Projecting a Dependent Loss Ratio Under Shifting Parameters

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CONTENTS

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Abstract

Reserving and ratemaking calculations are vital to how insurance companies perform financially. Current-day practices for predicting future loss ratio or pure premiums for reserving and ratemaking rates are not based on low prediction errors—mean squared error (MSE) criterion. The paper provides mathematically optimal weights that lead to minimum MSE prediction errors, and the formulas are validated using retro-testing with real-world datasets. The approach is applicable to all lines where triangulation of data is feasible and ratios are available (i.e., either loss ratios or pure premium).

In this report, two main theorems are discussed, each addressing prediction of future loss ratio or pure premium and improving upon current practices for how rates and reserves are calculated. The theorems provide a more stable and accurate way to calculate rates and reserves for immature (or recent) periods while utilizing all the data available to the fullest. Additionally, model stability is explored under time-dependent shifting parameters and prove an important theorem.

Section 1. Introduction

The paper has two goals, namely stable and accurate loss ratio projections for reserving and ratemaking.

1.1 BACKGROUND

Ratemaking, like many actuarial practices, is inherently risky. Actuaries are faced with the challenge of determining rates for a year with no emergent data. These rates must be fair such that they are competitive for marketability, as well as computationally sound. Undesirable consequences can arise due to predictive inaccuracies: If rates are too high, policyholders will take their business elsewhere; conversely, if rates are too low, companies will impose larger rate increases in subsequent years resulting in rate swings.

Naturally, this is not an exhaustive list of consequences that arise from imprecise ratemaking. Ultimately, the goal of any company is to minimize the discrepancy between projected and realized loss ratios. In this paper, a least squares approach is proposed to minimize this discrepancy. Table 1 summarizes the current ratemaking practice versus the proposed model presented in the paper.

Table 1

RATEMAKING: PROJECTING OVERALL FUTURE LOSS RATIO

No.	Current Ratemaking Practice	Proposed Model	Comment
1	Projection is done using a linear combination of previous accident year's developed loss ratios.	Projection is done using a linear combination of previous accident year's developed loss ratios.	There is no change.
2	Weights are set judgmentally. Approach may result in projected rates that are high or low compared to actual.	Weights on previous accident year's developed loss ratios are set using least squares technique that minimizes the distance between actual and projected loss ratios.	Improvement: Data is utilized to find the best possible weight structure, which is more precise and stable overall rate level.
3	This provides limited application to update experience weights in the projection of (future) loss ratio.	Proposed model can be used to update experience weights used in selecting ultimate estimates based on a least squares fit of selected ultimate loss ratios from historical accident years. Further, since the future loss ratio is projected as a linear combination of historical accident year loss ratios, these updates will also improve the projected loss ratio.	Proposed model enhances accuracy and stability of the projected loss ratio estimates and therefore, improves the accuracy of rates.

Additionally, adequate reserve amounts have to be set aside for past accident years. Predicting adequate reserve amounts imposes a vexing challenge: reserve changes due to inaccuracies in predicting the ultimate loss ratio subject companies to significant repercussions. If reserves are set too high, it will depress the annual year's profit and vice versa. If these unwanted reserve swings due to modeling errors occur excessively, stakeholders will be dissatisfied with the subsequent volatility, and shareholders will likely invest elsewhere. For example, in November 2017 American International Group (AIG) incurred a third-quarter loss and had to add $836 million to its reserves (Subba & Barlyn 2017). In response, many AIG shareholders sold their shares, resulting in a stock price decrease of 2.3%. While AIG's reserve increase may be justified, this example demonstrates the sensitivity of reserve swings to stock price. Table 2 summarizes the current reserving practice (using the Bornhuetter-Ferguson, B-F, method as an example) versus the proposed model presented in this paper.

Table 2

LOSS RESERVING COMPARISON TO B-F METHOD, PROJECTING AN IMMATURE ACCIDENT YEAR LOSS RATIO

No.	Current Reserving Practice	Proposed Model	Comment
1	For immature accident years (AYs), the a prior loss ratio can be judgement or experience-based.[1]	AY projections are calculated using a linear combination of B-F Method and older (prior to the AY) ultimate loss ratios.[2]	Improvement: This enables full utilization of data and, hence, provides stability and accuracy in AY ultimate loss ratio estimation.
2	B-F Method weights can lead to erroneous conclusions. For example, a long-tailed line with stable link ratios will result in most of the weight on the expected loss ratio for the most recent AY.	This corrects for the weight problem in the B-F Method. It spreads weight to older historical ultimate AY losses, expected loss ratio and chain ladder method for the most recent AY.	Improvement: The AY ultimate loss ratio is more accurate and stable.
3	B-F Method weights are based on judgement and not statistical properties.	Weights under the proposed model have least squares properties and are designed to minimize expected distance between actual and projected loss ratio.	Improvement: The AY ultimate loss ratio is more accurate.

[1] The Bornhuetter-Ferguson (B-F) Method takes a linear combination of chain ladder and expected loss ratio. The weight on the chain ladder estimate is (1/age-to-ultimate-factor) and the complement are the weight on expected loss ratio. Our paper considers this method in its most general sense—any selection of ultimate loss ratio can be justified by solving for the implied age to ultimate link ratio. Hence, the B-F Method justifies any ultimate loss selection and poses no restriction in this paper.

[2] In ratemaking, the future accident year loss ratio is projected using the previous historical developed loss ratios. Continuing this reasoning, in the reserving context, we should also use the previous historical loss ratio data to project the latest AY. However, we should also use partially emerged data for the accident year.

Actuaries have historically implemented various methods to project losses, with the B-F methods being two of the most recognizable. Despite having widespread application in valuation, those methods have evident inadequacies as explained by various researchers. For instance, the chain ladder method has an implicit assumption that future loss patterns mimic historical patterns, and thus, the chain ladder method does not function optimally in a volatile environment (Friedland 2010). Because of this potential periodic volatility, actuaries may turn to intuition and rely on more recent development patterns rather than focusing on patterns that have occurred throughout a lengthier period of time (e.g., only crediting the previous three years rather than the past decade) (Werner & Modlin 2016). This may be problematic due to the fact that such intuition considers less data. On the other hand, a notable drawback of the B-F method is its basis on an expected loss ratio that may be too high or too low (Arico et al. 2016). However, the critical drawback of the B-F method that we are focused on is its consideration of the abstract target loss ratio that has never been observed historically.

There are additional methods available, one of which is the approach proposed by Brehm (2006). In essence, he revises the B-F method by replacing the target loss ratio with a least squares loss ratio. However, the problem remains the same: This calculated loss ratio has never been observed historically. Additionally, we consider more sources of information by combining conventional approaches with a newly devised least-squares method that will be discussed later in this paper.

Ultimately, actuaries thus far have not optimally or completely addressed problems arising from inaccurate projections. In this paper, we enhance conventional actuarial methods using methodology that minimizes least-squared error. Project loss ratio and pure premium can be used in a variety of situations. B-F is one application and so is ratemaking. To clarify, our intent is not to address or resolve the problems that conventional methods such as chain ladder and B-F present; rather, we augment current methodology using mathematical properties and extend it for practical application in the industry.

To preface, the scope of our paper is international, with the United States being an example. A resounding strength of the model presented in this paper is its versatility and practicality: This approach is applicable to all lines where triangulation of data is feasible and ratios are available (either loss ratios or pure premium), whether focused on property and casualty, life and health, disability insurance, etc. Additionally, for illustrative reasons, we use accident year triangles, but we could have used any type of year, such as a policy, report or calendar year. Furthermore, our method can be applied to pure premium triangles. Typically, actuaries adjust the premium to current rate level or put the premium on-level (Werner & Modlin 2016) for ratemaking purposes; however, this approach can also be applied to a pure premium triangle, which would mean on-leveling would not be needed for ratemaking.

An important and related line of work relates to generalized linear modeling (GLM). There are various limitations that are identifiable when modeling with GLMs, and these are summarized in Table 3. These include the assumption of uncorrelated randomness in outcomes and the full credibility that data receive (Goldburd, Khare, & Tevet 2016). Accident year samples cannot be assumed to be independent; that is, they maintain some degree of correlation. This is attributable to the fact that several of the same policyholders from previous years may still constitute part of the sample in a current year. Our model does not encounter this complexity due to the fact that it does not assume independence. Second, insurance datasets involve newer accident years that have incomplete incurred or paid losses, and these data are not fully credible for parameter estimation purposes. Third, GLMs are concerned with least squared errors of historical data, and the actuarial problem of predicting future loss ratios relates to minimizing prediction errors.

Table 3
COMPARISON WITH GLM

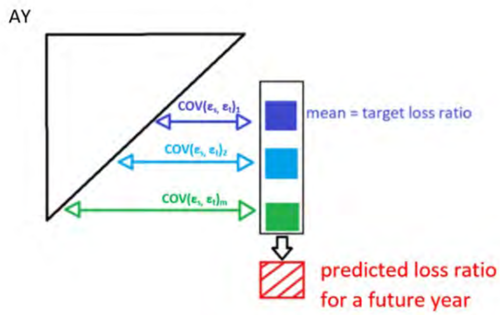
No.	GLM	Proposed Model	Comment
1	Observations are assumed to be independent.	Observations are not assumed to be independent.	GLM parameter estimates are inaccurate because AYs are not independent.
2	Model provides best historical data fit using least squares metric.	Model minimizes prediction error using least squares metric.	Proposed model achieves the true goal of future predictions.
3	Parameter estimation is carried out assuming that incurred loss for recent (immature) accident years is fully credible.	Parameter estimation is carried out without assuming that incurred loss for recent accident years is fully credible.	GLM parameter estimates are inaccurate when recent accident years are included in the calculation.

