B.A.J. **8**, I, 27-74 (2002)

A PRIMER IN FINANCIAL ECONOMICS

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[This paper was written following a presentation to a meeting of the Faculty of Actuaries on 1 October 2001, and is a slightly augmented version of the presentation.]

ABSTRACT

This paper is divided into three parts. Taken together, the three parts intend to provide the reader with an overview of the first 101 years of financial economics, with particular attention on those developments that are of special interest to actuaries. In Section 1, S.F. Whelan attempts to capture the flavour of the subject and, in particular, to give an overview or road map of this discipline, highlighting actuarial input. In Section 2, D.C. Bowie gives a concise and self-contained overview of the Modigliani and Miller insights (or MM Theorems, as they are often known). In Section 3, A.J. Hibbert considers the novel option pricing method proposed by Black, Merton, and Scholes. These two insights are highlights of this new science, and, in both cases, contradict our intuition.

T.S. Elliot, the mathematically trained poet, described the darkness that intercedes between the idea and the action as the 'shadow'. There is a shadow to be considered between these insights and their application. The demonstration of the results requires, of course, some idealised circumstances, and therefore the extent and degree of their applicability to the non-idealised problems encountered by actuaries requires some delicate considerations. An attempt is made to outline these further considerations.

KEYWORDS

Financial Economics; Options; Black & Scholes; Modigliani & Miller

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1. One Hundred and One Years of Financial Economics

1.1 *Three Landmarks in Financial Economics*

1.1.1 Convention has it that financial economics began as a separate discipline on 29 March 1900. On that day, Louis Bachelier defended one of his theses, *Théorie de la Spéculation*, to the Academy of Paris for his *Docteur en Sciences Mathématique*. Henri Poincaré, his famous supervisor, gave it an excellent report, but noted that his topic of modelling the French capital market as a fair game was: "somewhat remote from those our candidates are in the habit of treating".

1.1.2 Bachelier's work was widely read and appreciated by mathematicians with an interest in stochastic processes, and his results and methods were quickly disseminated amongst French actuaries. Yet, it took more than half a century for this thesis in mathematical physics to influence economic thinking, when, as the story goes, his work was rediscovered by Savage and Samuelson in the mid-1950s (Bernstein, 1992).

1.1.3 There was no such delay in economists appreciating the next highlight that came, in 1973. In that year, the trio of Merton, Black, and Scholes (Merton, 1973a; Black & Scholes, 1973) gave a novel and counter-intuitive method of option pricing. Though published in two separate papers in the spring and summer of 1973, the three researchers had earlier shared their ideas, and, as Fischer Black acknowledged:

> "A key part of the option paper I wrote with Myron Scholes was the arbitrage argument for deriving the formula. Bob [Robert Merton] gave us that argument. It should probably be called the Black-Merton-Scholes paper." Black (1988)

1.1.4 There is less of a consensus on what constitutes the third landmark in the development of this young science. We must wait a few decades more to gain the distance, and, with it, the hindsight, to judge. One possible candidate, at least on the basis of excitement generated on its publication, was the 600-plus-page book in 1944, *Theory of Games and Economic Behavior*, by John von Neumann and Oskar Morgenstern. The *New York Times* gave it front-page coverage, while the *Bulletin of the American Mathematical Society* was also premature in its assessment:

> "Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science — the science of economics. The foundation which they have laid is extremely promising."

In the event, even the prodigious talents of von Neumann could not quite extend the mathematical theory of non-cooperative games to embrace all of economics. Economics, at the beginning of the 21st century, is not regarded as a mathematical science.

1.1.5 Perhaps a less contentious choice for the third landmark of financial economics is the Modigliani & Miller paper of 1958, *The Cost of Capital, Corporation Finance and the Theory of Investments*. This paper introduced the now ubiquitous arbitrage argument into finance and applied it to show that the ideal capital structure of a firm — that is the optimum debt/equity mix — does not exist. How firms finance themselves is, they showed, largely irrelevant to the value of the firm.

1.2 *Importance to Actuaries*

1.2.1 Financial economics is, perhaps, the science that aligns itself most closely with the perpetual concern of actuaries. It attempts to model and price financial risks generally. In fact, financial economics has grown over the last 101 years to embrace three quite distinct branches — mathematical finance, asset pricing models, and corporate finance.

1.2.2 First, we have the original branch pioneered by Bachelier, the mathematical modelling of financial markets. The aim of this branch is to model the evolution of prices in (near) efficient markets, and thereby evaluate any function of these prices — options, guarantees, etc. This branch is the most mathematically sophisticated, relying on advances in the theory of stochastic processes, especially diffusion processes, to model prices and risk. Bachelier's work was extended and made rigorous by, *inter alia*, Wiener, Lévy, Kolmogorov, Cramér (the actuary), Khintchine, Feller, and Itô, on whose groundwork the Black-Merton-Scholes insight in finance rests. The actuarial functions of pricing, reserving, and quantifying mismatch reserves all fall into this branch. In fact, the press release announcing that Merton and Scholes were awarded the Nobel Prize in Economics in 1997 (Black had died earlier) put the overlap of interests in a somewhat more pointed way:

> "The method adopted by this year's laureates can therefore be used to value guarantees and insurance contracts. One can thus view insurance companies and the option market as competitors."

