two reasons for the discrepancy.

- 1) The sampling errors in stochastic valuation are big as we can see the length of 95 percent confidence interval is not small. Using more scenarios may reduce the error.
- 2) Closed-form valuation uses different models from stochastic valuation. In this example, it is based on the Black-Scholes model while the one-factor Hull-White model used in the stochastic valuation has different features such as mean reversion.

The pros and cons of those two valuation approaches are listed below.

Table 5.3.4 Closed-Form Valuation

Pros	Cons
This approach gives a quick estimation of the cost of guarantees provided in the pension plan.	The options provided to plan participants are complicated and some of them are based on the cumulative investment return or inflation with high-water mark features. Analytic solutions for those types of options have not been fully developed.
It is an easy tool for risk monitoring and risk assessment. Given an assumed or actual shock of the economic situation, the cost of embedded options can be estimated to evaluate the impact.	The options provided to plan participants are complicated and some of them are based on the cumulative investment return or inflation with high-water mark features. Analytic solutions for those types of options have not been fully developed.
	The approximation based on experiments may be suitable for one situation but not another. There is a risk of inappropriate usage of the parameter adjustment determined under a different economic situation.
	The analytical formula could be hard to understand.

Table 5.3.5 Stochastic Valuation

Pros	Cons
It can be used to determine the value of complicated embedded options that cannot	The run time is usually long and may not be able to generate timely results.

be readily solved analytically.	
Based on real-world scenarios, it can provide the distribution of the financial measures and the profile of the tail risk.	It requires a lot of market data and expertise in economic scenario generation.
	It might not be easy for people to understand the valuation with risk-neutral scenarios.
	Sampling risk is difficult to avoid although it can be mitigated by using more scenarios.

5.4 Replicating Portfolio

The main purpose of the replicating portfolio is not to value the embedded options but to understand the characteristics and risk profile of the options. With the replicating portfolio, it is an easy job to estimate the change in value given a real market event or a “what if” scenario.

In this section, a cash balance plan with a floor on the credit interest rate is used as an example. The credit interest rate is set to be the maximum of 10-year Treasury bond yield and 4 percent.

Input Files

Stochastic cash flows under 500 scenarios are generated with the pension plan cash flow projection model. Stochastic scenarios (discount factors, inflation rates, total equity returns and five-, 10-, 15- and 20-year bond yields) are generated by ESG tools. Those stochastic cash flows and scenarios are saved as .csv files and loaded to the replicating portfolio tool as input files. A list of the input files is given below.

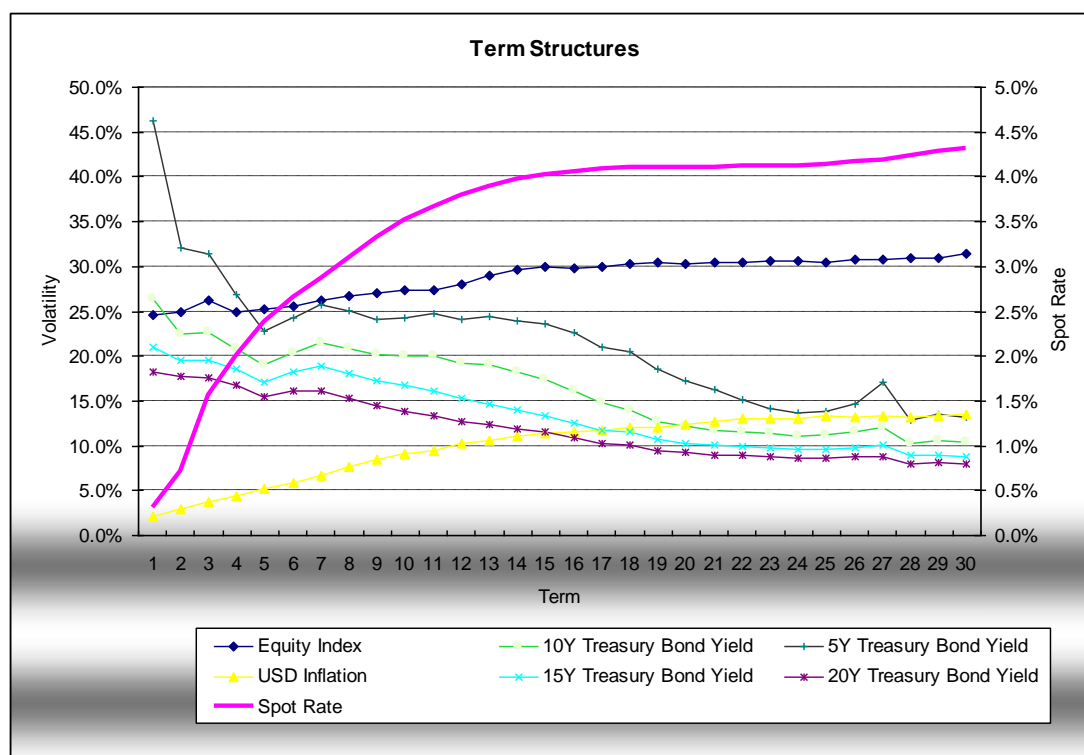
Table 5.4.1

No	Items	Economy	Type	Term	FileName
1	Cash Flows	USD			D:\SOAPPEO\RP files\cf.csv
2	Deflators	USD			D:\SOAPPEO\RP files\df.csv
3	Equity Index	USD	Index Return		D:\SOAPPEO\RP files\ter.csv
4	10Y Treasury Bond Yield	USD	Interest Rate	10	D:\SOAPPEO\RP files\py10.csv
5	5Y Treasury Bond Yield	USD	Interest Rate	5	D:\SOAPPEO\RP files\py5.csv
6	USD Inflation	USD	Index Return		D:\SOAPPEO\RP files\if.csv
7	15Y Treasury Bond Yield	USD	Interest Rate	15	D:\SOAPPEO\RP files\py15.csv
8	20Y Treasury Bond Yield	USD	Interest Rate	20	D:\SOAPPEO\RP files\py20.csv

Term Structures

Interest rate term structure and implied volatility term structure based on stochastic scenarios are calculated to have a high level view of the market situation. Figure 5.4.1 shows those term structures implied by the stochastic scenarios. They are quite consistent with the volatility assumption used in scenario generation.

Figure 5.4.1 Term Structures



Note: The implied volatility of the interest rate given below is not the absolute volatility of the interest rate but is expressed as a percentage of the interest rate.

Asset Instrument Candidates

The replicating portfolio is used to represent the embedded option. Theoretically, all market instruments in a liquid market can be used. In the replicating tool, the asset types listed below can be chosen as candidates in the portfolio.

- Zero coupon bond
- Forward-starting swap
- Equity index
- Equity index call/put
- Payer/receiver swaption

The pension plan credits accounts use a prescribed benchmark (10-year Treasury bond yield) with a floor of 4 percent. When the benchmark is below 4 percent, the embedded option is in the money and the plan sponsor has to apply a higher credit interest rate than the Treasury bond yield to the pension account. The resulting extra credits are similar to a long position in receiver swaptions. When the swap rate is below 4 percent, the option owner receives a rate better than the market rate. In that case, the receiver swaption will be exercised. A forward-starting swap is not a good choice here as it does not provide asymmetric payoffs like the floor on credit interest rates. Equity index and call/put options would be useful if the embedded option is linked to equity market performance. Zero coupon bonds are instrumental for replicating the interest rate sensitivity of the embedded option. Below is a list of asset

instrument candidates used in the case study.

Table 5.4.2

No	Instrument	Name	Economy	Term	Tenor	Strike
1	Zero Coupon Bond		USD	5		
2	Zero Coupon Bond		USD	20		
3	Zero Coupon Bond		USD	10		
4	Zero Coupon Bond		USD	25		
5	Zero Coupon Bond		USD	1		
6	Zero Coupon Bond		USD	15		
7	Receiver Swaption	10Y Treasury Bond Yield	USD	10	20	4%
8	Receiver Swaption	10Y Treasury Bond Yield	USD	20	10	4%
9	Receiver Swaption	10Y Treasury Bond Yield	USD	15	10	4%
10	Receiver Swaption	10Y Treasury Bond Yield	USD	1	20	4%
11	Receiver Swaption	10Y Treasury Bond Yield	USD	1	7	4%
12	Receiver Swaption	10Y Treasury Bond Yield	USD	1	5	4%
13	Receiver Swaption	10Y Treasury Bond Yield	USD	1	3	4%
14	Receiver Swaption	10Y Treasury Bond Yield	USD	1	10	4%

Note: For receiver swaption, the term is the term of the option while the tenor is the term of the underlying swap contract.

Stress Scenario Setting

Stress test results could be useful in two ways.

- (1) They can be used in the replicating process so that the replicated portfolio has the same change in value as embedded options in the pension plan under stress scenarios. This approach is used in the case study.
- (2) If the stress test results are not used in the replicating process, it can be a good reference point to check the effectiveness of the replicating portfolio regarding replication of the sensitivities and the risk profile.

Stress scenarios may include shocks to the initial yield curve, market value or implied volatility. The following is a list of possible stress tests.

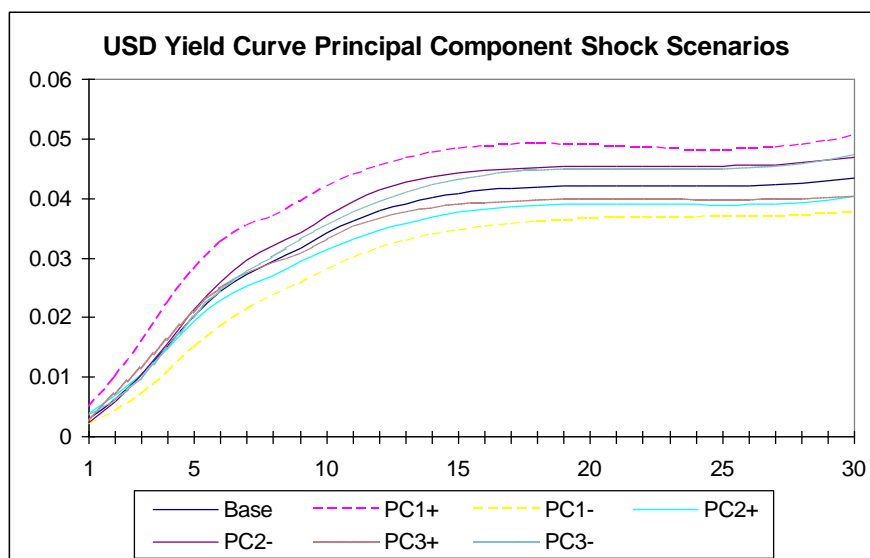
- Interest rate + 100bps
- Interest rate – 100bps
- Interest rate – 50bps
- Initial equity market value + 10 percent
- Initial equity market value – 10 percent
- Equity implied volatility + 25 percent
- Interest rate implied volatility + 25 percent
- Principal component¹⁹ shocks to the yield curve
 - 1st principal component plus
 - 1st principal component minus
 - 2nd principal component plus
 - 2nd principal component minus

¹⁹ Details about principal component analysis are included in [Appendix C](#).