1.2 THEOREMS

We now discuss the contents of our paper. These include two critical theorems (namely one for ratemaking and one for reserving), key definitions, supporting proofs, derivations and several mechanisms for calculation. First, it is important to identify and establish our incomplete accident year distribution. This distribution is derived in Appendix A and treats all years as partially developed with varying loss ratio means. It is important to note that the covariance calculation pertaining to accident year one will be zero due to the fact that the data is complete (fully developed) for that year.

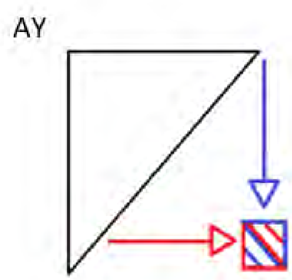
Our first theorem predicts the future loss ratio as a linear predictor of past ultimate loss ratio using least squares. The least squares approach minimizes the deviation between predicted and actual future loss ratio. In terms of Figure 1, we are projecting the future loss ratio downward using a linear combination of historically observed ultimate loss ratios for each accident year. The ultimate historical loss ratios are assumed to be provided by actuaries using a triangulation method such as the B-F method. While this may seem restrictive, the age to ultimate loss ratio in the B-F method can always be adjusted to match the actuary's selection, thus covering all possibilities, including hand-picked numbers that an actuary provides. Our approach is not changing the ultimate loss ratios results for the historical AYs. Rather, our approach is applying objective weights to these historical loss ratios in order to project a future loss ratio.

Figure 1
THEOREM 1 DEPICTION



In Theorem 2, we enhance the B-F method by considering these B-F estimates as starting points while also using vertical loss ratio projections, as shown in Figure 2. For example, suppose a data set is comprised of AY 2000 to 2010 loss ratios, and we want to estimate the 2006 ultimate loss ratio using Theorem 2.[3] Then we will use Theorem 1 to first calculate weights used in the linear combination of AY 2000–2005 (the blue line in Figure 2). We can choose which years we want to apply in the linear combination (i.e., choose weights that match with the year we are trying to predict).[4] Then we would use the B-F estimate for 2006 and this new estimate and take a linear combination of the two. The weights used in this linear combination are provided by Theorem 2.

Figure 2
THEOREM 2 DEPICTION



The horizontal loss ratio projections are assumed to be provided by the actuary, while the vertical projections are a new estimate provided in this paper. The vertical projections utilize Theorem 1; thus, combining two estimates stabilizes the B-F method estimates. The advantage in this approach is that it will likely result in a more accurate and stable estimate in a year with little to no emergent data. Previously, without the consideration of previous loss ratios, projection is subject to considerable and unwanted amounts of inherent randomness, which we are minimizing. We emphasize that actuaries may still use

[3] Note that if we wanted to predict AY 2006, then the weights would need to be recalculated using Theorem 1. This time, the AY used for the weight calculation would be 2000–2005. Hence, Theorem 1 would need to be used repetitively to project all historical AY, and there will be a set of weights for each estimates AY rather than a single fixed set.
[4] This is a critical issue in reserving. For any given AY and age combination, the selected historical AYs should be those that have a similar loss ratio emergence at that age. This would make such other AY similar to the current AY.

their selected ultimate loss ratios to project vertically using Theorem 1. Theorem 2 then uses these projections to stabilize loss reserves (horizontally).

Theorem 2 provides stability and accuracy to loss ratio estimates, partly because it uses actual observed loss ratios (vertical projections in Figure 1) as opposed to a target loss ratio, which we questioned earlier. By careful, weighted incorporation of both a horizontal and vertical approach, we consider the minimization of deviations. Thus, we arrive at a more accurate estimate. The weight given to vertical projections versus horizontal projections depends on the maturity of the year. More mature years will receive greater weight for horizontal projections and vice versa. Thus, the model adds great value to new accident years where loss ratio projections are difficult to make and future loss emergence leads to noise as the year converges to ultimate. This noise leads to reserve and income statement fluctuations that are undesirable to stakeholders. [5]

In the derivation of Theorems 1 and 2, we need the expected loss ratio for both the historical and projected accident year. Since these are not always available, we make the simplifying assumption that the target loss ratio equals the expected loss ratio and work with the target loss ratio. The assumption has merit, because rate changes are implemented each year based on the target loss ratio, and the company attempts to set rates with the goal to have the actual loss ratio equal the target loss ratio. However, this assumption can be problematic in cases where loss ratios in the future are too volatile to project. Appendix B provides two approaches to calculate the target loss ratio using commonly used loss ratio triangles. While readers may find newness in these results, our overarching goal was to make the paper as much self-contained as possible.

In Theorem 1, while no assumption is made about using on-level premiums, most actuaries commonly use on-level premiums for historical accident years. Second, this paper could be extended to project other quantities such as pure premiums. In this paper, we do not discuss such extensions, since actuarial ratemaking and reserving involving projections using historical datasets are widely carried out on loss ratios.[6]

1.3 DATA AND MODEL VALIDATION

To test the validity of our proposed model, we tracked loss ratio data triangles for 15 accident years across three business lines with varying volatility: Commercial Auto, stable volatility; Commercial General Liability, less stable volatility; and Commercial Umbrella, most volatility.[7] The AY triangles commence at year zero. Year zero is the target loss ratio (the target loss ratio is assumed to be the expected ultimate loss ratio. It can be replaced by expected loss ratio when available). Because the data starts at year zero, the model incorporates risk at time 0. After that interval, the model incorporates reserving risk.

The data utilized is obtained from an anonymous but real source; furthermore, the AYs were changed to ensure that the data cannot be used for commercial purposes. Also, the ultimate historical loss ratios were

[5] Reserve ranges are not discussed in this paper, because these have been discussed at length elsewhere. For example, the Rehman-Klugman Method provides a confidence interval for future ultimate loss ratios (Rehman & Klugman 2009) based on actuary's selected loss ratios. The calculated loss ratio provided by this model would also be covered by that paper.

[6] The Pure Premium Method for ratemaking does not require projecting pure premiums using historical data.

[7] It is important to note here that the data obtained will only be utilized to test Theorem 1. Model validation for both theorems in a single paper would make it too long, and a sequel to this paper is suggested to validate Theorem 2.

determined by the actuary using the B-F method. While utilizing the B-F method may seem restrictive, it is important to distinguish that the aforementioned method is general and broad in how it can be incorporated; the age to ultimate loss ratio was adjusted such that it was in accordance with the actuary's selection. Moreover, for the model, we utilize on-level premiums (i.e., premiums at the current rate level) along with trended loss ratios (i.e., loss ratios that have been trended to the current date). Furthermore, using a retrospective model validation, we assessed the strength of our model and compared it to the performance of two alternative methods: latest three-year average and an overall straight average. We quantitively evaluated the efficacy of each model using MSE. It is important to note that our model validation is a retrospective model validation, meaning our model predicts the loss ratio value first and then compares that to the actual future loss ratio. Thus, the loss ratio MSE is calculated as follows:

$$MSE = \left(Actual - \Pr ojected \right)^2 .$$

Section 2. Linear Predictor

The first part of the paper will assume constant parameters over time. We study shifting distributional parameters in the end and will alert the reader when that happens.

2.1 DISTRIBUTION OF EXPERIENCE YEAR LOSS RATIOS

Let the experience years (accident years or policy years) be $s = 1..M, M+1$. A marginal set of lognormal distributions for a random loss ratio $U_{s,M+1}$ with some known parameter set $\{\phi_s, \omega_s^2\}$ are defined as:

$$\ln U_{s,M+1} \sim N\left(\ln\phi_s - \frac{\omega_s^2}{2}, \omega_s^2\right) \qquad (1)$$

Appendix A discusses these distributions.

2.2 THEOREM 1: LINEAR PREDICTOR OF FUTURE YEAR

Define $\{U_{s,M+1} : s = 1, 2..M+1\}$, $\left\{\alpha_s : \sum_{s=1}^{M} \alpha_s = 1, 0 < \alpha_s \le 1\right\}$ and known B-F estimate,[8] $\phi_s > 0$.

Then the MSE predicted loss ratio is given by $U_{M+1}^P = \sum_{s=1}^{s=M} \alpha_s U_{s,M+1}$ whenever,

$$\alpha_s = \left(\sum_{s=1}^{s=M} \frac{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s, \varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s, \varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s, \varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s, \varepsilon_{M+1})}}\right)^{-1} \left(\frac{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s, \varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s, \varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s, \varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s, \varepsilon_{M+1})}}\right)$$

Further given known historical values $U_{s,M+1}^P$, linear predictor is given by,

- $EU_{M+1}^P = \sum_{s=1}^{s=M} \alpha_s EU_{s,M+1} = \sum_{s=1}^{s=M} \alpha_s U_{s,M+1}^P$

One reasonable choice is $U_{s,M+1}^P = \phi_s$.

2.2.1 PROOF

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[8] See Appendix B.

The table below is explained in Appendix 4 and we reproduce here convenience.