1.2.3 Asset pricing is the distinct branch that concerns itself with the factors that drive security prices. At its simplest, it incorporates the familiar dividend discount model and the expectation hypothesis of the term structure of interest rates. In the last half century more sophisticated models have been proposed, which attempt to account for some aspect of security returns — models which include the Capital Asset Pricing Model (CAPM), the generic Arbitrage Pricing Model (APT), the Consumption-Capital Asset Pricing Model (C-CAPM), and the Inter-Temporal Capital Asset Pricing Model (I-CAPM). This branch of financial economics is more statistical in nature, as the various posited models must be calibrated to the data and evaluated using statistical techniques. Actuaries employed as fund managers could point to this branch of financial economics for their theoretical underpinning.

1.2.4 Finally, the third prong of financial economics goes under the title 'corporate finance'. This concerns itself with the optimum financial management of companies, treating such diverse concerns as dividend policy, capital structure, pension fund investment, managerial remuneration. The topics treated tend to be more diverse and less coherent as a body than the other two branches, but important nonetheless. Actuaries in senior managerial positions in firms or advising such individuals, including advising on firm-sponsored pension schemes, would find this third branch especially stimulating, as, oftentimes, it challenges common practices and traditional views.

1.3 *Bachelier's Thesis: The Genesis of Financial Economics*

1.3.1 Bachelier's thesis was a remarkable piece of work. He delineated one of the major branches of the new science. First, he draws the boundary of the study:

> "The determination of these [stock price] fluctuations depends on an infinite number of factors; it is, therefore, impossible to aspire to mathematical prediction of it.... But it is possible to study mathematically the static state of the market at a given instant. If the market does not predict its fluctuations, it does assess them as being more or less likely, and this likelihood can be evaluated mathematically."

1.3.2 So he studied price changes as a stochastic process, recognising that 'the mathematical expectations of the buyer and the seller are zero' (i.e., form a martingale), coming up with what would later become known as the Chapman-Kolmogorov equation (homogenous version), identifying the equation describing the process as the diffusion equation, and showing that (what we now term) the Wiener process is a solution. In short, he mapped out the mathematical basis of modelling the price formation process. He considered some practical applications of the model, solving for the probability that a price will exceed (or, equivalently, fall below) a certain level within a given time period, and determining the distribution of the extremes of the price process. He accomplished all this in just 59 pages (in the American translation by Boness (Bachelier, 1900b)). He concludes his thesis:

> "It is evident that the present theory resolves the majority of problems in the study of speculation by the calculus of probability."

1.3.3 If we accept the convention that financial economics began with Bachelier's thesis, then, as Cootner (1964) put it: "we can say that the study of speculative prices has its moment of glory at its moment of conception." This held true up to the early 1970s. Even at the time when the thesis was read, its importance was recognised — Poincaré gave the thesis an excellent report

> "...one might fear that the author has exaggerated the applicability of Probability Theory as has often been done. Fortunately, this is not the case..."

and ensured that it was published in the prestigious *Annales Scientific de l'École Normale Supérieure*. Its methodology and results were widely disseminated and cited, although, as one can expect from the approach, the audience was largely mathematicians or probabilists. This audience paid little attention to the financial implications of the findings, evaluating it in the context of its contribution to stochastic processes, and often somewhat harshly critical of its lack of rigour. [For an excellent discussion of Bachelier and his work, see Taqqu (2001) or Courtault, J.-M., Kabanov, Y. *et al.* (2000). For an overview of the state of probability theory at that time, see, for instance, Cramér (1976).]

1.3.4 The main import from Bachelier's model of stock prices is the prediction that the standard deviation of the distribution of future price changes is directly proportional to the square root of elapsed time. This rule was not, in fact, novel to French actuaries. As Taqqu (2001) has recently highlighted, the *Journal des Actuaires Français* had noted this empirical rule more than two decades earlier, when Dormoy (1873) cites the work, a decade before that again, of Regnault (1863). Apparently this law was widely used on the French bourse, and would, in all likelihood, have been known to Bachelier from when, prior to pursuing his Ph.D., he worked in some capacity on the Paris Stock Exchange. [Incidentally, Lefèvre, another French actuary, was famous for his diagrammatic representation of trading, and these graphs are known to have influenced Bachelier's early work.]