- 3rd principal component plus
- 3rd principal component minus

Figure 5.4.2 shows the initial yield curve shocks based on the principal component analysis (PCA) applied in the case study. It is based on monthly historical U.S. Treasury bond yields from April 1996 to December 2010. Principal component shocks explain one standard deviation (a probability of 84.13 percent) of the historical yield curve volatility. However, it can be changed to a higher percentile like two standard deviations (a probability of 97.725 percent) or a probability of 99.95 percent by multiplying the components by the Z-score at the specified probability.

Figure 5.4.2 USD Yield Curve Principal Component Shock Scenarios



Due to the low interest rate environment assumed for the baseline in the case study, an interest rate volatility stress test is not conducted. Twenty-five percent additional interest rate volatility will generate many negative interest rates using the one-factor Hull-White model. Other more complicated interest rate models such as two-factor Black-Karasinski model and LIBOR market model are more appropriate for applying interest rate volatility shock. The following stress tests are included in the case study.

- Interest rate + 100bps
- Interest rate – 100bps
- Interest rate – 50bps
- Principal component shocks to the yield curve
 - 1st principal component plus
 - 1st principal component minus
 - 2nd principal component plus
 - 2nd principal component minus

For those stress tests, changes in liability value are needed as inputs. This would need repetition of the whole process of setting model parameters under stress scenarios,

regenerating economic scenarios using the changed model parameters, and calculating the embedded option value using either the closed-form valuation approach or the stochastic approach.

Replication Results

After applying the singular vector decomposition, the replicating portfolio is derived and the value and sensitivities are compared to those of the liability.

The table below shows that the replicating portfolio replicates the fair value of liability (stochastic) and the discounted value under the baseline scenario (deterministic) very well. Time value of financial options and guarantees (TVFOG) are the difference between stochastic and deterministic values, which represents the additional option cost caused by market volatility around the baseline projection. This terminology is frequently used in market-consistent embedded value topics. Replicating portfolio closed-form valuation²⁰ gives us the value of the resulting asset portfolio using the closed-form formula.

Table 5.4.3

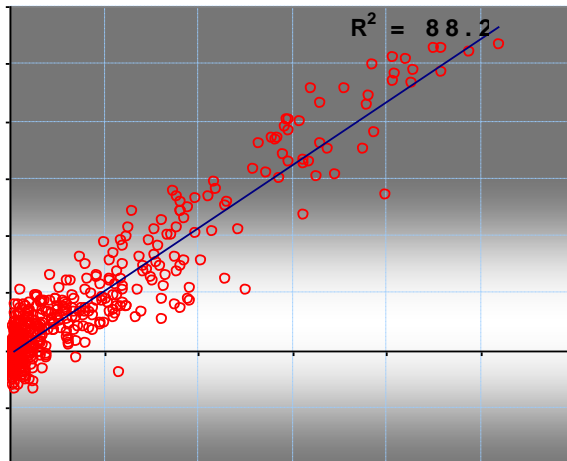
Value	Stochastic	Deterministic	TVFOG	Closed Form
Replicated Cash Flows	36,456.8	2,513.2	33,943.6	
Replicating Portfolio	36,456.8	2,513.2	33,943.6	44,146.7
Difference	0.0%	0.0%	0.0%	21.1%

In addition to making sure that the replicating portfolio does a good job replicating the average value, it is also important to check if it can replicate the volatility of the liability value. The liability value and replicating portfolio value under each scenario are compared. R-squared is calculated to measure the explanation power of the replicating portfolio. $R^2 = 88.2$ percent means that replicating portfolio can explain 88.2 percent volatility of the liability value. Figure 5.4.3 is a scatter plot of replicating portfolio value and liability value.

Figure 5.4.3 Present Value Fitness

²⁰ The stochastic interest rate scenarios in the case study were generated using the one-factor Hull-White model. On the other hand, the Black model is used for closed-form valuation in the replication tool. Therefore, some difference between the stochastic result and the closed-form result is expected. As the closed-form valuation result in the replication tool is only used to assess the sensitivity of the replicating portfolio, the difference does not matter much. This is further verified when comparing value change based on Greeks and that based on shocked stochastic scenarios on page 41.

Present Value F



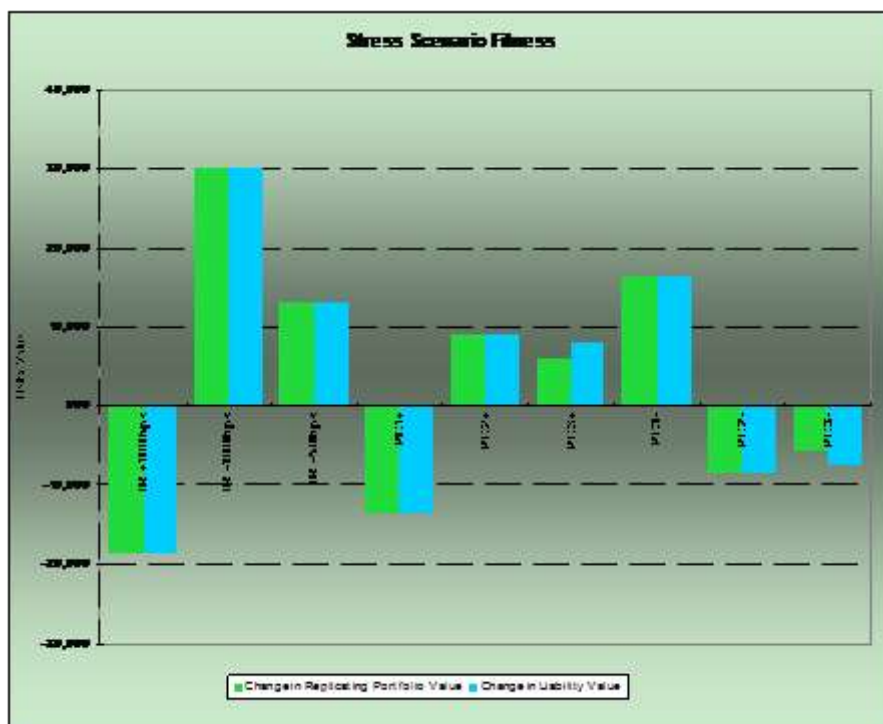
Descriptive statistics about the error distribution are given below. Skewness and kurtosis are close to zero. This implies the normality of the error distribution.

Table 5.4.4

Statistics	PV
Mean	0
Standard Deviation	21,456
Skewness	0.0
Excess Kurtosis	0.6

Value comparison under stress scenarios is also an effective way to make sure that the replicating portfolio is a good representation of the embedded option. Below is a summary of the comparison. As stress test results are included in the replicating process, an exact match is expected.

Figure 5.4.4 Stress Scenario Fitness



The replicating portfolio is given below. There are some long positions as well as short positions. As our purpose is to replicate the value instead of trading, we can still use them although in reality the company may have constraints on taking short positions due to regulation and market illiquidity.

Table 5.4.5

no	Instrument	Reference Rate	term	tenor	strike rate	Weight	Stochastic Valuation
1	Zero Coupon Bond		5			-787%	-286,977
2	Zero Coupon Bond		20			-263%	-95,915
3	Zero Coupon Bond		10			723%	263,713
4	Zero Coupon Bond		25			-2%	-612
5	Zero Coupon Bond		1			195%	71,066
6	Zero Coupon Bond		15			142%	51,600
7	Receiver Swaption	10Y Treasury Bond Yield	10	20	4%	26%	9,605
8	Receiver Swaption	10Y Treasury Bond Yield	20	10	4%	9%	3,396
9	Receiver Swaption	10Y Treasury Bond Yield	15	10	4%	60%	21,994
10	Receiver Swaption	10Y Treasury Bond Yield	1	20	4%	-133%	-48,471
11	Receiver Swaption	10Y Treasury Bond Yield	1	7	4%	-110%	-39,938
12	Receiver Swaption	10Y Treasury Bond Yield	1	5	4%	0%	-122
13	Receiver Swaption	10Y Treasury Bond Yield	1	3	4%	25%	8,998
14	Receiver Swaption	10Y Treasury Bond Yield	1	10	4%	214%	78,118

Sensitivities

A replicating portfolio provides information about the sensitivities of embedded options in pension plans. It can be used to quickly estimate the value changes given a change in market situation.

*Greeks*²¹ are the measures of sensitivities used in financial option pricing and valuation. The Greeks calculated in the model are rho, theta, delta, gamma and vega.

Table 5.4.6

Greeks (Closed Form Only)	Rho	Theta	Delta	Gamma	Vega
Total	(2,294,087)	(1,352)	(1,491,842)	49,553,310	262,451

Rho stands for the sensitivity of the value with respect to interest rate. A rho with a value of $-2,294,087$ means that if the interest rate drops by 1 percent, the value change of the replicating portfolio will be 22,941. This can be checked against the results for IR + 100bps and IR - 100bps. IR + 100bps causes a change of $-18,634$ and IR - 100bps causes a change of 30,129. The average magnitude of change is $(1/2) \times (18,634 + 30,129) = 24,382$. Rho is good for value estimation only for small change. The discrepancy of the absolute changes in the value under the shock-up and shock-down scenario shows the deficiency of using rho for estimation. As the direction of the change is normally assumed or known along with the magnitude of the change, interest rate sensitivities (IR +/- 100bps) would give more accurate estimation than using rho.

Theta stands for the sensitivity of the value with respect to time. As time passes, the value of the embedded option will change. If the market situation remains unchanged, it is a quite useful tool to project the embedded option value until the next full valuation date. However, it is only reasonable if the projection period is short.