Table 4
USING MSE PREDICTOR TO ESTIMATE ONE CELL

	1	2	…	$M-s+1$	$M-s+2$	$M-1$	M	$M+1$
1	$U_{1,1}=u_1$	$U_{1,2}=u_{1,2}$	…	$u_{1,M-s+1}$	$u_{1,M-s+2}$	$u_{1,M-1}$	$u_{1,M}$	$u_{1,M+1}$
2	$U_{2,1}=u_2$	$U_{2,2}=u_{2,2}$	…	$u_{2,M-s+1}$	$u_{2,M-s+2}$	$u_{2,M-1}$	$u_{2,M}$	
…	…	…	…	…	…			
s	$U_{s,1}=u_s$	$U_{s,2}=u_{s,2}$		$u_{s,M-s+1}$	$u_{s,M-s+2}$			
$s+1$	$U_{s+1,1}=u_{s+1}$	$U_{s+1,2}=u_{s+1,2}$		$u_{s+1,M-s+1}$				
…								
M	$U_{M,1}=u_M$	$U_{M,2}=u_{M,2}$						
$M+1$	$U_{M+1,1}=u_{M+1}$							U^P_{M+1}

Define the linear predictor used to predict the next year's loss ratio, $U_{M+1,M+1}$

$$U^P_{M+1}=\sum_{s=1}^{s=M}\rho e^{-|\theta_s|}U_{s,M+1}$$

The quantity ρ is some unknown constant. The uncertainty is due to the fact that $U_{M+1,M+1}$ is random for any year whenever it starts brand new. We will minimize MSE to estimate the set θ_s ,

$$E\left[\sum_{s=1}^{s=M}\rho e^{-|\theta_s|}U_{s,M+1}-U_{M+1,M+1}\right]^2$$

$$\frac{\partial}{\partial\theta_s}E\left[\sum_{s=1}^{s=M}\rho e^{-|\theta_s|}U_{s,M+1}-U_{M+1,M+1}\right]^2=0$$

CASE 1: $\left\{\theta_s : (\theta_s \ge 0), s \in \{1, 2 \ldots M\}\right\}$.

Some of the $\theta_s \ge 0$ but not necessarily all, and we restrict ourselves to such cases when $\theta_s \ge 0$. Proceeding with differentiation,

$$E\left\{\left[\sum_{s=1}^{s=M} \rho e^{-\theta_s} U_{s,M+1} - U_{M+1,M+1}\right](-\rho) e^{-\theta_s} U_{s,M+1}\right\} = 0$$

$$E\left(U_{s,M+1} \sum_{s=1}^{s=M} \rho e^{-\theta_s} U_{s,M+1}\right) = E\left(U_{M+1,M+1} U_{s,M+1}\right)$$

$$\rho e^{-\theta_s}\left[E\left(U_{1,M+1} U_{s,M+1}\right) + E\left(U_{2,M+1} U_{s,M+1}\right) + \ldots + E\left(U_{M,M+1} U_{s,M+1}\right)\right] = E\left(U_{M+1,M+1} U_{s,M+1}\right).$$

From equation (1),

$$\ln U_{s,M+1} U_{t,M+1} = \ln U_{s,M+1} + \ln U_{t,M+1} \sim N\left(\ln \phi_s - \frac{\omega^2}{2} + \ln \phi_t - \frac{\omega^2}{2}, \omega^2 + \omega^2 + 2\operatorname{cov}(\varepsilon_s, \varepsilon_t)\right).$$

Hence, $U_{s,M+1} U_{t,M+1}$ is also lognormal. Therefore,

$$E\left[U_{s,M+1} U_{t,M+1}\right] = \exp\left[\ln \phi_s - \frac{\omega^2}{2} + \ln \phi_t - \frac{\omega^2}{2} + \frac{\omega^2 + \omega^2 + 2\operatorname{cov}(\varepsilon_s, \varepsilon_t)}{2}\right]$$

$$E\left[U_{s,M+1} U_{t,M+1}\right] = \phi_s \phi_t e^{\operatorname{cov}(\varepsilon_s, \varepsilon_t)}.$$

Thus, we get,

$$\rho e^{-\theta_s}\left[\phi_s\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)}+\phi_s\phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)}+\ldots+\phi_s\phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}\right]=\phi_s\phi_{M+1}e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}$$

$$\theta_s=-\ln\frac{1}{\rho}\frac{\phi_{M+1}e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)}+\phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)}+\ldots+\phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}} \quad . \quad . \quad . \quad . \qquad (2)$$

CASE 2: $\left\{\theta_s:(\theta_s<0), s\in\{1,2\ldots M\}\right\}$.

Some of the $\theta_s<0$ but not necessarily all, and we restrict ourselves to such cases when $\theta_s<0$. Let $\theta_s=-\tau_s, \tau_s>0.$ Proceeding with differentiation,

$$E\left\{\left[\sum_{s=1}^{s=M}e^{-\tau_s}U_{s,M+1}-U_{M+1,M+1}\right](-\rho)e^{-\tau_s}U_{s,M+1}\right\}=0 .$$

The above is identical to case 1 except that τ_s replaces θ_s. We can proceed with minimizing the MSE with respect to τ_s to obtain a similar result as case 1,

$$\theta_s=-\tau_s=\ln\frac{1}{\rho}\frac{\phi_{M+1}e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)}+\phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)}+\ldots+\phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}} \quad . \quad . \quad . \qquad (4)$$

Combining both cases,

$$|\theta_s|=\left|-\ln\frac{1}{\rho}\frac{\phi_{M+1}e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)}+\phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)}+\ldots+\phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right|$$

$$|\theta_s|=\left|\ln\frac{1}{\rho}\frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)}+\phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)}+\ldots+\phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1}e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right| \quad . \quad . \quad . \quad . \qquad (5)$$

The unit sum constraint (both cases) requires,

$$\sum_{s=1}^{s=M} \rho e^{-|\theta_s|} = 1$$

$$\sum_{s=1}^{s=M} \rho e^{-\left|\ln \frac{1}{\rho} \frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right|} = 1$$

Note that since $\phi_s > 0$, we require $\rho > 0$. Additionally, since ϕ_{M+1} is of the same magnitude as ϕ_s, the only solution exists when

$$\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})} > \phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}.$$

$$\ln \frac{1}{\rho} \frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}} > 0$$

In this case, we require,

$$0 < \rho < \frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}. \qquad (6)$$

Then the solution is,

$$\rho = +\left[\sum_{s=1}^{s=M} \frac{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right]^{\frac{-1}{2}} \qquad (7)$$

An intuitive explanation of the formula is given below. Suppose that data is such that

$$\ln \frac{1}{\rho} \frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}} > 0$$

Then the weight,

$$\alpha_s = \rho e^{-|\theta_s|} = \rho e^{-\left|\ln \frac{1}{\rho} \frac{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right|}$$

$$\alpha_s = \frac{\left(\frac{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}\right)}{\sum_{s=1}^{s=M} \frac{\phi_{M+1} e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}{\phi_1 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_1)} + \phi_2 e^{\operatorname{cov}(\varepsilon_s,\varepsilon_2)} + \ldots + \phi_M e^{\operatorname{cov}(\varepsilon_s,\varepsilon_{M+1})}}}. \qquad \blacksquare$$

The above shows the “balance” the formula provides:

1. If all years are uncorrelated with the projected year M+1, then all years would get an equal weight of M^{-1}. This is the “default” situation.

2. Positively correlated years with the projected year M+1 will get a larger weight than the default case. These years carry the highest predictive power. The most recent years will have maximal impact, because the older years have decreasing covariance terms with the oldest year having 0 covariance terms, because it is fully developed and has no randomness.

3. Negatively correlated years with the projected year M+1 will get a lower weight than the default case. These reflect cyclicality years and carry the least predictive power. The most recent years will have maximal impact, because the older years have decreasing covariance terms with the oldest year having 0 covariance terms, because it is fully developed and has no randomness.

2.3 THEOREM 2

Suppose for a fixed $s>1$, we are given coefficients and weights $\left\{\alpha_t : t=1,2..s-1, \sum_{t=1}^{s-1}\alpha_t = 1\right\}$ under Theorem 1. Let the random latest emergent observation be $u_{s,M-s+2}$ for the s year (Appendix A) and B-F estimates ϕ_s given (Appendix B). Then an MSE predictor of $U_{s,M+1}$ is given by,

$$U_{s,M+1}^{P} = (1-\beta_s)\sum_{t=1}^{t=s-1}\alpha_t\phi_t + \beta_s\phi_s$$

$$\beta_s = e^{-|\mu_s|}$$

$$|\mu_s| = \left|\ln\frac{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}{\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t + \phi_s\sum_{t=1}^{t=s-1}\alpha_t e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}\right|$$

2.3.1 PROOF

The derivation is similar to Theorem 1 with the following situation: moving row-wise (see Table 5) for any single historical year $s\in\{1,..,M\}$. We will also use only two different loss ratios to be combined in the linear predictor. One is $\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}$ with coefficients α_t given under Theorem 1 and the other given by ϕ_s.

Table 5

COMBINING TWO PREDICTORS TO ESTIMATE ONE CELL

	1	2	…	$M-s+1$	$M-s+2$	$M-1$	M	$M+1$
1	$U_{1,1}=u_1$	$U_{1,2}=u_{1,2}$	…	$u_{1,M-s+1}$	$u_{1,M-s+2}$	$u_{1,M-1}$	$u_{1,M}$	$u_{1,M+1}$
2	$U_{2,1}=u_2$	$U_{2,2}=u_{2,2}$	…	$u_{2,M-s+1}$	$u_{2,M-s+2}$	$u_{2,M-1}$	$u_{2,M}$	
…	…	…	…	…	…			
s	$U_{s,1}=u_s$	$U_{s,2}=u_{s,2}$		$u_{s,M-s+1}$	$u_{s,M-s+2}$			$U_{s,M+1}$

$s+1$	$U_{s+1,1}=u_{s+1}$	$U_{s+1,2}=u_{s+1,2}$		$u_{s+1,M-s+1}$				
…								
M	$U_{M,1}=u_M$	$U_{M,2}=u_{M,2}$						

The linear predictor for $U_{s,M+1}$ under unit sum coefficient constraint is

$$U_{s,M+1}^{P}=\left(1-\beta_s\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+\beta_s\phi_s .$$