1.3.5 We can conclude that Bachelier's work was instrumental in laying the foundations of the mathematical study of the market, that he produced results of remarkable insight which it would take several decades for probabilists to make rigorous, and that his work had a direct influence on subsequent generations of probabilists, including those developments by Itô and others that would later underpin the Black-Merton-Scholes breakthrough. The main practical import of his model was known, prior to 1900, as an empirical rule (by French actuaries at any rate). Bachelier's work was quickly disseminated in a manner accessible to actuaries, including in a book first published in 1908 by the French actuary Barriol (which was to be issued in several editions over the following decades). Now, Bachelier's thesis is recognised as, not only a foundational paper in mathematical finance, but, with its inclusion unabridged in Haberman & Sibbett's 1995 collection of key papers in actuarial science, viewed also as a significant contribution to the scientific underpinning of our profession.

1.4 *Other Developments in the First Half of the 20th Century*

1.4.1 There is a gap of five decades between Bachelier's work and the next significant step forward in the theory of financial markets. It is of interest to speculate why the scientific study of financial markets was so muted over this period. Three plausible reasons can, we think, be given.

1.4.2 First, real returns from the capital markets were relatively modest in the first half of the 20th century (see, for instance, Dimson *et al.*, 2000) and, in particular, the equity risk premium was not very pronounced. This was especially true following 1929. With financial markets not generating or transferring much wealth in the economy, one can conclude that they did not generate the interest that they command today.

1.4.3 Second, the 1929 crash and the subsequent jailing of Richard Whitney, the President of the New York Stock Exchange, reinforced the view that, as Daniel Defoe put it over two centuries earlier:

> "...there is not a man but will own, 'tis [the Stock Exchange] a compleat System of Knavery; that 'tis a Trade founded in Fraud, born in Deceit, and nourished by Trick,

> Cheat, Wheedle, Forgeries, Falsehoods and all sorts of Delusions; Coining False News, this way good, that way bad; whispering imaginary Terrors, Fights, Hopes, Expectations, and they preying on the Weakness of those Imaginations they have wrought upon, whom they have either elevated or depress'd.'' Defoe (1719) as reproduced in Ellis (1997)

1.4.4 Thirdly, Bachelier had advanced the study of the price evolution process further than the mathematics of the time allowed. The theory of stochastic processes and, in particular, diffusion processes needed time to catch up.

1.4.5 However, it would be wrong to conclude that no foundational work was done in financial economics between 1900 and 1950. Financial economics has come to be a broader church than just mathematical finance, the branch initiated by Bachelier. So, while there was no other watershed publication in the first 50 years, some necessary spadework was done; the prosaic task of observing the markets and collecting data to fit and to evaluate any putative models required to be undertaken. Two initiatives in this regard deserve special mention.

1.4.6 *Data collection*

1.4.6.1 Douglas (1929) read before this body, the Faculty of Actuaries, a paper that excited the United Kingdom actuarial profession into gathering data on the U.K. market for statistical investigation. Murray's (1930) follow-up paper commenced our profession's long association with stock market indices. These U.K. price series were to provide the basis for Kendall's (1953) statistical investigation of the price formation process (see ¶1.7.9) and key empirical evidence for the efficiency of the markets.

1.4.6.2 We may also note, in passing, that U.K. actuaries had also made a pioneering contribution to the development of asset valuation models. Soldofsky (1966), citing Makeham (1869) and Todhunter's textbook of 1901, pointed out that British and Irish actuaries were familiar with, and regularly employed, the dividend discount model long before it was 'rediscovered' in the finance literature.

1.4.6.3 In the United States of America, Alfred Cowles III established the Cowles Commission in 1932, which collected prices and constructed indices of the U.S. Stock Market — the market that was to overtake the U.K. market to rank as the biggest capital market in the world as the 20th century wore on. Cowles was not content only to amass data; he also investigated it. He published several papers in a journal which he helped to establish financially, *Econometrica.* In particular, Cowles (1933) was a remarkable statistical investigation into the abilities of professional forecasters to predict future market movements and insurance companies to add value by active portfolio management. His study concluded that no professional adviser nor investor had demonstrated skill, but, intriguingly, that: "there is some evidence, on the other hand, to indicate that the least successful records are worse than could reasonably be attributed to chance."

1.4.7 *Data analysis*

The empirical aspects of financial economics did not flower with the meticulous amassing of data. It proved difficult to draw robust conclusions from the data. Price series, even if they were back-dated, had a short enough history (limited liability was only generally available in the U.K. since the 1850s, and the first investment trusts only appeared after 1884). Statistical testing procedures were rudimentary, and, without computers, it was only with painstaking toil that even elementary properties of the price series could be ascertained. The early empirical study of Working (1934) showed that speculative price series had some nasty statistical properties. He reports that: "a close student of stock-price behaviour" could differentiate between a random walk and a stock price series, and, moreover, that: "to the important extent that wheat prices resemble a random-difference series, they resemble most closely one that might be derived by cumulating random numbers drawn from a slightly skewed population of standard deviation varying rather systematically through time" (i.e., return series are heteroscedastic). These properties frustrate standard statistical tests, making it difficult to test theoretical models proposed.