Delta and **gamma** measure the sensitivities of the value with respect to the underlying asset/index/rate. Delta represents the first order impact and gamma represents the second order impact. In the case study, there is no equity or equity options allocated. Delta and gamma calculated here represent the sensitivity of the receiver swaptions in the replicating portfolio to the floating rate referenced in the swap. A delta with a value of $-1,491,842$ means that if the 10-year Treasury bond yield drops by 1 percent, the value of the replication portfolio will increase by 14,918, ceteris paribus. With gamma equal to 49,553,310, a refined estimation of the value change is 17,395.²²

Vega stands for the sensitivity of value with respect to the implied volatility. A 1 percent increase in volatility will increase the replicating portfolio value by 2,625. This can be checked against the stressed replicating portfolio value. A 25 percent increase in interest rate volatility has an impact of 59,055. Using vega, the impact will have an estimation of 65,625. Considering the nonlinear relationship between the price and the volatility, the

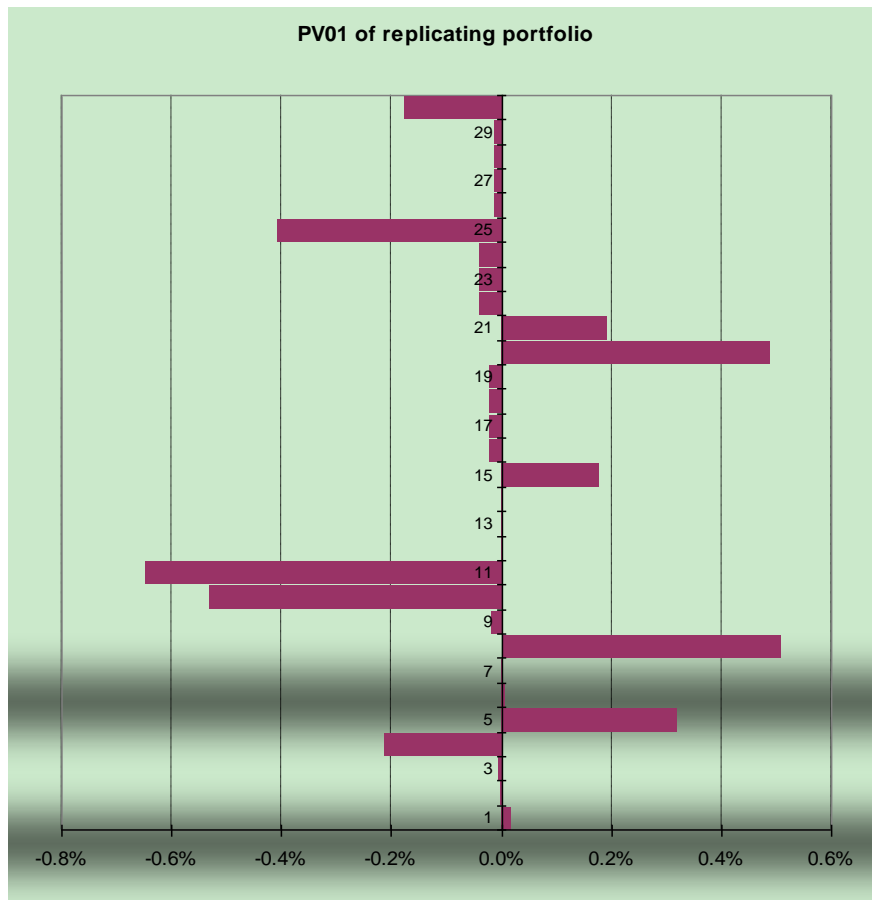
²¹ "Greeks are the quantities representing the sensitivities of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent." Wikipedia, s.v. "Greeks (finance)," [http://en.wikipedia.org/wiki/Greeks_\(finance\)](http://en.wikipedia.org/wiki/Greeks_(finance)). The method used to calculate Greeks in the replicating portfolio tool is described in [Appendix B](#).

²² Change in portfolio value can be approximated as $\Delta s \times \text{Delta} + 0.5 \times (\Delta s)^2 \times \text{Gamma}$.

difference is not big.

PV01 is the dollar value change given 1bp change of yield at a certain term. As rho only focuses on the parallel shift of the interest rate curve, PV01 helps identify the key terms on the yield curve that the embedded options are sensitive to. The values of PV01 at each term point are quite different as shown in Figure 5.4.5. We can see that the value of the replicating portfolio is sensitive to five-, eight-, 10-, 11-, 15-, 20-, 25- and 30-year interest rates. This is consistent with the term and tenor of the receiver swaptions in the replication portfolio.

Figure 5.4.5 PV01 of Replicating Portfolio



The pros and cons of the replicating portfolio approach for embedded options in pension plans are summarized below.

Table 5.4.7 Replicating Portfolio Approach

Pros	Cons
It replicates liability with an asset portfolio, which the investment community is familiar with.	Sometimes complicated embedded options cannot be replicated by existing asset securities in the market.
It helps understand the risk profile of embedded options.	Estimation of the change in value could be inaccurate if the risk profile is not perfectly replicated.
It helps make a quick estimation of embedded option values given a change in market situation.	Illiquid market can give inappropriate valuation results.
It helps identify the hedging instruments to neutralize the position by taking the opposite position of the replicating portfolio.	

6. Conclusion

This report illustrates the application of three valuation methods to the floor on credit interest rates for a cash balance plan. It is hoped that the readers can get both a high level understanding of the theory and process for estimating the fair value of embedded options in pension plans and gain some perspective on the quantitative works involved.

Pension plans have some more complicated options offered such as automatic cost-of-living adjustment and lump-sum subsidy at retirement. The practical valuation method and model for those options could be an area for further study.

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Appendix A. Scenario Validation Tool

The scenario validation tool is used to test whether the generated stochastic scenarios can be used to value the market instrument accurately or, in other words, can maintain model price market consistency. Two types of validation tests are shown in the illustration model: the martingale test (1=1 test) to check the mean and the implied volatility test to check the volatility.

A.1 Martingale Test

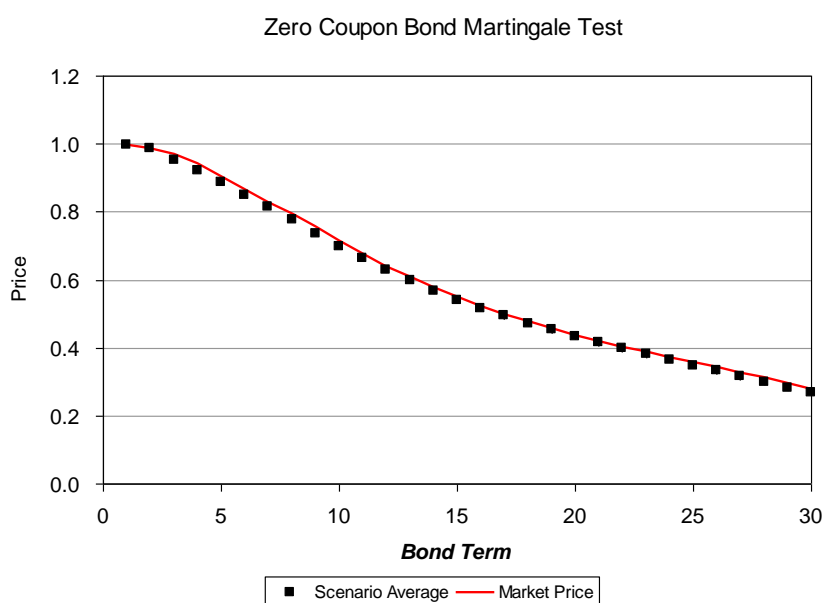
The martingale test is to evaluate whether the average discount value of cash flows using scenario-dependent rates is equal to the market price of the instrument. It is important to make sure that the price estimation based on stochastic scenarios is market consistent.

Zero Coupon Bond Price

Under risk-neutral scenarios, the average of scenario-dependent discount factors at each term point is compared to market observable zero coupon bond prices. If the average of scenario-dependent discount factors lies outside the confidence interval, the stochastic scenarios are considered to be market inconsistent.

Figure A.1.1 shows the martingale test results for zero coupon bond prices using 1,000 risk-neutral scenarios generated by the ESG tool. They match the initial yield curve well.

Figure A.1.1 Zero Coupon Bond Martingale Test



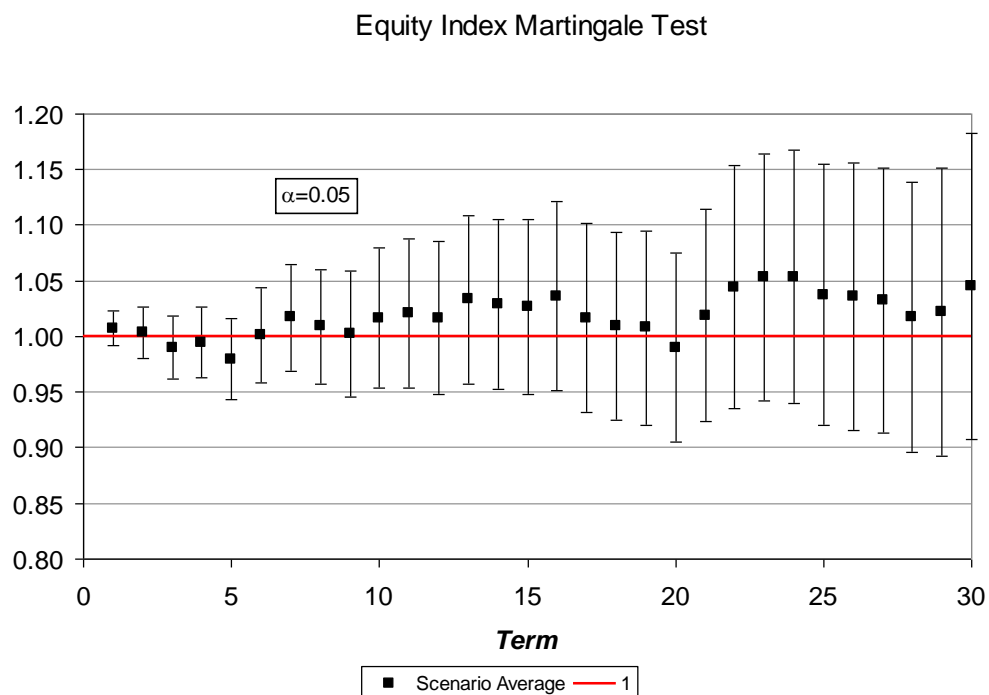
Discounted Equity Index Value

Under risk-neutral scenarios, the average of the scenario-dependent discounted equity index value at each term point is compared to one. A confidence interval is constructed to show the

statistically reliable range.

Figure A.1.2 shows the martingale test result for the discounted equity index value using 1,000 risk-neutral scenarios generated by the ESG tool. The generated equity index scenarios pass the martingale test at a significant level of 5 percent. The line with value equal to 1 lies in the 95 percent confidence interval for all terms.

Figure A.1.2 Equity Index Martingale Test



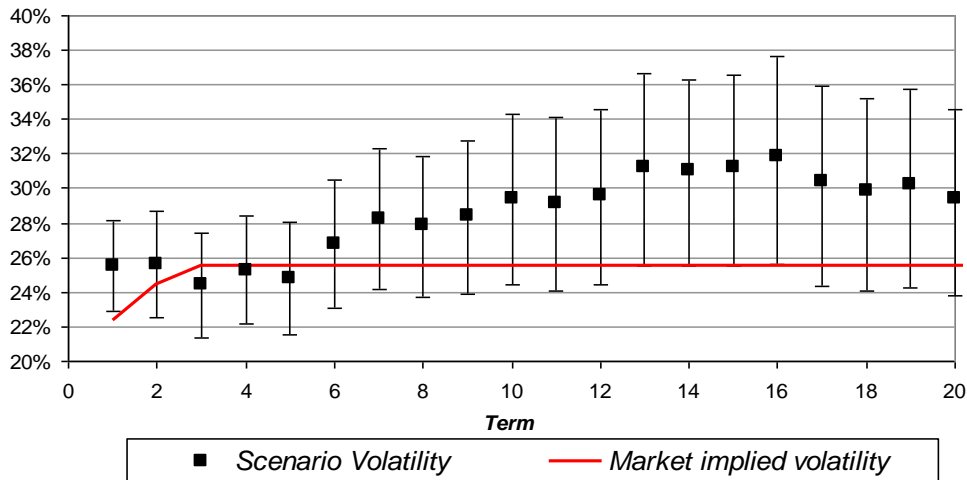
A.2 Implied Volatility Test

Implied volatility based on market price is compared to that based on model price calculated using stochastic scenarios. It can give us a clear view of how well the market volatility is captured in the scenarios.