Since it is desirable to have $0\leq\beta_s\leq 1$, we will setup the problem differently. Define,

$$\beta_s=e^{-\left|\mu_s\right|} .$$

The above model formulation ensures that $0\leq\beta_s\leq 1$. Setting up the MSE with respect to μ_s,

$$E\left[\left(1-e^{-\left|\mu_s\right|}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+e^{-\left|\mu_s\right|}\phi_s-U_{s,M+1}\right]^2$$

CASE 1 ($\mu_s\geq 0$):

$$E\left[\left(1-e^{-\mu_s}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+e^{-\mu_s}\phi_s-U_{s,M+1}\right]^2 .$$

Taking derivatives with respect to μ_s and setting it equal to zero,

$$2E\left\{\left[\left(1-e^{-\mu_s}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+e^{-\mu_s}\phi_s-U_{s,M+1}\right]\left[\left(e^{-\mu_s}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}-\lambda_s\phi_s e^{-\mu_s}\right]\right\}=0$$

$$E\left\{\left[\left(\frac{1-e^{-\mu_s\lambda_s}}{e^{-\mu_s\lambda_s}}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+\phi_s-\frac{U_{s,M+1}}{e^{-\mu_s\lambda_s}}\right]\left[\phi_s-\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right]\right\}=0$$

$$E\left\{\left[\left(e^{\mu_s}-1\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}+\phi_s-e^{\mu_s}U_{s,M+1}\right]\left[\phi_s-\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right]\right\}=0$$

$$E\left\{\left[\left(e^{\mu_s}-1\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right]\phi_s\right\}-E\left\{\left[\left(e^{\mu_s}-1\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right]\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right\}$$
$$+E\left\{\left[\phi_s-e^{\mu_s}U_{s,M+1}\right]\phi_s\right\}-E\left\{\left[\phi_s-e^{\mu_s}U_{s,M+1}\right]\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1}\right\}=0$$

$$e^{\mu_s}\phi_s\sum_{t=1}^{t=s-1}\alpha_t EU_{t,M+1}-\phi_s\sum_{t=1}^{t=s-1}\alpha_t EU_{t,M+1}-e^{\mu_s}\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r E\left(U_{t,M+1}U_{r,M+1}\right)+\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r E\left(U_{t,M+1}U_{r,M+1}\right)$$
$$+\phi_s^2-\phi_s e^{\mu_s}EU_{s,M+1}+\phi_s^2-e^{\mu_s}\phi_s EU_{s,M+1}-\phi_s\sum_{t=1}^{t=s-1}\alpha_t EU_{t,M+1}+e^{\mu_s}\sum_{t=1}^{t=s-1}\alpha_t E\left(U_{s,M+1}U_{t,M+1}\right)=0$$

Replacing, $EU_{t,M+1}=\phi_t$ and using Appendix C, $E\left[U_{s,M+1}U_{t,M+1}\right]=e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_s$,

$$e^{\mu_s}\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t-\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t-e^{\mu_s}\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r+\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r$$
$$+\phi_s^2-\phi_s e^{\mu_s}\phi_s+\phi_s^2-e^{\mu_s}\phi_s^2-\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t+e^{\mu_s}\sum_{t=1}^{t=s-1}\alpha_t e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_s=0$$

$$\mu_s=\ln\frac{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t-2\phi_s^2-\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}{\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t+\phi_s\sum_{t=1}^{t=s-1}\alpha_t e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t-2\phi_s^2-\sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}.$$

CASE 2 ($\mu_s < 0$):

Let $\mu_s = -\lambda_s, \lambda_s > 0.$ The MSE is

$$E\left[\left(1-e^{-\lambda_s}\right)\sum_{t=1}^{t=s-1}\alpha_t U_{t,M+1} + e^{-\lambda_s}\phi_s - U_{s,M+1}\right]^2.$$

The above is identical to CASE 1, and the solution is

$$\mu_s = -\lambda_s = -\ln\frac{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}{\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t + \phi_s\sum_{t=1}^{t=s-1}\alpha_t e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}.$$

Combining both cases,

$$|\mu_s| = \left|\ln\frac{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}{\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t + \phi_s\sum_{t=1}^{t=s-1}\alpha_t e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)}\phi_t\phi_r}\right|.$$

A special case arises when the year s is fully developed, and the best predictor should be $U^P_{s,M+1} = U_{s,M-s+2}$, implying $\beta_s = e^{-|\mu_s|} = 1$ and hence $\mu_s = 0$. We can recover this special case from the above formula by noting that years $t < s$ will also be fully developed. Hence, $e^{\operatorname{cov}(\varepsilon_s,\varepsilon_t)} = 1$. Then,

$$|\mu_s| = \left|\ln\frac{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t - 2\phi_s^2 - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r\phi_t\phi_r}{2\phi_s\sum_{t=1}^{t=s-1}\alpha_t\phi_t - \sum_{t=1}^{t=s-1}\sum_{r=1}^{r=s-1}\alpha_t\alpha_r\phi_t\phi_r - 2\phi_s^2}\right| = 0. \qquad \blacksquare$$

The above special case shows that the formula is most useful in projecting (subject) recent years as in those cases $\beta_s < 1$, and the older years will get more weight in the formula than the subject (recent) year because there is little emergence of data to make the B-F estimate credible. In loss reserving, it is indeed the case that recent years hold the largest reserves and require valuable insights.

The most important term in the formula is in the denominator $\sum_{t=1}^{t=s-1} \alpha_t e^{\operatorname{cov}(\varepsilon_s, \varepsilon_t)} \phi_t$. If the past $s-1$ years are uncorrelated with the year s (no pattern in past data), $\operatorname{cov}(\varepsilon_s, \varepsilon_t) = 0$ and $\beta_s = 1$ (shown above). However, if $\operatorname{cov}(\varepsilon_s, \varepsilon_t) = 0$ and there is positive correlation between past $s-1$ years and year s (pattern in the data), this displaces less weight on the B-F method and picks up the pattern in the historical loss ratios through $\beta_s < 1$. The magnitude of the displacement depends on the degree of the net correlation.

Section 3. Data Analysis and Findings

With regard to our findings, it is critical to start with a walkthrough of the model and how the predicted loss ratio is derived. Due to space restrictions, the data illustrated in the process will be limited to seven AYs. However, it is important to note that the model analysis was actually performed on a 15-by-15 triangle. To begin with, we will be walking through an example analysis and prediction of the Commercial Umbrella, which has the most volatility.

3.1 LOSS RATIO PREDICTION PROCESS

Starting off, the loss ratios are displayed in the first table below. Table 6 displays how the model encompasses both risk at time 0, displayed by the interval from AY zero to one, and reserving risk, displayed by the intervals after AY one. These ratios have been calculated by dividing the incurred loss by the on-level premium; but because these loss ratios aren't up to the current date, the ratios need to be trended to the level of year 15, which is displayed by the second table, Table 7. With the loss ratios being trended, the trended Ultimate B-F ratio can be produced, which we consider as our ϕ's.

Table 6

INCURRED LOSS RATIOS[9]

	Target LR	Incurred Loss Ratios						
Accident Loss Year	0	1	2	3	4	5	6	7
1	62.0%	2.9%	11.1%	11.1%	11.9%	12.3%	17.8%	19.4%
2	62.0%	0.0%	0.4%	0.1%	0.2%	3.5%	3.5%	3.5%
3	62.0%	2.3%	13.7%	13.8%	13.8%	13.8%	13.8%	14.4%
4	62.0%	1.2%	1.2%	2.4%	2.4%	2.4%	2.4%	2.4%
5	62.0%	19.0%	27.9%	26.7%	31.3%	31.4%	44.0%	44.0%
6	62.0%	0.0%	0.0%	0.0%	0.0%	1.0%	0.0%	2.5%
7	62.0%	0.0%	18.3%	23.6%	36.6%	57.6%	59.0%	59.0%

[9] Target loss ratio can be different for AYs. In this case, it happens to be equal.

Table 7
TRENDED INCURRED LOSS RATIOS

	Target LR	Trended Incurred Loss Ratios						
Accident Loss Year	0	1	2	3	4	5	6	7
1	0.62	4.5%	17.2%	17.1%	18.3%	18.9%	27.4%	30.0%
2	0.62	0.1%	0.6%	0.1%	0.4%	5.4%	5.4%	5.4%
3	0.62	3.5%	20.8%	20.8%	20.8%	20.8%	20.8%	21.8%
4	0.62	1.8%	1.8%	3.6%	3.6%	3.6%	3.6%	3.6%
5	0.62	28.2%	41.4%	39.6%	46.4%	46.5%	65.2%	65.2%
6	0.62	0.0%	0.0%	0.0%	0.0%	1.5%	0.0%	3.7%
7	0.62	0.0%	26.6%	34.2%	53.2%	83.6%	85.6%	85.6%

Following the calculation of the trended Ultimate B-F ratios, which will be utilized in the latter steps, the calculation of the errors[10] can be performed, which is displayed by the Table 8.

Table 8
ERRORS

	Errors						
Accident Loss Year	1	2	3	4	5	6	7
1	−2.61910	1.33444	−0.00099	0.06494	0.03398	0.37031	0.08883
2	−6.77586	2.09647	−1.69420	1.24223	2.69356	0.00000	−0.00467
3	−2.88681	1.79343	0.00050	0.00000	−0.00017	0.00000	0.04870
4	−3.54946	0.00359	0.68955	0.00000	0.00265	−0.00265	0.00000
5	−0.78731	0.38351	−0.04394	0.15685	0.00413	0.33649	0.00110
6	−9.04828	0.26159	1.57880	−1.57880	5.04171	−5.30330	6.21661

[10] Errors are calculated by performing the natural log of the trended loss ratios divided by the target loss ratio; and if the natural log cannot be performed, the error will result in zero.

We can then proceed to produce our covariance from the errors table.[11] Table 9 and Table 10 display covariance. In Table 10, it is imperative to note that the covariance is zero for the first accident year because, as explained in the introduction, the data are complete, so there is no covariance.