1.5 *Developments from 1950 to 1973*

1.5.1 A fertile line of investigation was initiated by Markowitz (1952), and developed in Markowitz (1959). Markowitz took return as the reward for investment (as usual), but defined risk as the variance of returns. Under this framework, each security is a vector consisting of expected reward, expected risk and the covariance of returns between the security and every other available security. Portfolio choice is now reduced to an optimisation problem — to minimise the risk (variance of the portfolio) for a given level of reward or, alternatively, to maximise the reward for any given level of risk. He showed, assuming a quadratic utility function of wealth (or returns follow a Normal distribution), that portfolio construction reduced, in this framework, to a quadratic programming problem that was, in theory if not in practice, soluble.

1.5.2 Markowitz's insight of identifying risk with the variance of returns was, of course, implicit in Bachelier's model. Bachelier, however, modelled the time evolution of risk, whereas Markowitz modelled the cross-sectional returns and risks of many securities over the same single time step. Markowitz's mean-variance framework, with the further assumption of Normality of returns or a quadratic utility function of wealth, gives a powerful first-order model for understanding portfolio construction, as it formulates the trade-off between risk and return, and, thereby, *inter alia*, quantifies the cost and benefits of diversification.

1.5.3 Another flash of insight into the problem of portfolio construction was given in Tobin (1958). Here he pointed out the unique role of the risk-free asset to come up with the Separation Theorem. The Separation Theorem, which can be seen as a generalisation of Keynes' Liquidity Preference Theory, states that the proportion of a portfolio held in the risk-free asset depends on risk aversion; the composition of the risky part of the portfolio is independent of the attitude to risk. That is, construction of a portfolio is a two stage process: first, the level of risk is determined, which gives the split between the riskless (or matching asset) and the risky asset; and, second, the portfolio of risky assets is selected, independent of the first step. The validity of this theorem can easily be seen by simple geometry in Markowitz's mean-variance space.

1.5.4 Markowitz's elegant reduction of the problem of portfolio choice to a quadratic optimisation problem began a rapid succession of developments that grew, with the contributions, in particular, of Treynor, Sharpe, Lintner and Mossin in the 1960s, into the Capital Asset Pricing Model (CAPM). CAPM, in the form usually presented in text-books, posits that the *ex ante* excess return (over the risk-free rate) expected from security i over the next time interval is related to the excess return on the market portfolio (over the risk-free rate) by:

$$\mathrm{E}[R_i] - r = \beta_i(\mathrm{E}[R^m] - r)$$

where:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R^m)}{\mathrm{Var}(R^m)}$$

where:
— R_i is a random variable denoting the *ex ante* return from security i;
— r is the return from the riskless or matching asset over same time interval; and
— R^m is a random variable denoting the *ex ante* return from holding the full universe of risky assets over the same time step.

1.5.5 The contention that the expected excess return of each security is a linear function of its covariance with the market portfolio appears an empirically rich theory. Roll (1977) made the important point that CAPM is not, in fact, directly testable, as any test is, in reality, a joint test of CAPM and that the 'market portfolio' assumed in the test was really the market portfolio. This criticism has been lessened by Stambaugh's (1982) finding that the results of empirical tests are not sensitive to the constitution of the 'market portfolio', even when widened to include bonds, property, and consumer durables. Extensive tests have been performed on the CAPM model over the years, and the general conclusion is that the single beta factor does not adequately explain cross-sectional variation in stock returns (see, for instance, Hawawini & Keim (2000) for a recent synthesis of this literature). In fact, valuation measures, such as the price-to-earnings ratio or the book-to-market value, commonly employed by fund managers, have been shown to have more predictive ability than a firm's beta. There is also evidence of a strong, and yet unexplained, seasonality to returns, especially the abnormally high return often observed in January. CAPM is thus an important theoretical model of the market, but fails to represent it adequately in all its sophistication.

1.5.6 Our treatment of CAPM, above, envisages it as a tool to help construct or, more accurately, to understand the principles of construction of a portfolio that maximises reward for a given level of risk. Sharpe (1964) made a subtle, but important, contribution when he re-interpreted CAPM so that it was not simply a portfolio construction tool, but a theory that could account for the relative prices of capital assets at a given time. Sharpe made clear the equilibrium relationships between risk and reward in markets, distinguishing between diversifiable risk, which is not rewarded, and undiversifiable risk, whose reward is proportional to its beta.

1.5.7 Markowitz and Sharpe were to share a Nobel Prize in 1990 (with Miller) for their pioneering work in financial economics. The passage of more than a quarter of a century was necessary to put their developing discipline and their contribution to it in perspective. At the time when the research was done, both Markowitz and Sharpe had an uphill struggle to convince others of its worth. Sharpe's 1964 paper was originally rejected by the *Journal of Finance*, on the grounds that it made 'preposterous' assumptions, and Markowitz's defence of his (Nobel Prize winning) work for his doctorate in economics was frequently interrupted by complaints from the one-time actuarial student Milton Friedman that the work was not in economics.