Figure A.2.1 shows the implied volatility test result for 1,000 stochastic equity return scenarios generated by the ESG tools. The equity return volatility based on the scenarios matches our implied volatility assumption well. The implied volatility assumption lies within the 95 percent confidence interval of the implied volatility based on the stochastic scenarios for all terms except the one-year volatility. As the implied volatility based on the stochastic scenarios includes both the volatility of excess equity return and that of the interest rate, it is expected that it is higher than the excess equity return volatility based on the market price.

Figure A.2.1 Equity Implied Volatility Test

Equity Implied Volatility Test



A similar test for interest rate volatility can be conducted based on the implied volatility of caps/floors or swaptions but is not included in the scenario validation tool.

Appendix B. Financial Instruments in the Replication Tool

Zero Coupon Bond

A zero coupon bond will pay the bond owner the face value of the bond at maturity date T . There is no coupon payment during the life of the bond and the investment yield is reflected in the discount of the price.

Assume face value = 1 and the value $V = \left(\frac{1}{1 + r_T} \right)^T$

Greeks:

$$Theta \Theta = -V \left[\ln(1 + r_T) + \frac{\partial r_T}{\partial T} \cdot \frac{T}{(1 + r_T)} \right]; \text{ here } \frac{\partial r_T}{\partial T} \text{ is estimated as } \frac{r_{T+\varepsilon} - r_{T-\varepsilon}}{2\varepsilon}$$

$$Rho \rho = -\frac{T \cdot V}{1 + r_T}$$

$$Delta \Delta = 0$$

$$Gamma \Gamma = 0$$

$$Vega \nu = 0$$

Forward-Starting Swap

The forward-starting swap is an interest rate swap starting at a future specified date, t_1 . The

fixed leg receiver, at the end of each period, will receive a fixed amount and pay a floating amount determined at time t_1 . The floating rate is $t_2 - t_1$ year forward rate at time t_1 .

Let k be the fixed rate of the swap;
 τ is the length of each payment period; and
 n is the number of payment periods of the swap. $n = t_2/\tau$

Assume the notional amount is 1.

$$\text{value } V = k \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} + \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}} - \frac{1}{(1+r_{t_1})^{t_1}}$$

Here the floating rate s ²³ corresponding to fixed rate k is expressed as

$$\frac{\frac{1}{(1+r_{t_1})^{t_1}} - \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}}}{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}$$

We can then rewrite the value as $(k - s) \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}$ for the fixed rate receiver.

As the value is the combination of a series of zero coupon bonds, theta and rho can be easily derived as those for zero coupon bonds are already known. Delta, gamma and vega are derived against the floating rate s .

$$\text{Delta } \Delta = - \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}$$

$$\text{Gamma } \Gamma = 0$$

$$\text{Vega } v = 0$$

Receiver Swaption

The swaption is to give the option holder the right to enter into a swap at a future time, t_1 . The receiver swaption gives the holder the right to receive a fixed rate and pay a floating rate if the option is exercised. At time t_1 , if the strike rate k is higher than the then-current swap rate with term t_2 , the receiver swaption holder will exercise the option because he/she can receive more than what is available in the market at that time. During the life of the swap, the floating rate is assumed unchanged once it is determined at time t_1 for simplicity, which is consistent with the special forward-starting swap mentioned above.

Assume the notional amount of the instrument is 1;
 σ is the implied volatility of a receiver swaption of maturity t_1 with a tenor of t_2 .

²³ The floating rate that makes the value of the swap equal to zero at the start of the forward-start swap

τ is the length of each payment period; and
 n is the number of payment periods of the swap. $n = t_2/\tau$

As has been derived in the forward-starting swap, the floating rate s is determined as

$$\frac{1}{(1+r_{t_1})^{t_1}} - \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}} = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}$$

Using the Black model, value $V = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} [-s\Phi(-d_1) + k\Phi(-d_2)]$

where

$$d_1 = \frac{\ln\left(\frac{s}{k}\right) + \frac{t_1\sigma^2}{2}}{\sigma\sqrt{t_1}} \quad d_2 = d_1 - \sigma\sqrt{t_1}$$

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$Theta \Theta = \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial t} [-s\Phi(-d_1) + k\Phi(-d_2)] - \frac{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \sigma \cdot s \cdot \phi(d_1)}{2\sqrt{t_1}}$$

$\frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial t}$ is the sum of the Thetas of zero coupon bonds.

$$Rho \rho = -\Phi(-d_1) \frac{\partial \left[s \cdot \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \right]}{\partial r_i} + \Phi(-d_2) k \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial r_i}$$

$$= -\Phi(-d_1) \frac{\partial \left(\frac{1}{(1+r_{t_1})^{t_1}} - \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}} \right)}{\partial r_i} + \Phi(-d_2) k \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial r_i}$$

Therefore, similar to the theta, the rho of receiver swaption can be calculated based on those of zero coupon bonds. Delta, gamma and vega are calculated against the floating rate s .

$$Delta \Delta = - \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \Phi(-d_1)$$

$$Gamma \Gamma = \frac{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \phi(d_1)}{s \cdot \sigma \cdot \sqrt{t_1}}$$

$$Vega v = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \cdot s \cdot \sqrt{t_1} \cdot \phi(d_1)$$

Payer Swaption

The payer swaption is the same as the receiver swaption except that the holder has the option to pay a prescribed fixed rate k and receive the then-current floating rate with term t_2 at time t_1 . If the then-current swap rate is higher than the prescribed strike rate, the option is in the money and will be exercised.

Assume the notional amount of the instrument is 1;

σ is the implied volatility of a payer swaption of maturity t_1 with a tenor of t_2 ;

τ is the length of each payment period; and

n is the number of payment periods of the swap. $n = t_2/\tau$.

As has been derived for the forward-starting swap, the floating rate s is determined as

$$\frac{1}{(1+r_{t_1})^{t_1}} - \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}}$$

$$\frac{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\quad}$$

Using the Black model, value $V = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} [s\Phi(d_1) - k\Phi(d_2)]$

$$d_1 = \frac{\ln\left(\frac{s}{k}\right) + t_1\sigma^2/2}{\sigma\sqrt{t_1}} \quad d_2 = d_1 - \sigma\sqrt{t_1}$$

where

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$Theta \Theta = \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial t} [s\Phi(d_1) - k\Phi(d_2)] - \frac{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \sigma \cdot s \cdot \phi(d_1)}{2\sqrt{t_1}}$$

$\frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial t}$ is the sum of the Thetas of zero coupon bonds.

$$Rho \rho = \Phi(d_1) \frac{\partial s \cdot \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial r_i} - \Phi(d_2) k \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial r_i}$$

$$= \Phi(d_1) \frac{\partial \left(\frac{1}{(1+r_{t_1})^{t_1}} - \frac{1}{(1+r_{t_1+t_2})^{t_1+t_2}} \right)}{\partial r_i} - \Phi(d_2) k \frac{\partial \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i}}{\partial r_i}$$

Therefore, similar to the theta, the rho of payer swaption can be calculated based on that of zero coupon bonds. Delta, gamma and vega are calculated against the floating rate s .

$$\text{Delta } \Delta = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \Phi(d_1)$$

$$\text{Gamma } \Gamma = \frac{\sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \cdot \phi(d_1)}{s \cdot \sigma \cdot \sqrt{t_1}}$$

$$\text{Vega } v = \sum_{i=t_1+\tau}^{t_1+n\tau} \frac{1}{(1+r_i)^i} \cdot s \cdot \sqrt{t_1} \cdot \phi(d_1)$$

Equity Index

The equity index reflects the accumulative equity portfolio value. The index reflects total return and assumes all the dividends are reinvested in the same equity portfolio.

Assume the notional amount is 1.

$$\text{value } V = 1$$

$$\text{Theta } \Theta = 0$$

$$\text{Rho } \rho = 0$$

$$\text{Delta } \Delta = 1$$

$$\text{Gamma } \Gamma = 0$$

$$\text{Vega } v = 0$$

Equity Index Call Option

An equity call option gives the option holder the right to buy an equity index at a certain price (exercise price K) at prescribed date(s). The owner of European equity call options can exercise the option only at option maturity date T. If the equity index price S_T at maturity is higher than the exercise price, the option will be exercised to gain a payoff of $S_T - K$.

σ_t is the implied volatility of an equity call option with a remaining term of t; and r_t is the continuous interest rate with a term of t.

$$\text{value } V = S\Phi(d_1) - Ke^{-r_{T-t} \cdot T-t} \Phi(d_2)$$

$$\text{where } d_1 = \frac{\ln\left(\frac{S}{K}\right) + (T-t)\left(r_{T-t} + \frac{\sigma_{T-t}^2}{2}\right)}{\sigma_{T-t} \sqrt{T-t}} \quad d_2 = d_1 - \sigma_{T-t} \sqrt{T-t}$$

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$\text{Theta } \Theta = -\frac{S\sigma_{T-t}}{2\sqrt{T-t}} \phi(d_1) - r_{T-t} K e^{-r_{T-t}(T-t)} \phi(d_2)$$

$$\text{Rho } \rho = (T-t) K e^{-r_{T-t}(T-t)} \Phi(d_2)$$

$$\text{Delta } \Delta = \Phi(d_1)$$

$$\text{Gamma } \Gamma = \frac{\phi(d_1)}{S\sigma_{T-t}\sqrt{T-t}}$$

$$\text{Vega } v = S\sqrt{T-t}\phi(d_1)$$

Equity Index Put Option

Equity put option gives the option holder the rights to sell the equity index at a certain price (exercise price K) at prescribed date(s). The owner of European equity put options can exercise the option only at option maturity date T . If the equity index price S_T at maturity is lower than the exercise price, the option will be exercised to gain a payoff of $K - S_T$.