Table 9

COVARIANCE

	Covariance - COV(εs,q,εt,r)						
	1	2	3	4	5	6	7
1	1783.84519%	−1166.28926%	153.07308%	151.12301%	−540.42889%	319.85186%	−428.60678%
2	−1166.28926%	1507.98589%	−288.33474%	117.50124%	−315.78544%	17.44929%	−25.55556%
3	153.07308%	−288.33474%	825.45940%	156.08415%	−400.57156%	−56.61313%	39.03332%
4	151.12301%	117.50124%	156.08415%	337.23832%	−720.71137%	56.54713%	−113.83959%
5	−540.42889%	−315.78544%	−400.57156%	−720.71137%	1633.88366%	-162.58023%	286.04787%
6	319.85186%	17.44929%	−56.61313%	56.54713%	−162.58023%	263.22387%	−330.26139%
7	−428.60678%	−25.55556%	39.03332%	−113.83959%	286.04787%	−330.26139%	380.25163%

Table 10

COVARIANCE POST 1ST ACCIDENT YEAR

	Covariance - COV(εs,εt)							
	1	2	3	4	5	6	7	8
1	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%
2	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%
3	0.00000%	0.00000%	0.25275%	1.43752%	1.43574%	1.43217%	1.67770%	2.18789%
4	0.00000%	0.00000%	1.43752%	7.21418%	6.85006%	6.83218%	8.07903%	10.49200%
5	0.00000%	0.00000%	1.43574%	6.85006%	6.60632%	6.58917%	7.78336%	10.12463%
6	0.00000%	0.00000%	1.43217%	6.83218%	6.58917%	6.70101%	8.04769%	10.44597%
7	0.00000%	0.00000%	1.67770%	8.07903%	7.78336%	8.04769%	9.70782%	12.42276%
8	0.00000%	0.00000%	2.18789%	10.49200%	10.12463%	10.44597%	12.42276%	15.74629%

[11] For example, the covariance for row two, column two in Table 9 is calculated by taking the covariance between the error in column one and the error in column two in Table 8.

After the covariances are produced, the next step is producing the product of the ϕ's, which are the trended Ultimate B-F ratios that were calculated in a former step and the exponentials of the covariances as displayed in Table 11. Finally, after the products are calculated, the ratios and the weights, α's, can be derived. With the weights produced, we can finally derive next year's predicted loss ratio by taking the summation of each ϕ multiplied by the corresponding α . In Table 12, it is vital to point out all 15 years have been displayed to demonstrate the predicted 16th year loss ratio.

Table 11
PRODUCT OF B-F RATIO AND COVARIANCES

	Product of ϕ's and Exponential of Covariances -- ϕ·EXP(Cov(ε's))						
	1	2	3	4	5	6	7
1	0.28096334	0.80940946	0.27216902	0.21040947	1.21357299	0.26509313	1.00828807
2	0.28096334	0.80940946	0.27216902	0.21040947	1.21357299	0.26509313	1.00828807
3	0.28096334	0.80940946	0.27285779	0.21345601	1.2311224	0.26891703	1.02534686
4	0.28096334	0.80940946	0.27610977	0.22614974	1.29961686	0.28383781	1.09312897
5	0.28096334	0.80940946	0.27610483	0.22532777	1.29645309	0.2831489	1.08990172
6	0.28096334	0.80940946	0.27609498	0.22528748	1.29623072	0.28346576	1.09278649
7	0.28096334	0.80940946	0.27677373	0.22811407	1.31180298	0.28730896	1.11107959

Table 12
FINAL OUTPUTS

	Ratios	α_s	Ultimates
1	0.07038	0.13603	28.1%
2	0.07038	0.13603	80.9%
3	0.06727	0.13002	27.2%
4	0.05467	0.10566	21.0%
5	0.05576	0.10777	121.4%
6	0.05750	0.11114	26.5%
7	0.05973	0.11545	100.8%
8	0.05655	0.10929	29.4%
9	0.01966	0.03800	60.2%
10	0.00021	0.00041	102.0%
11	0.00524	0.01013	119.5%
12	0.00000	0.00000	92.9%
13	0.00000	0.00000	133.2%
14	0.00004	0.00007	124.1%
15	0.00000	0.00000	102.5%
	AY 16 Predicted LR		55.0%

3.2 SUMMARY AND FINDINGS

[12]Throughout our data analysis and model validation process, the overarching conclusion was that the model proposed in this paper performed considerably better than a common industry practice, the three-year average. Comparing across all three business lines, during stable volatility (Commercial Auto), all three methods performed relatively well. As shown in Table 13, the MSE of the three-year average, .0243, is slightly better than the MSE of the other two methods, .02816 and .02816 but within striking distance.

[12] Reserve ranges are not discussed in this paper, because these have been discussed at length elsewhere. For example, the Rehman-Klugman Method provides a confidence interval for future ultimate loss ratios (Rehman & Klugman 2009) based on actuary's selected loss ratios. Hence, the calculated loss ratio provided by this model would also be covered by that paper.

Table 13
COMMERCIAL AUTO—STABLE DATA

		Predicted LR		
Acc Year	Actual LR	Proposed Model	Straight Average	Last 3 years
16	0.88531	0.94704	0.94705	0.89917
17	0.87954	0.93618	0.93618	0.89618
18	0.84673	0.93040	0.93040	0.86886
19	0.85557	0.91474	0.91474	0.85630
20	0.87175	0.89871	0.89872	0.84334
21	0.85501	0.88587	0.88587	0.82222
22	0.85073	0.88211	0.88211	0.82504
23	0.84064	0.88047	0.88048	0.82236
24	0.83851	0.87268	0.87269	0.80122
25	0.84527	0.85971	0.85971	0.79866
26	0.85507	0.85448	0.85448	0.79915
27	0.88052	0.85201	0.85200	0.79957
28	0.86128	0.83457	0.83457	0.79056
29	0.82942	0.82930	0.82930	0.79236
30	0.74238	0.80140	0.80140	0.77140

MSE		
Proposed Model	Straight Average	Last 3 years
0.00381	0.00381	0.00019
0.00321	0.00321	0.00028
0.00700	0.00700	0.00049
0.00350	0.00350	0.00000
0.00073	0.00073	0.00081
0.00095	0.00095	0.00108
0.00098	0.00099	0.00066
0.00159	0.00159	0.00033
0.00117	0.00117	0.00139
0.00021	0.00021	0.00217
0.00000	0.00000	0.00313
0.00081	0.00081	0.00655
0.00071	0.00071	0.00500
0.00000	0.00000	0.00137
0.00348	0.00348	0.00084
0.02816	0.02816	0.02430

But once the business lines start to have more volatility, there is a stark difference in model accuracy, because the three-year average lags behind. When analyzing the Commercial GL business line with less stable volatility, it is quite lucid that the three-year average begins to underperform. Table 14 reaffirms this fact, and it is clear that the MSE of the three-year average, .27614, is underperforming quite significantly when compared to the MSEs of the other two methods, .18830 and .18814.

Table 14

COMMERCIAL GL—LESS STABLE DATA

		Predicted LR			MSE		
Acc Year	Actual LR	Proposed Model	Straight Average	Last 3 years	Proposed Model	Straight Average	Last 3 years
16	0.60675	0.74226	0.74218	0.85545	0.01836	0.01834	0.06185
17	0.66135	0.76336	0.76324	0.94771	0.01041	0.01038	0.08200
18	0.89079	0.71109	0.71104	0.86244	0.03229	0.03231	0.00080
19	0.76316	0.71528	0.71531	0.82324	0.00229	0.00229	0.00361
20	0.87433	0.72310	0.72316	0.86119	0.02287	0.02285	0.00017
21	0.74887	0.71014	0.71024	0.90708	0.00150	0.00149	0.02503
22	0.84500	0.70577	0.70593	0.87133	0.01938	0.01934	0.00069
23	0.81658	0.70991	0.71011	0.88932	0.01138	0.01134	0.00529
24	0.94813	0.73783	0.73792	0.89841	0.04423	0.04419	0.00247
25	0.74547	0.74928	0.74944	0.93342	0.00001	0.00002	0.03532
26	0.78772	0.75000	0.75014	0.87572	0.00142	0.00141	0.00774
27	0.66745	0.76108	0.76111	0.86517	0.00877	0.00877	0.03909
28	0.69416	0.76385	0.76395	0.78330	0.00486	0.00487	0.00795
29	0.74921	0.78878	0.78885	0.76952	0.00157	0.00157	0.00041
30	0.67586	0.77049	0.77056	0.73667	0.00896	0.00897	0.00370
					0.18830	0.18814	0.27614

Furthermore, during the most volatile situations, which is the Commercial Umbrella business line, our proposed method truly begins to distinguish itself from the other two methods. We summarize how the proposed method and the other methods perform over the experience years based on the average MSE. In Table 15, the MSE of our proposed model, 1.3495, compared to the MSE of the other two methods, 2.25526 and 4.30800, illustrates the efficacy of our proposed model and reaffirms the notion that the current industry practice falls short quite remarkably.