1.5.8 The development of CAPM, and with it the clarification of elementary concepts like risk, the distinction of what risks should command a reward, and an equilibrium theory of pricing risky assets, was the dominant theme in financial economics from the 1950s up to the early 1970s. There were, of course, other themes, of which we mention, briefly, just two.

1.7.9 First, in 1953 Kendall read a paper on the statistical properties of stock and commodity prices to the Royal Statistical Society. He used the weekly Actuaries Index of Industrial Share Prices and its 18 sub-sectors over the period 1928-1938, augmented with the weekly price of wheat over a half century and the monthly price of cotton over a hundred year period. He concluded, from his analysis, that there were no patterns in the price series that can be profitably exploited, or, as he put it:

> "Investors can, perhaps, make money on the Stock Exchange, but not, apparently, by watching price movements and coming in on what looks like a good thing."

1.5.10 This study provided, in current terminology, the early empirical basis for the weak form of the Efficient Market Hypothesis (EMH) or, in mathematical finance, empirical support for the assumption that the price formation process forms a martingale, as originally postulated by Bachelier. Cowles work of 1933, mentioned in ¶1.4.6.3, provided evidence that even professional investors could not systematically beat the market average return or, in current terminology, evidence supporting the semi-strong form of the EMH. Fama (1970) gives an important synthesis and organisation of empirical research up to that time, concluding: "the evidence in support of efficient markets is extensive, and (somewhat uniquely in economics) contradictory evidence is sparse".

1.5.11 The second theme would not usually be noted in a short review of the history of financial economics, but is mentioned here, as it is of special importance to actuaries. Actuaries require a good model of the return distribution, so that mismatch risks can be quantified and reserved for in a prudent manner. The form of the distribution of returns was known to be non-Normal from as soon as anybody bothered to perform a test of Normality (and certainly the non-Normality of returns was well-documented by the late 1920s). Mandelbrot (1963) speculated that the family of symmetric stable distributions is a likely candidate for the form of the distribution of returns, the motivation coming from a generalisation of the central limit theorem. Mandelbrot's conjecture also provided a challenge to the accepted definition of investment risk, as stable distributions (other than the Normal) do not have a finite variance. This family of distributions captured well the thick tails of the data, and it would take a decade or so for this conjecture to be toppled by statistical tests (see, Hsu *et al.*, 1974).

1.6 *Developments from 1973*

1.6.1 The highlight of financial economics is undoubtedly the Black-Merton-Scholes method of option pricing. As we develop in Section 3, the concept of 'self-financing strategy' is a generalisation of the immunisation concept familiar to actuaries. The mathematics is more sophisticated, but the underlying concepts are straightforward once one overcomes the counter-intuitive conclusion to which they lead.

1.6.2 Financial economics did not, of course, end in 1973. All of the themes mentioned earlier have continued to be investigated, insights deepened over the following decades, new lines have opened, and theory is now giving a better fit to reality than it did then. An excellent survey of such developments is given in Dimson & Mussavian (1998, 1999), and a three volume collection of foundational papers is promised shortly, Dimson & Mussavian (2002). Most, though, of the fundamental questions of financial economics are not satisfactorily answered, or, where the theory is especially convincing, the fit to reality is too loose. This science retains youthful excitement.

1.6.3 For instance, CAPM is a static model where an optimum portfolio is constructed for the next time step, which, as we mentioned, fails in its attempt to account for the cross-sectional variation in returns of securities. Merton (1973b) and Breeden (1979) attempt to move it into a dynamic setting, where now time flows, to make it a more realistic and, it is hoped, better fit experience. However, even these generalisations are not broad enough to capture the full problem; they model only the demand side for securities, they do not yet address the supply side of firms issuing securities, which, clearly, is significant to their equilibrium price. Also, CAPM and its extension into an inter-temporal setting take as given the term structure of interest rates. Cox, Ingersoll, & Ross (1985) make a pioneering attempt to explain the yield curve.

1.6.4 There has also been some drawback from the rather dogmatic espousing of the EMH since the 1980s. Financial economists are now more willing to accept that innovation can be rewarded in the financial sphere as elsewhere, and now, for instance, the *Journal of Finance*, which was the original outlet for Fama's exposition of the EMH, entertains articles purporting to demonstrate the existence of exploitable market opportunities (or 'anomalies', as they are often called). In recent years, for instance, Lo, Mamayasky & Wang (2000) contest Kendall's conclusions in 1953, and with it the EMH in its weakest form, when they demonstrate that certain stock price patterns "provide incremental information and may have some practical value." Brown, Goetzmann & Kumar (1998) review some of Cowles' (1933) evidence with modern statistical methods, and come to the opposite conclusion to his paper's title question 'Can Stock Market Forecasters Forecast?'. These recent papers add to the already voluminous literature on exploitable anomalies (e.g., the January effect and other seasonalities in returns). In fact, as an illustration of how little we know of the price formation process, even after 101 years' study, a forthcoming paper by Bouman & Jacobsen (2002) makes the extraordinary claim that the old stock market saying, of 'sell in May and go away, but buy back by St Leger day', removes almost half the risk of equity investing without significantly altering returns. The claim is backed by impressive empirical support — the trading strategy is shown to work in almost all national markets over the last few decades, and, further, demonstrated to work in ten out of 11 markets studied for the full length of their historic record (including the U.K. market since 1694). While such anomalies are often dismissed as instances of data-mining (see, for instance, Sullivan *et al.*, 2001), this charge cannot be maintained against the 'sell in May' rule, as has been shown by Lucey & Whelan (2001).