σ_t is the implied volatility of an equity put option with a remaining term of t ; and r_t is continuous interest rate with a term of t .

$$\text{value } V = -S\Phi(-d_1) + K e^{-r_{T-t}(T-t)} \Phi(-d_2)$$

$$\text{where } d_1 = \frac{\ln(S/K) + (T-t)\left(r_{T-t} + \frac{\sigma_{T-t}^2}{2}\right)}{\sigma_{T-t}\sqrt{T-t}} \quad d_2 = d_1 - \sigma_{T-t}\sqrt{T-t}$$

$$\phi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$

$$\text{Theta } \Theta = -\frac{S\sigma_{T-t}}{2\sqrt{T-t}} \phi(d_1) + r_{T-t} K e^{-r_{T-t}(T-t)} \Phi(-d_2)$$

$$\text{Rho } \rho = -(T-t) K e^{-r_{T-t}(T-t)} \Phi(-d_2)$$

$$\text{Delta } \Delta = -\Phi(-d_1)$$

$$\text{Gamma } \Gamma = \frac{\phi(d_1)}{S\sigma_{T-t}\sqrt{T-t}}$$

$$\text{Vega } v = S\sqrt{T-t}\phi(d_1)$$

Appendix C. Principal Component Analysis

The sensitivities in the replication portfolio tool reflect equity, volatility and interest rate risks. Principal component shock of the yield curve can help assess the risk due to the yield curve volatility and the correlation between different term points. It considers changes in the shape of the yield curve in addition to the parallel shift, which is measured by duration. It is usually based on historical data and normally the first three principal components can explain over 97 percent of the yield curve volatility. The first three principal components are:

- (1) Parallel shift
- (2) Steepening of the yield curve
- (3) A butterflying yield curve (short-term and long-term shocks have the opposite direction of medium term)

The procedure is listed below.²⁴

- X is the dataset as a $m \times n$ matrix (m is the number of historical time points in the data set while n is the number of term points of the historical yield curves at each time point).
- Covariance matrix is estimated as $\frac{1}{n-1} XX^T$ (symmetric matrix).
- For any symmetric matrix, it is possible to find n orthogonal eigenvectors and associated eigenvalues.
- For a symmetric matrix A , $A = EDE^T$, where D is a diagonal matrix and E is a matrix of eigenvectors of A arranged as columns.
- Let P be the matrix that includes all the eigenvectors, i.e., the i^{th} row is the i^{th} eigenvectors.
- Variance of each component is the diagonal of $\text{cov}(PX)$

$$\begin{aligned}
 & \text{let } A = XX^T \\
 & \text{cov}(PX) \\
 &= \frac{1}{n-1} (PX)(PX)^T \\
 &= \frac{1}{n-1} PXX^T P^T \\
 &= \frac{1}{n-1} PAP^T \\
 &= \frac{1}{n-1} P(EDE^T)P^T \\
 &= \frac{1}{n-1} P(P^T DP)P^T \text{ note : } P^T = E \\
 &= \frac{1}{n-1} P(P^{-1} DP)P^{-1} \text{ note : } P^T = P^{-1} \\
 &= \frac{1}{n-1} D
 \end{aligned}$$

²⁴ Shlens, "A Tutorial," 6, 7, 11–12.

- PCA is conducted on the continuous change of monthly zero rates. After determining eigenvector and eigenvalues, the shocked interest rate curve is calculated as below.

$$\begin{pmatrix} r_{Shocked1yr} \\ r_{Shocked2yr} \\ \dots \\ r_{Shockednyr} \end{pmatrix} = \begin{pmatrix} r_{Current1yr} \\ r_{Current2yr} \\ \dots \\ r_{Currentnyr} \end{pmatrix} \cdot \exp(\sqrt{\lambda_{PC_x}} \cdot e_{PC_x} \cdot CI)$$

λ_{PC_x} : eigenvalue for PCx

e_{PC_x} : eigenvector for PCx

CI : confidence level

Those shocked yield curves could be one of the sensitivities when replicating the liability cash flow.

Appendix D. Cubic Spline Interpolation²⁵

The assumed form used in cubic polynomial curve fit for each segment (x_i, x_{i+1}) is

$$y = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i.$$

The cubic spline constrains the function value $y(x)$, first derivative $y'(x)$ and second derivative $y''(x)$. $y(x)$, $y'(x)$ and $y''(x)$ are equal at the interior node points for adjacent segments.

Substituting a variable S for the polynomial's second derivative into the constraints give us an equation of S for each segment.

For the i^{th} segment, the equation is,

$$h_{i-1}S_{i-1} + (2h_{i-1} + 2h_i)S_i + h_iS_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right)$$

where the difference between successive data points is $h_i = x_{i+1} - x_i$.

In matrix form, the governing equations reduce to a tridiagonal form.

²⁵ Mathews and Fink, *Numerical Methods*, 281–87. The algorithm used in this report based on the reference is described here.

$$\begin{bmatrix} 2(h_1 + h_2) & h_2 & & & \\ h_2 & 2(h_2 + h_3) & \dots & & \\ & \dots & \dots & h_{n-2} & \\ & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) \end{bmatrix} \begin{bmatrix} s_2 \\ s_i \\ \dots \\ s_{n-1} \end{bmatrix} = 6 \cdot \begin{bmatrix} \frac{y_3 - y_2}{h_2} - \frac{y_2 - y_1}{h_1} \\ \dots \\ \dots \\ \frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}} \end{bmatrix}$$

Through the matrix equation above, the value of the $n-2$ second derivative (S) in the middle can be solved. The value of S_1 and S_n still needs to be determined. The extrapolating method specifying these two variables assumes that the end cubic is an extension of the adjacent cubic. In other words, a single cubic curve is formed over the interval $[x_0, x_2]$ and another one over the interval $[x_{n-2}, x_n]$.

The value of a , b , c and d for each segment can be solved using the following formula.

$$\begin{aligned} a_i &= (S_{i+1} - S_i) / 6h_i \\ b_i &= S_i / 2 \\ c_i &= \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6} \\ d_i &= y_i \end{aligned}$$

Appendix E. One-Factor Hull-White Model

One-Factor Hull-White Model²⁶

The interest rate evolves according to the diffusion process given below.

$$dr = (\theta(t) - ar)dt + \sigma dz$$

The mean of the short rate r reverts to $\theta(t)/a$ at rate a .

An annual step discrete version of the one-factor Hull-White model is implemented to generate stochastic scenarios. Euler approximation is used in the case study, although the Mil'shtein method,²⁷ adding the second order term, may be used in practice. This method has a higher order of convergence.

- (1) Assume a model with annual time steps. The basic building block is 1-year rate $R(t)$.
- (2) Assume the following stochastic process for $R(t)$:

²⁶ Hull and White, "One-Factor Interest Rate," 235–54. This section describes the details of the one-factor Hull-White interest rate model used for the case study and the economic scenario generation tool based on this reference.

²⁷ Mil'shtein, "A Method," 396–401.

$$R(t+1) = bR(t) + \theta_t + \tau\varepsilon_t$$

ε_t is the random shock $\sim N(0,1)$

τ is the volatility parameter

b is the auto correlation

$$b = e^{-a}$$

$$\tau = \frac{\sigma(1 - e^{-a})}{a} \sqrt{\frac{1 - e^{-2a}}{2a}}$$

$R(t+1)$ follows $N(\theta_t + bR(t); \tau^2)$, a normal distribution with specified mean and variance

$$R(T) = b^{T-t} R(t) + \sum_{k=t}^{T-1} b^{T-1-k} (\theta_k + \tau\varepsilon_k)$$

(3) Probability distribution of $R(T)$ at time t is a normal distribution.

$$\text{Mean: } R(T) = b^{T-t} R(t) + \sum_{k=t}^{T-1} b^{T-1-k} \theta_k$$

$$\text{Variance: } \text{Var} \left(\sum_{k=t}^{T-1} b^{T-1-k} \tau\varepsilon_k \right) = \tau^2 \left(\frac{1 - b^{2(T-t)}}{1 - b^2} \right)$$

(4) Under risk-neutral valuation, the bond price is calculated as

$$P(t, T) = \mathbf{E}_t^* \left[\exp \left\{ - \sum_{k=t}^{T-1} R(k) \right\} \right]$$

$$\begin{aligned} \sum_{k=t}^{T-1} R(k) &= \sum_{k=t}^{T-1} \left(b^{k-t} R(t) + \sum_{l=t}^{k-1} b^{k-1-l} (\theta_l^* + \tau\varepsilon_l) \right) \\ &= \left(\frac{1 - b^{T-t}}{1 - b} \right) R(t) + \sum_{l=t}^{T-2} \left(\frac{1 - b^{T-l-1}}{1 - b} \right) (\theta_l^* + \tau\varepsilon_l) \end{aligned}$$

The bond price can be written as follows.

$$\begin{aligned} P(t, T) &= \mathbf{E}_t^* \left[\exp \left\{ - \left(\frac{1 - b^{T-t}}{1 - b} \right) R(t) - \sum_{l=t}^{T-2} \left(\frac{1 - b^{T-l-1}}{1 - b} \right) (\theta_l^* + \tau\varepsilon_l) \right\} \right] \\ &= \exp \left\{ - B(T-t) R(t) + A(t, T) \right\} \end{aligned}$$

$$B(S) = \left(\frac{1 - b^S}{1 - b} \right)$$

$$A(t, T) = - \sum_{l=t}^{T-2} \left(\frac{1 - b^{T-l-1}}{1 - b} \right) \theta_l^* + \frac{1}{2} \tau^2 C(T-t-1)$$

$$C(S) = \frac{1}{(1-b)^2} \left(S - 2b \left(\frac{1 - b^S}{1 - b} \right) + b^2 \left(\frac{1 - b^{2S}}{1 - b^2} \right) \right)$$

(5) Term structure of interest rates (zero rates) at time t is given by solving $P(t,T) = \exp\{-Z(t,T)(T-t)\}$:

$$Z(t,T) = \frac{-\ln P(t,T)}{T-t} = \left(\frac{B(T-t)}{T-t}\right)R(t) - \left(\frac{A(t,T)}{T-t}\right)$$

Fit Initial Yield Curve

Interest rate model parameters $\theta_t, t = 0,1,2,\dots$ need to be calibrated to the initial yield curve.

This can be achieved by solving the following function. $Z(0,T)$ is the T -year zero rate at time zero.

$$-Z(0,T)T = -\sum_{l=0}^{T-2} \left(\frac{1-b^{T-l-1}}{1-b}\right)\theta_l^* + \frac{1}{2}\tau^2 C(T-1) - B(T)R(0)$$

This equation can be solved iteratively. For $T = 2$,

$$\theta_0^* = Z(0,2)2 + \frac{1}{2}\tau^2 C(1) - B(2)R(0)$$

The values for θ_1^*, θ_2^* , etc. can be found for $T = 3, 4, \dots$

$$\theta_{T-2}^* = Z(0,T)T - \sum_{l=0}^{T-3} \left(\frac{1-b^{T-l-1}}{1-b}\right)\theta_l^* + \frac{1}{2}\tau^2 C(T-1) - B(T)R(0)$$

Appendix F. Additional ESG Functions

Some economic variables are not used in the case study in Section 5, such as equity return, credit spread and inflation rate. They are useful for other common embedded options available in pension plans such as guaranteed cost-of-living adjustment and plans where the credit interest rate depends on actual investment performance of the portfolio including equity and/or corporate bonds. The economic scenario generation tool and assumption calibration tool have some models built in for them.

Credit Spread Model (One-Factor Hull-White Model)

To model credit spread stochastically, two types of models are available: intensity and rating-based term structure. The intensity model describes default events based on the hazard rate. Its analytical tractability makes it easy to be incorporated into term structure modeling. A rating-based term structure model uses the rating migration process to model the distribution of credit ratings for defaultable bonds.

The one-factor Hull-White model is built into the economic scenario generation tool. The mean is the current credit spread curve and the volatility is the credit option implied volatility. In practice, this simple model is likely to be replaced with a more advanced stochastic credit

spread model when multiple credit ratings need to be considered. An example is the extended version of the Jarrow-Lando-Turnbull model,²⁸ a rating migration stochastic model.