Table 15
COMMERCIAL UMBRELLA—VOLATILE DATA

		Predicted LR			MSE		
Acc Year	Actual LR	Proposed Model	Straight Average	Last 3 years	Proposed Model	Straight Average	Last 3 years
16	0.49662	0.55021	0.77984	1.19923	0.00287	0.08021	0.49366
17	0.05442	0.53565	0.79813	1.06617	0.23159	0.55311	1.02364
18	0.08553	0.46989	0.75034	0.95359	0.14773	0.44197	0.75352
19	0.67143	1.01800	1.02891	0.86594	0.12011	0.12779	0.03783
20	0.05997	0.53104	0.70886	0.86492	0.22191	0.42106	0.64795
21	0.72346	0.65532	0.73263	0.85197	0.00464	0.00008	0.01651
22	0.65134	0.63677	0.68840	0.92844	0.00021	0.00137	0.07679
23	0.58267	0.59113	0.64774	0.92687	0.00007	0.00423	0.11848
24	0.46779	0.49795	0.59621	0.90427	0.00091	0.01649	0.19052
25	1.00313	0.50457	0.65056	0.87413	0.24856	0.12430	0.01664
26	0.84827	0.66540	0.72758	0.96983	0.03344	0.01457	0.01478
27	0.28580	0.71183	0.78574	0.97223	0.18150	0.24994	0.47120
28	0.73447	0.68999	0.74269	0.89522	0.00198	0.00007	0.02584
29	0.48957	0.69368	0.72947	0.82586	0.04166	0.05755	0.11309
30	0.25914	0.59425	0.66225	0.81370	0.11230	0.16250	0.30754
					1.34950	2.25526	4.30800

Overall, after analyzing each business line, the performance of the current industry practice fails to perform adequately. Instead, utilizing the proposed model will enhance the way ratemaking is performed.

Section 4. Shifting Parameters Over Time

All parameters—the calculated coefficients in Theorems 1 and 2—have been assumed fixed so far. In practice, this never happens, and these parameters change or shift over time. In practice, time-dependent parameters are updated as new data emerges in data triangles, possibly annually and no explicit formulas for such time dependent parameters exist.

4.1 MODEL STABILITY

For the weights to be stable under Theorems 1 and 2, we would require the entire covariance matrix to remain unchanged over time. This is unlikely to happen, and the weights should be re-derived at each such study. Instead, we explore the changes in the total variance, namely sum of entries of the covariance matrix, ω^2. Additionally, this leads to the stability of target loss ratio. We call this phenomenon "Model Stability."

Second, we will need to invoke arguments that involve some basic understanding of loss reserving and loss emergence. To explore model stability, we need to develop the concept of reserve function.

4.1.1. DEFINITION OF RESERVE FUNCTION

Let the written premiums be denoted by $p_k(t)$ at time t for year k. Denote cumulative reserve (due to newly added year only) to premium ratio function, resulting from newly written premiums by $c_{M+1}(t)$. Then reserve function $\theta_k(t)$ is defined by the differential equation for some function $\pi_k(t)$, and all functions are assumed differentiable and $t \in (0,1)$.

$$\sum_{k=1}^{M+1} \theta_k^{'}(t)\left\{1-\pi_k(t)\right\} = p_{M+1}^{'}(t)c_{M+1}(t) \qquad \blacksquare$$

The above definition is motivated as follows: Over the course of a year $t \in (0,1)$, the insurance company issues policies at different times, and policies age continuously, supporting development of a continuous model that explicitly incorporates time. In a small but finite time interval $\delta t > 0$, the reserve function changes due to aging of existing years and grows due to addition of a new year. In a time interval δt, a new set of policies are written with premium volume $p_k(t+\delta t) - p_k(t)$ at a reserve to premium ratio function $c_k(t)$. Let $\left\{\theta_k(t+\delta t) - \theta_k(t)\right\}\pi_k(t)$ to represent reserve change due to aging, where $\pi_k(t)$ represents some multiplicative function for aging. Thus,

$$\sum_{k=1}^{M+1}\left\{\theta_k(t+\delta t)-\theta_k(t)\right\}=\sum_{k=1}^{M+1}\left\{p_k(t+\delta t)-p_k(t)\right\}c_k(t)+\sum_{k=1}^{M+1}\left\{\theta_k(t+\delta t)-\theta_k(t)\right\}\pi_k(t).$$

The function $\pi_k(t)$ accounts for growth or decline in premiums relative to reserve function. For example, if $p_k(t+\delta t)=p_k(t)\ \forall k$, then $\pi_k(t)=1$. Assuming[13] that $\theta_k(t+\delta t)\neq\theta_k(t)$,

$$\sum_{k=1}^{M+1}\frac{\left\{\theta_k(t+\delta t)-\theta_k(t)\right\}}{\delta t}\left\{1-\pi_k(t)\right\}=\sum_{k=1}^{M+1}\frac{\left\{p_k(t+\delta t)-p_k(t)\right\}c_k(t)}{\delta t}.$$

Taking limits $\delta t\to 0$ and assuming $\theta_k(t)$ and $p_k(t)$ are differentiable,

$$\sum_{k=1}^{M+1}\theta_k^{'}(t)\left\{1-\pi_k(t)\right\}=\sum_{k=1}^{M+1}p_k^{'}(t)c_k(t).$$

Now $p_k^{'}(t)=0$ for $k=1..M$ as cumulative premiums for fixed for older years. Hence,

$$\sum_{k=1}^{M+1}\theta_k^{'}(t)\left\{1-\pi_k(t)\right\}=p_{M+1}^{'}(t)c_{M+1}(t).$$

4.1.2 DEFINITION OF STABILITY (REVISITED)

We define stability as a condition whenever $\pi_k(t)=\pi_k$ (constant decay) and $c_{M+1}(t)=c_{M+1}$ for $t>0$. The motivation is as follows. From our derivations for priori distributions, we know that under a constant c_{M+1} and ω^2, the parameter u_{M+1} will remain unchanged. The condition $\pi_k(t)=\pi_k$ implies that error triangle in Figure 2 will be "stable". Hence ω^2 will be stable.

Theorem 3

A necessary and sufficient condition for stability is given by

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[13] In that case, $p_{M+1}(t+\delta t)=p_{M+1}(t)$ and $\pi_k(t)=1$.

$$\sum_{k=1}^{M+1} \theta_k(t)\left\{1-\pi_k(t)\right\} = p_{M+1}(t)c_{M+1}(t) .$$

Hence, additional reserve required due to a new premium at any time $t > 0$ equals release of reserve due to aging older years.

Proof

From the definition of reserve function, we can solve the first order differential equation using second fundamental theorem of calculus,

$$\sum_{k=1}^{M+1} \int_0^t \theta_k'(t)\left\{1-\pi_k(t)\right\} dt = \int_0^t p_{M+1}'(t)c_{M+1}(t) .$$

Since $p_{M+1}(0) = 0$ and $\pi_k(0) = 1$ as a boundary condition and integrating by parts,

$$\sum_{k=1}^{M+1} \left[\theta_k(t)\left\{1-\pi_k(t)\right\}\right]_0^t + \sum_{k=1}^{M+1} \int_0^t \theta_k(t)\pi_k'(t)dt = \left[p_{M+1}(t)c_{M+1}(t)\right]_0^t - \int_0^t p_{M+1}(t)c_{M+1}(t)dt$$

$$\sum_{k=1}^{M+1} \theta_k(t)\left\{1-\pi_k(t)\right\} + \sum_{k=1}^{M+1} \int_0^t \theta_k(t)\pi_k'(t)dt = p_{M+1}(t)c_{M+1}(t) - \int_0^t p_{M+1}(t)c_{M+1}'(t)dt . \quad (8)$$

From the definition of stability $\pi_k(t) = \pi_k$ (constant decay) and $c_{M+1}(t) = c_{M+1}$ for $t > 0$. Hence, $\sum_{k=1}^{M+1} \theta_k(t)\left\{1-\pi_k(t)\right\} = p_{M+1}(t)c_{M+1}(t)$. For sufficiency, suppose $\dfrac{\sum_{k=1}^{M+1} \theta_k(t)\left\{1-\pi_k(t)\right\}}{c_{M+1}(t)} = p_{M+1}(t)$.

Then from (8), $\sum_{k=1}^{M+1} \int_0^t \theta_k(t)\pi_k'(t)dt = -\int_0^t \dfrac{c_{M+1}'(t)}{c_{M+1}(t)} \sum_{k=1}^{M+1} \theta_k(t)\left\{1-\pi_k(t)\right\}dt$. One solution to this equation is: $\pi_k(t) = \pi_k$ (constant decay) and $c_{M+1}(t) = c_{M+1}$.

Section 5. Conclusion

This paper provides a stable and accurate weighting structure using minimum mean square prediction error criterion to predict loss ratio or pure premiums for ratemaking and loss reserving.

The modeled ratemaking weights were validated using retro-testing and real-world datasets using loss ratio triangles. Three different models (the current industry practice, the straight average and our proposed model) were tested along three different business lines with varying volatility. Compared with a common industry practice being the latest three-year average, the results of the data analysis conclusively illustrate the deficiency of current industry practice. Over all three business lines, the current industry practice performed relatively effectively only for the Commercial Auto business line. As volatility increased across the business lines, the industry practice significantly underperformed, whereas the model proposed in this paper conclusively performed at the worst on par with the industry practice and at best performed remarkably better than current industry practice. The proposed model, even under shifting parameters (i.e., time), still performed at a more stable and accurate rate compared to the current industry practice. While the proposed model doesn't obviate all errors, it enjoys minimum squared prediction errors, and the industry can and should incorporate it into the ratemaking process, due to its superior ability to predict a quality loss ratio. As prediction of loss estimates improves, the proposed model serves to provide a more theoretical correct rates or loss, but it should be noted this proposed model is not the only method; it is just one of the reasonable methods among others.

Appendix A: Distribution for Incomplete Year

Lowercase letters denote realized values; these are known and fixed. Uppercase letters denote random variables. In Table 16, half the rectangle is observed, and the remaining is unobserved. Our interest lies in the unobserved values for future policy year $i = M + 1$. The unobserved errors $\{\varepsilon_{M+1,q}\}_{1 \le q \le M}$ are generally postulated to be multivariate normally distributed.[14] More important, for estimation purposes, we make the following two assumptions (Table 17):

(i) Entries in any given column have the same marginal variance. This permits sample variance as an estimate of population variance $Var(\varepsilon_{M+1,q})$.