1.6.5 Finally, the statistical properties of returns and, in particular, the characterisation of the return distribution have been studied recently with very large data sets (some exceeding 50 million data points), and some empirical regularities have been documented. In particular, it is becoming clear that the second moment of the return distribution exists, but the fourth moment fails (see, for instance, Loretan & Phillips, 1994; Müller *et al.*, 1998; Plerou *et al.,* 1999a, 1999b), and that return series are not covariance stationary (Loretan & Phillips, 1994). These statistical regularities appear to apply across equity, bond, and foreign exchange markets, and are reported when the time interval over which the return is measured is minutes, hours, days, weeks, or months. These facts complicate statistical estimation of model parameters and the testing of putative theories. However, as empirical regularities, they can be used to model returns (at least by simulation), and thereby, for instance, stress test mismatch reserves.

1.6.6 Irish and U.K. actuaries kept pace with the broad developments insofar as they affected their practical work, but, as neither financial economics nor stochastic processes were part of the formal education of actuaries until the last few years, the initiative rested with the individual actuary. There were, though, singularly important individual contributions. For instance, Fagan (1977), in a remarkable paper presented to the Society of Actuaries in Ireland, proposed, independently of Black-Merton-Scholes, a dynamic hedging of the reserves to meet investment guarantees, sketched a proof of the existence of such a strategy that would significantly reduce reserves otherwise needed, and then investigated, by simulation, whether a practical hedging strategy could be found to effect the safe release of reserves otherwise required. Collins (1982), at the request of the Institute of Actuaries, produced a detailed assessment of whether the approach could be made to work reliably. Collins concluded that such a hedging strategy "compares unfavourably with the conventional strategy", and that a "disturbing reason for the poor performance of the immunization strategy was that from time to time (e.g. early in 1975) the unit price was subject to sudden large fluctuations which were inconsistent with the continuous model assumed in deriving it."

1.6.7 This brief, and highly selective, digression into actuarial literature perhaps makes the intended point that the U.K. and Irish professions were not slow in appreciating the potential impact of developments in financial economics on their practice areas. This is as well, as there is a surprisingly short gestation period from such developments to application — witness the rise of passive management of funds as a consequence of the EMH and its empirical support, witness the growth of derivatives markets subsequent to the publication of the Black-Merton-Scholes formula. [For a fuller, and often amusing, account of the close link between theory and practice, see Bernstein, 1992.]

1.6.8 Actuaries must obviously heed theoretical developments, but must also be mindful of the shadow that falls between the idea and reality. The recent emphasis on financial economics in the education of the next generation of actuaries will ensure that the profession can quickly adapt developments to our practice areas, can use its fundamental concepts to maintain our unity across specialisations, and, perhaps, even maintain the profession's innovative contribution to the science. One cannot witness the birth, childhood, and early youth of financial economics and not get excited for the future of a profession that promises to use the science ably to price and transfer risk in society. Keynes put it succinctly, more than three-quarters of a century ago, when he urged us to embrace financial theory:

> "It is a task well adopted to the training and mentality of actuaries, and not less important, I fancy, to the future of the insurance industry than the further improvement of Life Tables."
> Keynes (1925) quoted in Soldofsky (1966)

1.7 *Suggested further Reading*

Cootner (1964) is a collection of foundational papers in financial economics up to 1963, with insightful commentary. Bernstein (1992) gives a more informal account of the development, with colourful anecdotes, and, uniquely, stresses the link between the development of financial economics and investment practice. Dimson & Mussavian (1998, 1999), and their promised collection of foundational papers, Dimson & Mussavian (2002), give a recent and comprehensive perspective on the development of the discipline.

2. The Modigliani-Miller Theorems and their Relevance to Actuaries

2.1 In a pair of papers in 1958 and 1961, Franco Modigliani and Merton Miller introduced rigorous arbitrage arguments into modern corporate financial theory. In their first indifference (or irrelevance) proposition, they demonstrated how the financing (the debt to equity ratio) of a firm was irrelevant to how the market would value it. In the second proposition, they showed that investors would be indifferent to the dividend distribution policy of the company, i.e. changing dividend distribution policy would not affect the value of the company.