A missing piece in the ESG tool is the default process. Using risk-neutral scenarios where every asset earns a risk-free rate, on average, the additional yield related to a nonrisk-free rating is totally offset by the loss due to default events (default probability \times (1 – loss recovery rate)). In the real world, credit spread is normally higher than the expected loss due to default. Many behavior finance theories have explanations for this, such as conservatism of investors. For pension plans that have high allocations to corporate bonds, a rating migration stochastic model that can handle both credit spread and default for each credit rating would be proper.

Equity Return Model (Lognormal Model)

Equity total return is modeled as the sum of short rate and equity excess returns. A lognormal model is used to generate stochastic scenarios of equity excess return.

$$\text{Log} \frac{\text{Excess Return Index}_t}{\text{Excess Return Index}_0} \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma\sqrt{t}\right)$$

Here, the index has a return equal to the excess return. The drift parameter is zero for risk-neutral scenario generation and expected equity risk premium for real-world scenario generation. The volatility parameter σ can have a flat or nonflat term structure. In the lognormal model, the Black-Scholes formula can be used to derive the implied volatility of the equity return based on market price of equity index options.

The mean parameter for excess equity return (equity total return in excess of the short rate) needs to be determined (zero under risk-neutral world and nonzero under realistic world). The volatility parameter needs to be calibrated using equity index call or put options. At-the-money equity index put options are used as the calibration target.

- (a) Collect market value of at-the-money put options with different terms.
- (b) Calculate implied volatility assuming the lognormal model at each term point. If a nonlevel volatility term structure is assumed, it can be used for volatility assumption. For missing term points, linear interpolation or other interpolation/extrapolation techniques can be applied. Nonlevel equity volatility term structure is assumed here.
- (c) On the other hand, the volatility term structure can be assumed to be flat. The parameter can be solved by minimizing the sum of the square of the difference between market and model prices.

Take equity index put options, for example. The market price of a three-year at-the-money option is \$0.16 for a notional amount of \$1. The implied volatility is 26 percent. The optimal

²⁸ Jarrow-Lando-Turnbull model is a reduced form credit model. For details, refer to Jarrow, Lando and Turnbull, “A Markov Model,” 481–523.

value of level volatility is 21.6 percent, which minimizes the squared error between market price and model price.

Table F.2.1

At-the-money Equity Index Put Option				Constant Volatility Term	21.6%
Term	Market	Zero Rate	Implied Volatility	Model Price	Squared difference
0.0833333	0.012	0.30%	0.11	0.02	3.987E-27
0.17	0.023	0.30%	0.14	0.03	
0.25	0.031	0.30%	0.16	0.04	
0.33	0.039	0.30%	0.17	0.05	
0.50	0.055	0.30%	0.20	0.06	
0.75	0.073	0.30%	0.21	0.07	
1.00	0.087	0.30%	0.223	0.08	
1.50	0.110	0.47%	0.23	0.10	
2.00	0.130	0.64%	0.24	0.11	
3.00	0.157	1.04%	0.26	0.13	

Inflation Rate Model (Lognormal Model)

The lognormal model is used to generate stochastic inflation rate scenarios.

$$\text{Log} \frac{\text{Inflation Index}_t}{\text{Inflation Index}_0} \sim N\left(\left(r_N - r_R - \frac{1}{2}\sigma^2\right)t, \sigma\sqrt{t}\right)$$

The drift parameter can be set to the difference between nominal yield curve and real yield curve or a flat rate equal to the forward-looking long-term inflation rate. The volatility parameter σ can be a flat rate or a nonlevel term structure based on caps or floors on the inflation rate.

Like the calibration of excess equity volatility, the volatility parameter for the inflation rate can be determined using the same techniques except that the market price of caplets/floorlets on the inflation rate is now the replication target. The process is similar to equity volatility calibration.

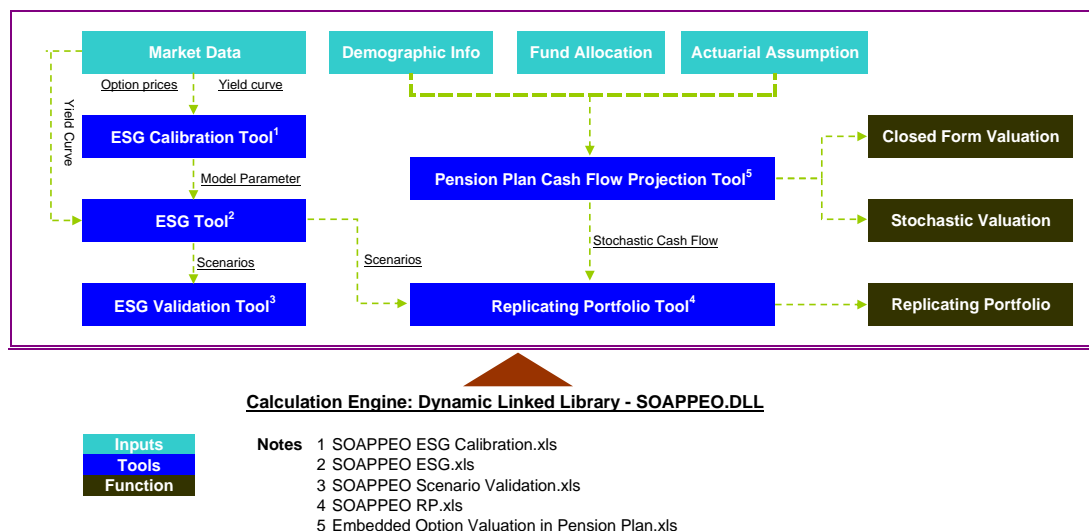
- (a) Collect the market value of inflation index caplets, nominal zero coupon bond price and real zero coupon bond price.
- (b) Calculate implied volatility assuming the lognormal model at each term point. If a nonlevel volatility term structure is assumed, it can be used for volatility assumption. For missing term points, linear interpolation or other interpolation/extrapolation techniques can be applied.
- (c) On the other hand, the volatility term structure can be assumed to be flat. The parameter can be solved by minimizing the sum of square error between market price and model price.

In practice, more sophisticated approaches may be used to model inflation such as modeling it as the difference between nominal interest rate and real interest rate. Often stochastic volatility is used to allow more flexibility and better calibration results.

Appendix G. Valuation Tool Quick Start Guideline

G.1 Flow Chart of Valuation Process

Figure G.1.1 Flow Chart of Valuation Process



The chart above describes the relationship among the inputs, tools and outputs.

G.2 Model Setup

Files

1. SOAPPEO ESG Calibration.xls: Economic scenario model calibration tool
2. SOAPPEO ESG.xls: Economic scenario generation engine
3. SOAPPEO Scenario Validation.xls: Stochastic scenario validation tool
4. SOAPPEO RP.xls: Replicating portfolio tool
5. Embedded Option Valuation in Pension Plan.xls: Liability cash flow projection and embedded option valuation tool
6. RP source files (folder): Input files used in the case study for the replicating portfolio approach including stochastic cash flows and scenarios

Installation/Directory Setup

1. Put files 1–5 in the same folder. Example directory: “D:\SOAPPEO.”
2. File “SOAPPEO ESG.xls” → tab “Short Rate” → cell “A15”: Input the directory where all the generated stochastic scenarios will be stored as .csv files. The stored .csv file contains the input files needed for the replication portfolio tool. Example directory: “D:\SOAPPEO\RP source files\.”
3. File 6: RP source files (folder) are the stochastic scenarios and cash flows input files used by file “SOAPPEO RP.xls.” In the case study, if different assumptions or a different

pension plan is used, the files need to be regenerated using “SOAPPEP ESG.xls” and “Embedded Option Valuation in Pension Plan.xls.” In addition, the files need to be saved in the directory specified in file “SOAPPEO RP.xls” for portfolio replication.

4. File “SOAPPEO RP.xls” → tab “InputFile” → cell “C4”: Input the directory that has all the RP source files (stochastic cash flows and scenarios). A back slash is needed at the end of the path. Example directory: “D:\SOAPPRO\RP source files\.”
5. Make sure the Excel add-in “Solver” is installed and referred to the VBA project in file 1 and file 2. Details are described in [Appendix G.4](#).
6. When using file 5 for embedded option valuation, make sure iteration is disabled in the calculation option. Details are described in [Appendix G.5](#).

G.3 Key Steps/Inputs/Outputs

A. Economic Assumption Setting

1. Yield Curve Extension

File: SOAPPEO ESG.xls

Tab: Curve_Data

Interpolation: Input terms and government bond yields/swap rates (cells A7:B36). Press button “Cubic Spline Interpolation.”

Tab: NS

Extrapolation: Press button “Curve fitting with price difference objective” or “Curve fitting with weighted price diff objective.”

Tab: Curve Extension

Extended curve:

- a. Extended yield curve using linear interpolation and constant forward rate extrapolation (cells F3:H103)
- b. Extended yield curve using cubic spline interpolation and Nelson-Siegel extrapolation (cells K3:M103)

2. HW Calibration—Caps & Floors

File: SOAPPEO ESG Calibration.xls

Tab: HW Calibration—Caps & Floors

Inputs

- a. Initial term structure (cells B7:B16); this is based on extended yield curve in Step 1
- b. Caps & Floors payment frequency (cell E6)
- c. Caps & Floors notional amount (cell E7)
- d. Caps & Floors maturities, exercise rates and market prices (cells D18:G29 and D35:G46)

Press button “HW max fit” (maximization of goodness of fit to market price of caps and floors) or “HW min error” (minimization of squared error).

Outputs

One-factor Hull-White interest rate model parameters:

- a. Continuous version: sigma (cell B8) and a (cell B9)
- b. Discrete version: b (cell B11) and tau (cell B12)

Parameters in the discrete version are used in the ESG tool to generate stochastic short rates.

3. Equity Volatility Calibration

File: SOAPPEO ESG Calibration.xls

Tab: Equity Vol Calib

Inputs

- a. At-the-money equity index put option terms and prices (cells D6:E17)
- b. At-the-money equity index call option terms and prices (cells D22:E33)

Press button “eqvol solve put” (solve level volatility parameter using put option price) or “eqvol solve call” (solve level volatility parameter using call option price).

Outputs

- a. Constant volatility based on equity put options (cell J4)
- b. Time varying volatility based on equity put options (cells G6:G17)
- c. Constant volatility based on equity call options (cell J20)
- d. Time varying volatility based on equity call options (cells G22:G33)

Equity volatility is used as the volatility parameter for excess equity return in the ESG tool to generate excess equity return.

4. Inflation Volatility Calibration

File: SOAPPEO ESG Calibration.xls

Tab: Inflation Vol Calib

Inputs

- a. Terms and prices of inflation caps/floors on CPI (cells A5:A15 and E5:E15)
- b. Strike rates of forward caps (cells C6:C15)
- c. Nominal zero coupon bond prices (cells F5:F15)
- d. Real zero coupon bond prices (cells G5:G15)

Outputs

Inflation volatility (cells D16:D15)

Inflation volatility is used in the ESG tool to generate stochastic inflation rates.