(ii) Entries in any two columns have the same covariance. This permits sample covariance as an estimate of population covariance $Cov\left(\varepsilon_{M+1,q}, \varepsilon_{M+1,p}\right)$ with $p \ne q$.

The last two postulates are necessary for calibration of the parameters. However, note the following caveats. First, as new data come each year, the sample estimates are updated. Second, the old “completed rows” (Table 16) greatly enhance accuracy in estimation and should be retained. Reference to these postulates will not be provided, henceforth in the paper, and all estimation is done using these postulates.

The premiums underlying the different years in Table 16 are usually on-level; they are recalculated at current rate levels (including past years). This is necessary to make all years identical except for different random loss experience.

By construction, the first column in Figure 3 is always the target loss ratio of the respective policy years. For year M+1, the company uses target loss ratio in its rate filings $U_{M+1,1} = u_{M+1}$. We will assume that u_{M+1} is available, and its measurement is discussed in Appendix B.

Table 16
DATA TRIANGLE OF LOSS RATIO [MXM+1]

	1	2	…	$M-s+1$	$M-s+2$	$M-1$	M	$M+1$
1	$U_{1,1} = u_1$	$U_{1,2} = u_{1,2}$	…	$u_{1,M-s+1}$	$u_{1,M-s+2}$	$u_{1,M-1}$	$u_{1,M}$	$u_{1,M+1}$
2	$U_{2,1} = u_2$	$U_{2,2} = u_{2,2}$	…	$u_{2,M-s+1}$	$u_{2,M-s+2}$	$u_{2,M-1}$	$u_{2,M}$	
…	…	…	…	…	…			

[14] Theoretical rationale and empirical support for this is found in Rehman, Klugman (2010).

s	$U_{s,1}=u_s$	$U_{s,2}=u_{s,2}$		$u_{s,M-s+1}$	$u_{s,M-s+2}$			
$s+1$	$U_{s+1,1}=u_{s+1}$	$U_{s+1,2}=u_{s+1,2}$		$u_{s+1,M-s+1}$				
…								
M	$U_{M,1}=u_M$	$U_{M,2}=u_{M,2}$						
$M+1$	$U_{M+1,1}=u_{M+1}$							

Table 17

ERROR TRIANGLE OF LOSS RATIO [MXM]

	1	2	…	$M-s$	$M-s+1$	…	$M-1$	M
1	$\varepsilon_{1,1}$	$\varepsilon_{1,2}$	…	$\varepsilon_{1,M-s}$	$\varepsilon_{1,M-s+1}$	…	$\varepsilon_{1,M-1}$	$\varepsilon_{1,M}$
2	$\varepsilon_{2,1}$	$\varepsilon_{2,2}$	…	$\varepsilon_{2,M-s}$	$\varepsilon_{2,M-s+1}$	…	$\varepsilon_{2,M-1}$	
…	…	…	…	…	…			
s	$\varepsilon_{s,1}$	$\varepsilon_{s,2}$	…	$\varepsilon_{s,M-s}$	$\varepsilon_{s,M-s+1}$			
$s+1$	$\varepsilon_{s+1,1}$	$\varepsilon_{s+1,2}$	…	$\varepsilon_{s+1,M-s}$				
…	…	…	…					
M	$\varepsilon_{M,1}$							
$M+1$								

A.1 STATISTICAL MODEL

For any year s, $\varepsilon_s = \ln\left(\frac{U_{s,M+1}}{U_{s,M-s+2}}\right)$ and $\omega_s^2 = Var \ln U_{s,M+1}$. If ε_s is normal, then $U_{s,M+1}$ is lognormal. Further, we can write:

$$\ln U_{s,M+1} \sim N\left(\ln EU_{s,M+1} - \frac{\omega_s^2}{2}, \omega_s^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

This leads to

$$\ln U_{s,M+1} \sim N\left(\ln \phi_s - \frac{\omega_s^2}{2}, \omega_s^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

A.2 ESTIMATION OF ω_s^2

We will rely on notation introduced in Tables 16 and 17. For $s = 1, 2 \ldots M+1$

$$\varepsilon_{s,M-s+1} = \ln \frac{u_{s,M-s+2}}{u_{s,M-s+1}}$$

$$\varepsilon_s = \sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q} = \sum_{q=1}^{s-1} \ln\left(\frac{U_{s,M-s+2+q}}{U_{s,M-s+1+q}}\right) = \ln\left(\frac{U_{s,M+1}}{u_{s,M-s+2}}\right)$$

$$\varepsilon_s = \ln\left(\frac{U_{s,M+1}}{u_{s,M-s+2}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Note that due to the multivariate normality of the errors $\left\{\varepsilon_{s,M-s+1+q}\right\}_{1\le q\le s-1}$, $s = 1 \ldots M+1$ the cumulative error $\varepsilon_s = \sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q}$ is also normally distributed. Then for $s = 1 \ldots M+1$,

$$\omega_s^2 = \operatorname{cov}(\varepsilon_s, \varepsilon_s) = \operatorname{cov}\left(\sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q}, \sum_{p=1}^{s-1} \varepsilon_{s,M-s+1+p}\right) = \sum_{q=1}^{s-1}\sum_{p=1}^{s-1} \operatorname{cov}\left(\varepsilon_{s,M-s+1+q}, \varepsilon_{s,M-s+1+p}\right) \quad (4)$$

Where $\operatorname{cov}\left(\varepsilon_{s,M-s+1+q}, \varepsilon_{s,M-s+1+p}\right)$ can be estimated from Figure 2 using postulates.

A.3 MEAN CALCULATION INVOLVING PARTIALLY COMPLETED YEAR

Recall equation (2),

$$\ln U_{s,M+1} \sim N\left(\ln \phi_s - \frac{\omega_s^2}{2}, \omega_s^2\right).$$

For a partially completed year, we can proceed as in Appendix A and define

$$\varepsilon_s := \sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

We now determine covariance calculation involving $EU_{s,M+1}U_{t,M+1}$, $t < s$. Hence,

$$\ln U_{t,M+1} \sim N\left(\ln \phi_t - \frac{\omega_t^2}{2}, \omega_t^2\right)$$

$$\ln U_{s,M+1} \sim N\left(\ln \phi_s - \frac{\omega_s^2}{2}, \omega_s^2\right)$$

$$\ln U_{s,M+1}U_{t,M+1} = \ln U_{s,M+1} + \ln U_{t,M+1}$$

$$\sim N\left(\ln \phi_t - \frac{\omega_t^2}{2} + \ln \phi_s - \frac{\omega_s^2}{2}, \omega_t^2 + \omega_s^2 + 2\operatorname{cov}\left(\varepsilon_s, \varepsilon_t\right)\right)$$

Hence, $U_{s,M+1}U_{t,M+1}$ is also lognormal. Therefore,

$$E\left[U_{s,M+1}U_{t,M+1}\right] =$$

$$\exp\left[\ln\phi_t - \frac{\omega_t^2}{2} + \ln\phi_s - \frac{\omega_s^2}{2} + \frac{\omega_t^2 + \omega_s^2 + 2\operatorname{cov}\left(\varepsilon_s, \varepsilon_t\right)}{2}\right]$$

$$= \phi_t \phi_s e^{\operatorname{cov}(\varepsilon_s, \varepsilon_t)}$$

$$E\left[U_{s,M+1} U_{t,M+1}\right] = \phi_t \phi_s e^{\operatorname{cov}(\varepsilon_s, \varepsilon_t)} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (6)$$

A.4 COVARIANCE CALCULATION INVOLVING PARTIALLY COMPLETED YEAR

Equation (6) requires $\operatorname{cov}(\varepsilon_s, \varepsilon_t)$, where $t < s$. We are given that $\varepsilon_s = \ln\left(\frac{U_{s,M+1}}{u_{s,M-s+2}}\right)$ and s is a partially completed year, and year t is priori. From equation (5),

$$\varepsilon_s = \sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q}$$

Hence,

$$\operatorname{cov}(\varepsilon_s, \varepsilon_t) = \operatorname{cov}\left(\sum_{q=1}^{s-1} \varepsilon_{s,M-s+1+q}, \sum_{q=1}^{t-1} \varepsilon_{t,M-t+1+q}\right) = \sum_{r=1}^{t-1} \sum_{q=1}^{s-1} \operatorname{cov}\left(\varepsilon_{s,M-s+1+q}, \varepsilon_{t,M-t+1+r}\right) \qquad . (7)$$

Appendix B: B-F Estimate & Target Loss Ratio

As before, a typical triangle data set with years $k = 1..M, M+1$ is shown in Appendix A along with standard lognormal model postulates. This section will also show derivation of the target loss ratio u_k.

B.1 BORNHUETTER- FERGUSON ESTIMATE

Define random age to ultimate factor $A_{s,M-s+2}$,

$$U_{s,M+1} = \frac{A_{s,M-s+2} u_{s,M-s+2}}{A_{s,M-s+2}} + \left(1 - \frac{1}{A_{s,M-s+2}}\right) u_s$$

$$U_{s,M+1} = u_{s,M-s+2} + \left(1 - \frac{1}{A_{s,M-s+2}}\right) u_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

We can replace the realized quantity $a_{s,M-s+2}$ to obtain the B-F estimate,

$$\phi_s = u_{s,M-s+2} + \left(1 - \frac{1}{a_{s,M-s+2}}\right) u_s \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

We will assume that $a_{s,M-s+2}$ is fixed and known age to ultimate factor for year s. This appendix will discuss the determination of u_s. For now, we note that we can replace u_s with other quantities. This makes ϕ_s much less restrictive than it appears. For example, if $u_s = u_{s,M-s+2} * a_{s,M-s+2}$, then $\phi_s = u_{s,M-s+2} * a_{s,M-s+2}$, which is the chain ladder estimate.