2.2 The assumptions under which these indifference propositions hold true are, admittedly, heroic. The propositions are groundbreaking because of:

(a) the way in which they were argued, using no-arbitrage arguments based on 'homemade' financial engineering; and
(b) the demonstration that focusing too narrowly on any one corporate financial feature could result in unintentional destruction of value.

The Modigliani-Miller analysis shows how apparently second-order effects, such as taxation or the probability of bankruptcy or agency effects, can actually be the main drivers of strategy.

2.3 In Section 2.4 we present a short illustration of the Modigliani-Miller line of argument and try to flesh these out into general principles that may be important in actuarial application. We also illustrate the importance of second-order effects. This enables us to demonstrate how the rather theoretical insights can end up affecting practice. Thereafter, we provide some examples of where their reasoning has affected actuarial work to-date, and how it may affect actuarial thinking in the future.

2.4 *Modigliani-Miller Illustrated*

2.4.1 We consider a very simple economy in which two firms, A and B, operate with identical success and with perfect correlation. We assume that there are no taxes, inefficiencies or other 'second-order' effects. The difference between the two firms is their capital structure (see Table 1); A is entirely equity financed and B is financed 50% with equity and 50% by debt. B's debt is assumed to be a perpetuity with interest payments of 5% p.a. paid annually in arrear.

2.4.2 Book and market values are assumed to coincide. The two firms, being identical other than in capital structure, return a profit of 10% on their operating assets over the course of the year. Table 2 shows the profit and loss accounts. The analysis in Table 2 demonstrates that, on the basis of an expected rate of return of 10% on capital employed, B's structure offers a higher rate of return to shareholders, and might, therefore, be considered preferable on the basis of expected return on equity.

2.4.3 However, if the operating profit over the course of the year turned out to be lower than expected, say 2.5%, then B's shareholders would suffer more than A's (see Table 3). B is therefore higher risk than A, but is expected to produce a higher rate of return. At first sight, it would therefore appear

Table 1. Capital structure

	Firm A	Firm B
Equity	100	50
Debt	0	50
Total liabilities	100	100
Operating assets	100	100

Table 2. Profit and loss account (1)

	Firm A	Firm B
Operating profit	10	10
Interest	-	(2.5)
Profit after interest	10	7.5
Rate of return to equity holders	10%	15%

Table 3. Profit and loss account (2)

	Firm A	Firm B
Operating profit	2.5	2.5
Interest	-	(2.5)
Profit after interest	2.5	0
Rate of return to equity holders	2.5%	0%

that the optimal financing structure for the firm would depend on the expected rate and volatility of the return on assets employed, as well as on the investors' risk-return preferences.

2.5 *Home-made Leverage*

2.5.1 The important step that Modigiliani & Miller made was to look at the shareholders who were bearing the risks and taking the returns rather than at the firm. They recognised that, for most shareholders, the firm was only one holding in a portfolio of many. Moreover, shareholders could borrow or lend money on their own account to offset any positions taken by individual firms. For example, if the company management estimates that the return on capital for the company is close to 10% and that there is low volatility, they may choose to finance a company like B, with a high debt to equity ratio. If the shareholders disagree and estimate a higher level of volatility, they can invest less in the company and invest more in debt in order to achieve their desired level of risk and return.

2.5.2 In fact, this argument is more powerful, and can be used to show how the market will force companies A and B to have exactly the same value, i.e. their share prices will be the same. Suppose that the shareholders of B find an investor who mistakenly thinks that the equity in B is more valuable than the equity in A, and is willing to pay 75 (say) to buy the whole equity stake (previously valued at 50). The (ex-)shareholders of B would be foregoing an expected income stream of 7.5 from their holding in B, but would now hold 75. In other words, they have given up a rate of return of 10% on their capital.

2.5.3 They could borrow an additional 25 (at 5%), and use the total of 100 to buy all of A's equity. They would now be entitled to an expected income stream (with exactly the same risk as B's income stream) of 10, less the cost of their borrowing of 1.25 (5% of 25). The shareholders would have been able to substitute an income stream of 7.5 with one of 8.75 (a rate that is greater than 10% on their capital) at no extra cost and no extra risk.

2.5.4 Since perfect (and indeed most actual) markets will not permit two identical risks to be priced with different expected rates of return, the value of A's equity must rise (to 150), or the value of B's equity must fall (back to 50), or some combination. If we assume that the shareholders have not revised their estimates of the operating risks and returns on the

Table 4. Incomes streams

Action	Holds	Owes	Income stream
Holds B's equity	B's equity	0	7.5
Sells B for 75	75	0	0
Borrows 25	100	25	$-5\% \times 25 = -1.25$
Buys A's equity	A's equity	25	$10 - 1.25 = 8.75$

companies, it is going to be B's share price that reacts. Whatever the case, the new owner of B will have lost out significantly, either because of the drop in value of his/her investment or in terms of opportunity cost, as shown in Table 4.