5. Stochastic Interest Rate Generation

File: SOAPPEO ESG.xls

Tabs: Short Rate, DF, n-year zero rate, Par_Yield

Inputs

- a. Discrete version of one-factor Hull-White model parameters: b and τ (cells B3 and B4, tab "Short Rate") calibrated using ESG calibration tool
- b. Number of scenarios to be generated (cell B5, tab "Short Rate")
- c. Projection years (cell B6, tab "Short Rate")
- d. Random number seed (cell B7, tab "Short Rate")
- e. .csv file output directory (cell A15, tab "Short Rate")
- f. Term (cell B1, tab "n-year zero rate") used for n-year zero rate
- g. Term (cell F1, tab "Par_Yield") used for par yield and annuity factor
- h. The variable to be generated (cell E1, tab "Par_Yield"):
 - 1 → Par Yield
 - 2 → Annuity factor

Press button "Hull White Short Rate," tab "Short Rate" (interest rate scenario generation) and/or "Save as .csv" (save the result as an input file for the replicating portfolio tool).

Press button "Zero Rate," tab "n-year zero rate" to generate n-year zero rates. This needs to be done after interest rate model parameters are changed.

Press button "Hull White Par Yield," tab "Par_Yield" to generate par yields and "Save as .csv" to create input files used for the replicating portfolio tool. For example, in the case study, five-, 10-, 15- and 20-year par yields are used for the replicating portfolio tool. The user would need to change cell F1 in tab "Par_Yield" for each term required and press "Hull White Par Yield" and "Save as .csv" to get the files.

Outputs

- a. Generated stochastic short rates (cells D3:CX1002, tab "Short Rate")
- b. Stochastic discount factors (cells B3:CW1002, tab "DF")
- c. Stochastic n-year zero rate (cells B5:CW1004, tab "n-year zero rate")
- d. Stochastic par yield or annuity factor (cells B3:CX1002, tab "Par_Yield")

Note that discount factors, zero rates, par yields and annuity factors are based on the stochastic short rates in tab "Short Rate."

6. Stochastic Credit Spread Generation

File: SOAPPEO ESG.xls

Tab: Credit Rate

Inputs

- a. One-factor Hull-White intensity model parameters: b and τ (cells B3 and B4)
- b. Number of scenarios to be generated (cell B5)
- c. Projection years (cell B6)

- d. Random number seed (cell B7)
- e. Expected credit rate at each term point (cells D4:CZ4)

Press button “Hull White Intensity Model” to generate stochastic credit spreads.

Outputs

Generated stochastic credit rates (cells D7:CZ1006)

7. Stochastic Equity Return Generation

File: SOAPPEO ESG.xls

Tabs: Equity Excess Return, Equity Total Return

Inputs

- a. Risk premium (cell B3)
Risk-neutral scenarios for fair valuation: zero
Real-world scenarios for capital and risk analysis: positive value based on expected market risk premium and correlation between the asset portfolio and the market
- b. Number of scenarios to be generated (cell B4)
- c. Projection years (cell B5)
- d. Random number seed (cell B6)
- e. Equity volatility term structure (cells B9:CW9)

Press button “Excess equity return” to generate stochastic excess equity returns and “Save as .csv” (save the result as an input file for the replicating portfolio tool).

Outputs

- a. Generated stochastic equity excess return (cells B12:CW1011, tab “Equity Excess Return”)
- b. Total equity return (Short rate + Equity excess return) (cells B5:CW1004, tab “Equity Total Return”)

8. Stochastic Inflation Rate Generation

File: SOAPPEO ESG.xls

Tab: Inflation

Inputs

- a. Number of scenarios to be generated (cell B3)
- b. Projection years (cell B4)
- c. Random number seed (cell B5)
- d. Expected inflation rate term structure (cells B8:CW8)
- e. Inflation volatility term structure (cells B9:CW9)

Press button “Inflation rate” to generate stochastic inflation rates and “Save as .csv” (save the result as an input file for the replicating portfolio tool).

Outputs

Generated stochastic inflation rate (cells B11:CW1010)

9. Holistic ESG

File: SOAPPEO ESG.xls

Tabs: ESG, Scenario Table

Inputs

- a. Number of scenarios (cell B7)
- b. Projection years (cell B8)
- c. Random number seed (cell B9)
- d. Correlation matrix (cells N2:Q5)

Interest rate, credit spread, equity and inflation rate model parameters are set in the tabs discussed above.

Press button “ESG” to generate stochastic scenarios for correlated economic variables.

Outputs

- a. Generated stochastic holistic scenarios (cells D12:CY4011, tab “ESG”)
- b. Deterministic scenario (cells C4:P104, tab “Deterministic”) used as input for the pension plan embedded option valuation tool
- c. Stochastic scenarios (cells C4:P50003, tab “Stochastic”) used as input for the pension plan embedded option valuation tool. To get the n-year bond yield for all the terms in columns C:K, cell “B1” in tab “n-year zero rate” needs to be changed for all the terms one by one to get the stochastic n-year zero rates.

B. Scenario Validation

1. Martingale Test

File: SOAPPEO Scenario Validation.xls

Tabs: Martingale Test, Discount factor, Equity Index, PV of Equity Index

Inputs

- a. Discount factors generated by ESG tools (cells A1:AE1001, tab “Discount factor”)
- b. Equity indices generated by ESG tools (cells A1:AE1001, tab “Equity Index”)

Outputs

- a. Discounted average equity index (cells D9:AG9, tab “Martingale Test”)
- b. Average zero coupon bond price (cells D16:AG16, tab “Martingale Test”)

2. Equity implied option test

File: SOAPPEO Scenario Validation.xls

Tabs: Equity Option, PV of Equity Option Payoff

Outputs

- a. Estimated option prices based on stochastic scenarios (cells D15:AG15, tab “Equity Option”)
- b. Implied volatility based on estimated option prices (cells D22:AG22, tab “Equity Option”)

C. Closed-Form/Stochastic Valuation

1. Input deterministic and stochastic scenarios
Columns C:P, tab “Scenario” (deterministic and correlated stochastic scenarios generated by SOAPPEO ESG.xls)
2. Input demographic information: columns P:V, tab “Input”
3. Input pension plan assumption: tab “Input”
 - a. Employee ID (cell C5) and scenario number (cell E7); they can be changed to calculate the result for the specified individual and scenario
 - b. General Pension Plan Assumption: salary growth rate, contribution rate and vesting period (cells C14:C16)
4. Input embedded option parameters, tab “Input”
Cash balance plan floor on the credit interest rate (cell E5)
5. Input actuarial assumption: tab “Input”
 - a. Mortality rates (cells H8:I118)
 - b. Mortality shock (cell I5)
 - c. Termination assumption (cells L8:L57)
 - d. Termination shock (cell L5)
6. Input valuation assumption: tab “Input”

Inputs

For closed-form valuation:

Volatility of Treasury bond yield (cell E9)

For stochastic valuation:

- a. Number of scenarios to use (cell E11)
- b. Number of employees to be included (cell E12)
- c. CTE percentile (cell E14), which is only used for real-world analysis

Press button “Stochastic Analysis,” tab “Input”: Generate stochastic cost of embedded options

Press button “Portfolio—Closed Form Valuation,” tab “Input”: Generate option cost using closed-form valuation method for the pension plan

Outputs

- a. Tab “Output,” option cost using either closed-form valuation or stochastic valuation method (columns B:E)

- b. Tab “Stochastic Results,” option cost and present value of benefit without options for each employee and scenario (columns A:E)
- c. Tab “Stochastic Results,” average cost and CTE cost analysis for each employee and for the pension plan (columns G:L)
- d. Tab “Stochastic Results,” option cost and present value of benefit without options for scenario (columns Q:T)

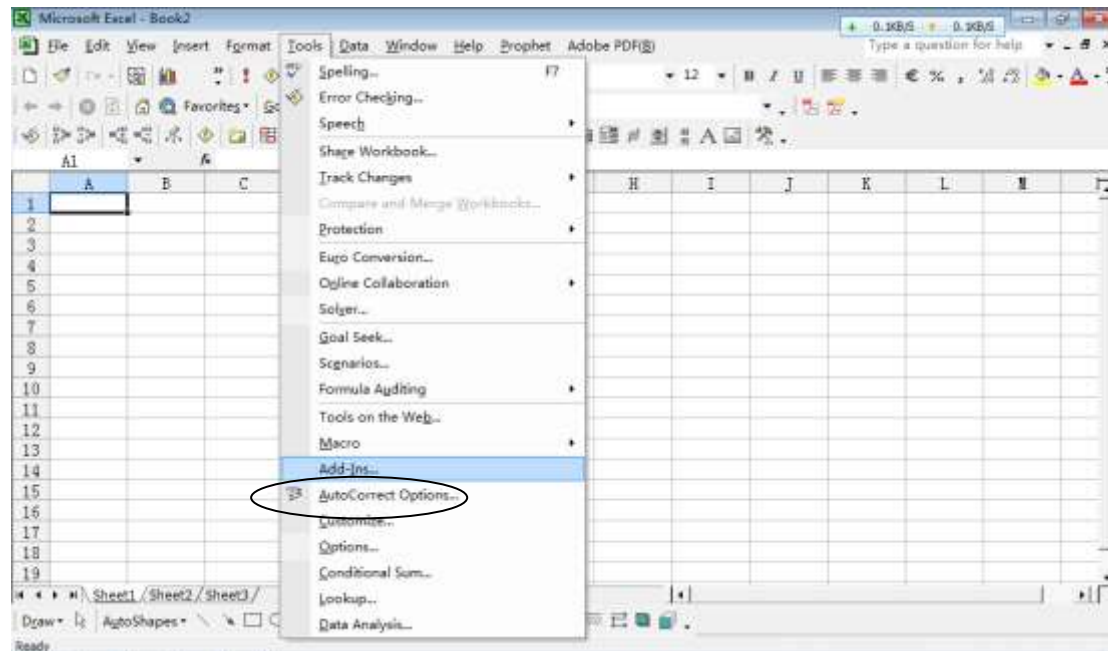
D. Replicating Portfolio Tool

1. Load stochastic cash flow and stochastic scenarios: Tab “InputFile”
 - a. Input files directory (cell C4): This is where the stochastic cash flow and scenario files should be stored in .csv format.
RP source files (folder):
 The files below are used as input to the replication tool.
 - cf.csv: stochastic cash flow
 - df.csv: scenario dependent discount factor
 - if.csv: stochastic inflation rate
 - py*.csv: n-year par yield
 - ter.csv: total equity return
 - b. The content, type, term and economy of input files (cells C7:F19)
 - c. Input the number of scenarios (cell E4) and projection years (cell G4)
 - d. File names (cells G7:G19): They need to be refreshed if the directory and/or the file names are changed.
2. Term structure analysis: Tab “Term Structure”
 Press button “Term Structure” to generate term structure based on the stochastic scenarios. The results are output in cells I7:AL18.
3. Input replicating portfolio asset candidates information (columns C:I, tab “Asset Instruments”): instruments, linked index name, economy, term, tenor, strike rate/price and whether to include in the asset portfolio or not.
4. Input scenario threshold (cell E3, tab “scen result”): If the square of the cash flows for a particular scenario is larger than X percent of the sum of squared cash flows for all scenarios, this scenario will be excluded from the replicating process.
5. Input stress scenarios details: tab “Stress Test”
 - a. Whether to consider stress test results in the replication (cell C13)
 - b. Stress scenario name, change in liability value and whether to consider this stress scenario or not (cells G7:Z9)
 - c. Change in the spot rate for each stress scenario (cells G12:Z47)
 - d. Change in the initial market value for each stress scenario (cells G50:Z60)
 - e. Change in the implied volatility for each stress scenario (cells G63:Z73)
6. Replication: Press button “Replication” in tab “Asset Instruments,” to replicate liability using selected asset instruments
7. Detailed results about the replicating portfolio composition, its value, Greeks, PV01 and other sensitivities are output in tabs “Asset Instruments,” “Replication,” “scen result” and “FI Results.”