B.2 ESTIMATION OF TARGET LOSS RATIO u_k USING VAR APPROACH

Since this section will only deal with the projected year M+1, we will drop the subscript for the development year. For example, $\omega = \omega_{M+1}$.

Target loss ratios are a subject in themselves and widely used in actuarial ratemaking as well as rate regulation. Insurance companies and regulators adjust rates to ensure that long-term average loss ratios equal these target loss ratios. In exchange, the target loss ratios are based on fairness from a consumer and insurer standpoint. The model presented in this paper leads to elegant closed form results for target loss ratio and is easy to calculate due to availability of data.

The parameter u_k is also called the permissible loss ratio (PLR), and the motivation will be made obvious in this section. We present two distinct approaches. The first, value at risk (VaR), provides a percentile measure of risk tolerance α. The second, conditional value at risk (CVaR), specifies risk tolerance at a given conditional expected excess loss. The condition here is typically the VaR loss amount. Hence, CVaR is always more conservative than VaR for the same specified percentile α. Conditional value at risk possesses a property of risk measures known as coherence (Artzner, Delbaen, et al. 1999), implying compliance with a set of common-sense axioms. VaR, lacking this property, can lead to perverse consequences in some circumstances. From Appendix A, define[15]

$$U_k^{VaR} := \exp\left(\ln u_k^{VaR} - \frac{\omega^2}{2} + z_\alpha \omega\right) \quad . \quad . \quad . \quad . \quad . \quad .. \qquad (3)$$

Here z_α is $1-\alpha$ percentile from a standard normal distribution, and U_k^{VaR} is the $1-\alpha$ percentile under VaR approach. Since the insurance company will collect u_k^{VaR} in premium, we can think of the excess $U_k^{VaR} - u_k^{VaR}$ as underwriting capital to premium ratio to obtain

$$c_k^{VaR} = u_k^{VaR}\left[\exp\left(-\frac{\omega^2}{2} + z_\alpha \omega\right) - 1\right] \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Suppose that we know the expected pre-tax cost of underwriting capital[16, 17] y_k. The expected underwriting profit provision is defined as

$$r_k = c_k y_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Hence from (4),

---

[15] $\phi_k^{VaR} = u_k^{VaR}$ for the projected year. This is due to the definition of Bornhuetter Ferguson formula. The super script VaR is used to emphasize that the calculation of target loss ratio is under the VaR approach.

[16] Using pre-tax returns simplifies calculations. Secondly, because underwriting capital is defined to include only insurance risks, the cost of underwriting capital is technically due to insurance risk. Thirdly, since r_k includes investment income offset, y_k should be reduced to reflect this fact. To illustrate, suppose the investment income offset is 6% (as a % of written premium) then at a capital to premium ratio of 0.5, the offset as a % of underwriting capital equals 12%. If the total target underwriting ROE is 10%, then y_k = 10%-12% = -2%.

[17] It is the return that investors desire for providing capital to the company to underwrite the line of business. One way to do this is to determine firm-wide cost of capital for a mono-line insurance company that writes the same line of business. There is considerable literature available on determining pre-tax cost of capital for the insurance company as a whole (such as CAPM based models). We do not delve into this discussion in this paper.

$$r_k^{VaR} = u_k^{VaR} y_k \left[\exp\left(-\frac{\omega^2}{2} + z_\alpha \omega \right) - 1 \right] \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

The components of the premium must add up to 1 (called totality constraint) and thus for an expense ratio e_k,

$$u_k^{VaR} = 1 - e_k - r_k^{VaR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Incorporating the totality constraint in (6) leads to

$$r_k^{VaR} = \frac{(1-e_k) y_k \left\{ \exp\left(-\frac{\omega^2}{2} + z_\alpha \omega \right) - 1 \right\}}{1 - y_k + y_k \exp\left(-\frac{\omega^2}{2} + z_\alpha \omega \right)} \quad . \quad . \quad . \quad . \qquad (8)$$

From (5),

$$c_k^{VaR} = \frac{(1-e_k) \left\{ \exp\left(-\frac{\omega^2}{2} + z_\alpha \omega \right) - 1 \right\}}{1 - y_k + y_k \exp\left(-\frac{\omega^2}{2} + z_\alpha \omega \right)} . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

This completes the VaR approach. ■

B. 3 ESTIMATION OF TARGET LOSS RATIO u_k USING CVAR APPROACH

We can also determine the parameter u_k using CVaR approach. Given a non-negative amount β_k,

$$c_k^{CVaR} = E\left(U_k \middle| U_k > \beta_k \right) - u_k^{CVaR} \quad . \quad . \quad .. \quad . \quad . \quad . \qquad (10)$$

Setting up the initial integrals and using lognomality of U_k in Appendix A, we find:

$$c_k^{CVaR} = \frac{\int_{\beta_k}^{\infty} x f_{U_k}(x)dx}{1 - \int_0^{\beta_k} f_{U_k}(x)dx} - u_k^{CVaR}$$

$$\int_{\beta_k}^{\infty} x f_{U_k}(x)dx = u_k - \int_0^{\beta_k} x f_{U_k}(x)dx = u_k^{CVaR} - u_k^{CVaR}\Phi\left(\frac{\ln\beta_k - \ln u_k^{CVaR} + \frac{\omega^2}{2} - \omega^2}{\omega}\right)$$

$$\int_{\beta_k}^{\infty} x f_{U_k}(x)dx = u_k^{CVaR} - u_k^{CVaR}\Phi\left(\frac{\ln\beta_k - \ln u_k^{CVaR} - \frac{\omega^2}{2}}{\omega}\right)$$

Also,

$$\int_0^{\beta_k} f_{U_k}(x)dx = \Phi\left(\frac{\ln\beta_k - \ln u_k^{CVaR} + \frac{\omega^2}{2}}{\omega}\right).$$

Thus,

$$c_k^{CVaR} = \frac{u_k^{CVaR} - u_k^{CVaR}\Phi\left(\frac{\ln\beta_k - \ln u_k^{CVaR} - \frac{\omega^2}{2}}{\omega}\right)}{1 - \Phi\left(\frac{\ln\beta_k - \ln u_k^{CVaR} + \frac{\omega^2}{2}}{\omega}\right)} - u_k^{CVaR} \quad . \quad . \quad .. \qquad (11)$$

Let $\alpha_k = \dfrac{\ln\beta_k - \ln u_k^{CVaR} + \frac{\omega^2}{2}}{\omega}$.

Using totality constraint (7),

$$\alpha_k = \frac{\ln \beta_k - \ln\left(1 - e_k - r_k^{CVaR}\right) + \frac{\omega^2}{2}}{\omega} \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Then from (11) and (11),

$$\frac{c_k^{CVaR}}{u_k^{CVaR}} = \frac{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)}{1 - \Phi\left(\alpha_k\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Using (5),

$$\frac{r_k^{CVaR}}{u_k^{CVaR}} = \frac{y_k c_k^{CVaR}}{u_k^{CVaR}} = \frac{y_k \left\{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)\right\}}{1 - \Phi\left(\alpha_k\right)} .$$

Using the totality constraint $u_k^{CVaR} = 1 - e_k - r_k^{CVaR}$ to remove u_k^{CVaR},

$$r_k^{CVaR} = \frac{y_k (1 - e_k) \left\{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)\right\}}{1 - \Phi\left(\alpha_k\right) + y_k \left\{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)\right\}} \quad . \quad . \quad . \quad . \qquad (14)$$

Recall that α_k was defined previously in equation (12) and contains r_k^{CVaR}. Since the right-hand side of equation (14) also includes r_k^{CVaR}, the two equations must be solved iteratively. We conclude this section with the equation for c_k^{CVaR},

$$c_k^{CVaR} = \frac{r_k^{CVaR}}{y_k} = \frac{(1 - e_k) \left\{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)\right\}}{1 - \Phi\left(\alpha_k\right) + y_k \left\{\Phi\left(\alpha_k\right) - \Phi\left(\alpha_k - \omega\right)\right\}} . \quad . \quad .. \qquad (15)$$

The CVaR approach can be made necessarily more conservative than VaR if equation (3) is used to set β_k ,[18]

$$\beta_k = U_k^{VaR} = \exp\left(\ln u_k^{VaR} - \frac{\omega^2}{2} + z_\alpha \omega \right) \qquad . \qquad . \qquad . \qquad . \qquad . \qquad .$$

(16)

A numerical comparison of VaR and CVaR is provided below for $e_k = 30\%$, $y_k = 15\%$, $z_{1-\alpha} = 2.33$ an expense ratio of 30%.

Table 18
COMPARISON OF VAR AND CVAR RESULTS

	Value at Risk			Conditional Value at Risk		
Percentile	Underwriting Capital Ratio	Underwriting Profit	PLR	Underwriting Capital Ratio	Underwriting Profit	PLR
95.00%	0.22	3.30%	66.70%	0.3	4.50%	65.50%
95.50%	0.23	3.40%	66.60%	0.31	4.70%	65.30%
96.00%	0.24	3.60%	66.40%	0.32	4.80%	65.20%
96.50%	0.25	3.70%	66.30%	0.33	4.90%	65.10%
97.00%	0.26	3.90%	66.10%	0.34	5.10%	64.90%
97.50%	0.27	4.10%	65.90%	0.35	5.20%	64.80%
98.00%	0.29	4.30%	65.70%	0.36	5.40%	64.60%
98.50%	0.3	4.60%	65.40%	0.38	5.70%	64.30%
99.00%	0.33	5.00%	65.00%	0.4	6.10%	63.90%

[18] We do not formally prove this assertion.

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