G.4 Excel Add-in—Solver (Installation and Reference)

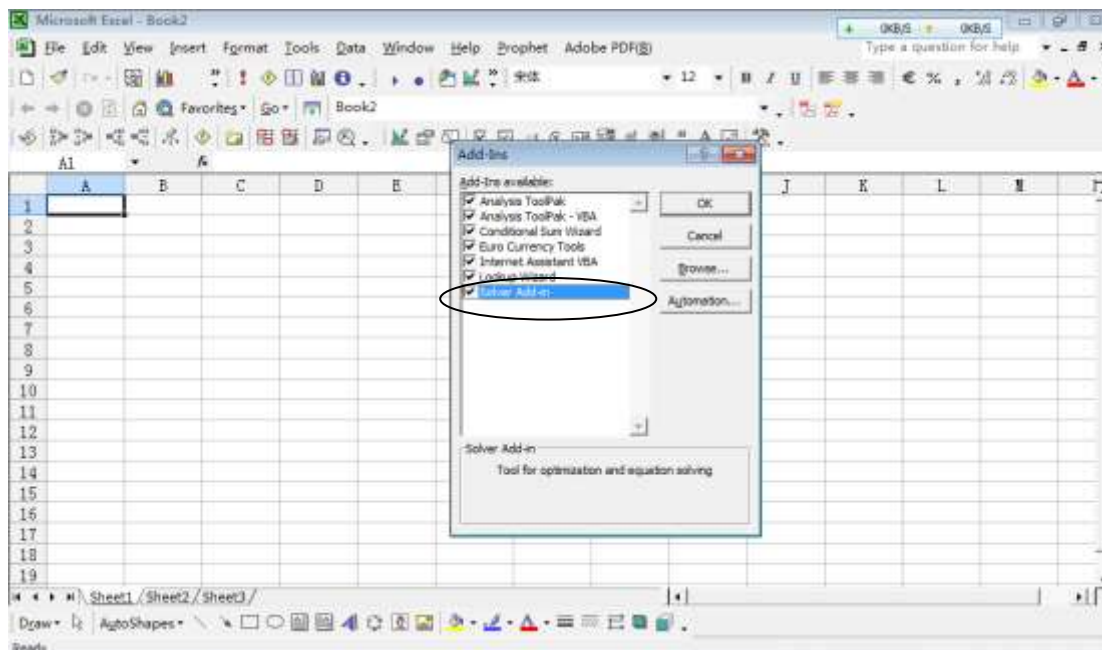
1. Check if the Solver is installed on your computer: Tools → Add-Ins.

Figure G.4.1



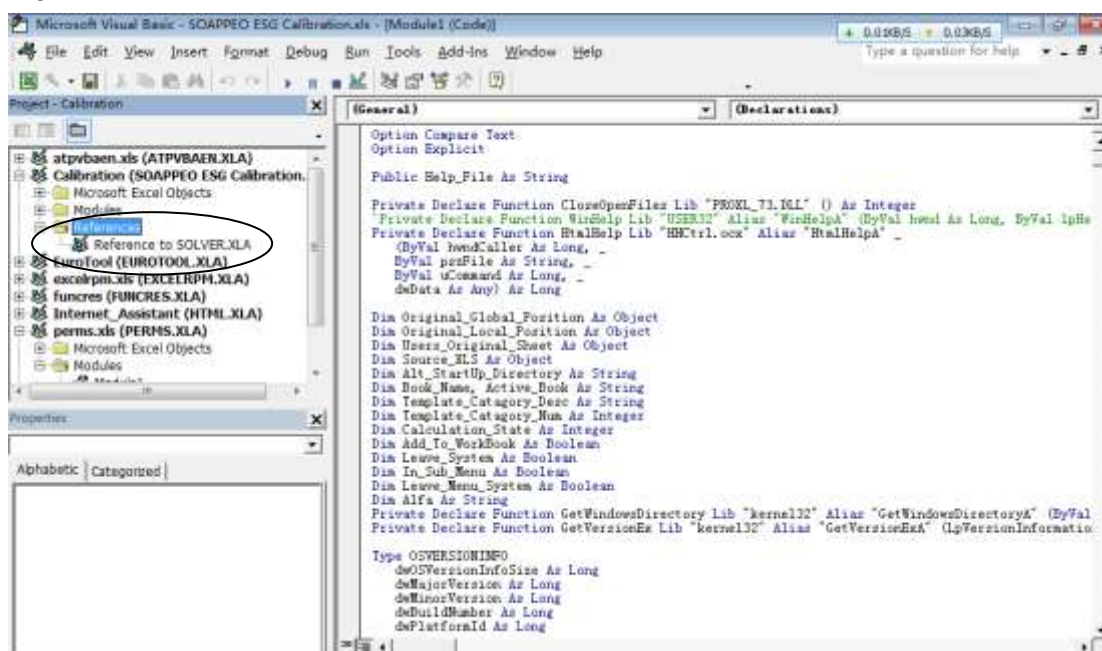
- If the checkbox “Solver Add-in” is not checked, check it and press “OK” to install it.

Figure G.4.2



- Check if the reference in VBA projects includes “Solver.” Take “SOAPPEO ESG Calibration.xls” as an example. Go to Reference and check if “Reference to Solver.xla” is shown. If you cannot even find Reference, then the reference is not built.

Figure G.4.3



4. Add the reference as shown below if necessary.

Figure G.4.4

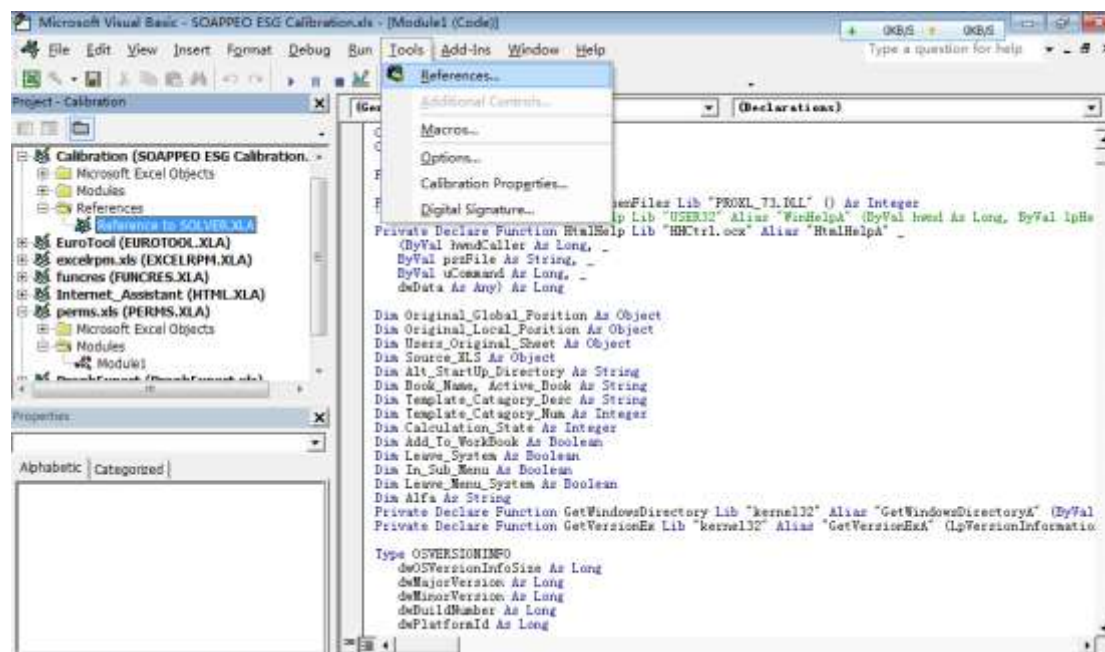
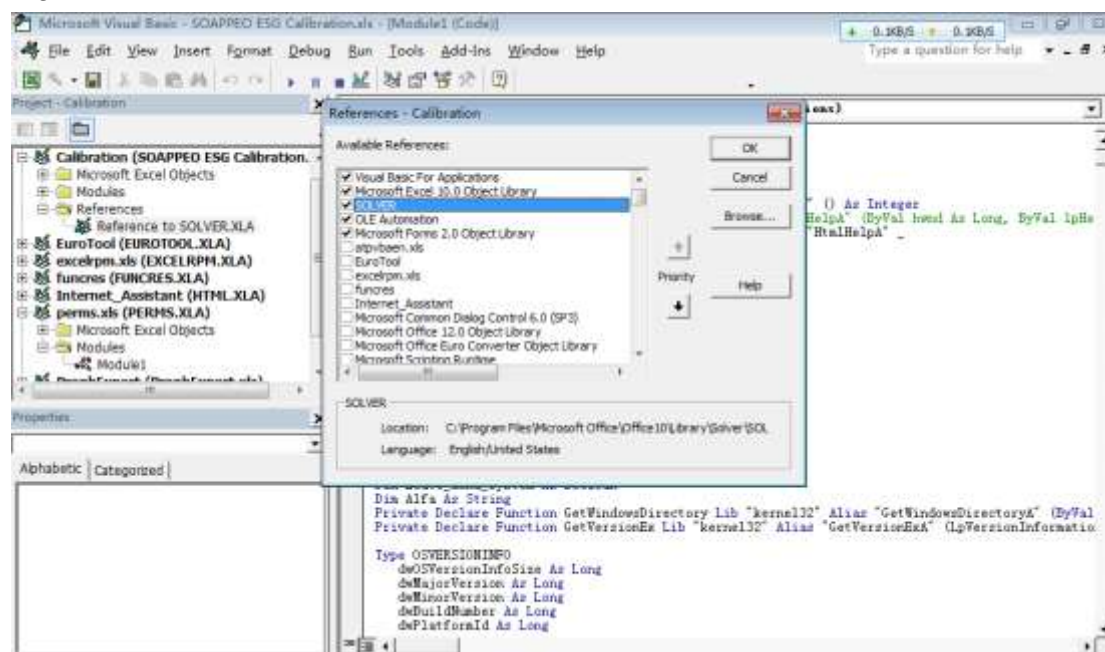


Figure G.4.5



G.5 Calculation Property Setting

In the option valuation tool, to reduce the calculation time, iteration needs to be disabled in the calculation property.

Figure G.5.1