2.5.5 Market pricing will, therefore, eliminate any impact that capital structure might have on value (under the assumptions). The aggregate investor in company B (who holds 50 of debt and 50 of equity) has a total rate of return of $50\% \times 15\% + 50\% \times 5\% = 10\%$, which is identical to the aggregate investor in company A (all equity).

2.5.6 Perhaps the most important insight from Modigliani & Miller is that it is a mistake to look at a company as a corporate entity in isolation. It is only when one looks at the people (referred to as stakeholders) who are involved in the company that the blinkers are removed, and one can assess the true impact of decisions. After all, a company is (economically-speaking) not a person, despite any legal incorporation.

2.5.7 The other major insight that corporate finance theory (also via Modigliani & Miller) gives us is that when we take away the unrealistic assumptions that make the debt-equity split (the 'first-order' decision) irrelevant, we find a raft of 'second-order' effects that can make capital structuring very important.

2.6 *Second-order Effects*

2.6.1 In the context of capital structuring, second-order effects include issues such as taxation, agency effects and bankruptcy costs. Additionally, regulations and legislation will also constrain the extent to which companies can use debt or equity financing.

2.6.2 Some of the discussions surrounding these second-order effects on capital structuring have parallels in how they might impact on more actuarial issues, such as the investment strategy for defined benefit pension schemes.

2.7 *Taxation*

2.7.1 If corporate taxation is introduced into the simple example given earlier, the conclusion that we reach is dramatically different from the irrelevance proposition. Because corporate taxation is deducted only after

Table 5. Effect of taxation

	Firm A	Firm B
Operating profit	10	10
Interest	-	(2.5)
Profit after interest	10	7.5
Profit after corporate tax	6.7	5.0
Rate of return to equity holders	6.7%	10%

interest on debt is paid, debt financing becomes attractive. Table 5 shows the profit and loss account if corporate taxation, at a rate of 33%, is applied to companies A and B.

2.7.2 The aggregate investor in A has a rate of return of 6.7%, whereas the aggregate investor in B has a rate of return of $50\% \times 10\% + 50\% \times 5\% = 7.5\%$. This difference cannot be offset by homemade leverage without altering the risk characteristics of the income streams. The equity in B is genuinely worth more than that of A in a world where there are only corporate taxes.

2.7.3 The introduction of corporate taxation makes debt financing more attractive than equity financing, as it increases the value of the firm by virtue of the value of a corporate tax shield. The introduction of this second-order effect implies that the optimal capital structure is all debt.

2.7.4 Other research (notably Miller, 1977) has explored how this result changes with the introduction of personal taxation, especially differential taxation between debt and equity holders. The value of debt changes dramatically depending on the relative sizes of the corporate debt holders' and corporate equity holders' rates of taxation. In some plausible cases debt financing reduces value; in others it further increases the value of debt; in yet others it returns the value of the tax shield to zero, i.e. back to the irrelevance proposition. Miller, for example, argues that, in equilibrium, tax rates are typically set so that the capital structure is largely irrelevant.

2.8 *Agency Effects*

2.8.1 The people making the decisions about capital structure are not always the people who own the company. The company management's interests are not perfectly aligned with those of the shareholders. The company represents only a small portion of a shareholder's wealth, but is a significant player in the employees' lives. Broadly, management will prefer company growth to growth in shareholder value (see, for example, Jensen & Meckling, 1976).

2.8.2 For example, it may be difficult for management to find jobs similar to their current ones in locations that suit them, and, moreover, they may have significant personal value built into keeping the company viable because of their remuneration structure. Management may, therefore, be inclined to take lower risks (especially company-specific, 'diversifiable' risks), or to diversify the risks that the company takes on, in a way that is different from that which benefits shareholders.

2.8.3 Another example is that management may value the prestige of a large and well-appointed office more than that it generates economic value for shareholders. Shareholders may worry that management might be able to pursue their own goals at their expense, because management typically has more information than the providers of the capital. Consequently, shareholders will usually value structures that force some of that information to be distributed by management to them.

2.8.4 High levels of debt will reduce discretion in management control over assets, and will force on management a reliance on external finance. So, for example, management will have to come to the market to raise additional capital in order to fund a new office block or to acquire a 'diversifying' asset. By forcing management to come to the market, the shareholders will obtain better information about their investment. In this context, it can be argued that shareholders will value higher levels of debt because they reduce the cost of monitoring management activities and decisions.

2.9 *Bankruptcy Costs*

2.9.1 The phrase 'bankruptcy costs' covers a multitude of costs that are associated with avoiding financial distress, as well as the (relatively small) direct costs of bankruptcy that will reduce residual shareholder value, should it occur.

2.9.2 Should a firm become financially distressed, it will often not be able to raise additional capital, or, if it can, it might be at a higher cost. Shareholders will therefore place value on any actions that reduce the probability of financial distress and the extent of any associated costs.

2.9.3 High levels of debt that reduce management flexibility and discretion may trigger financial distress or exacerbate it. If management is forced to seek external finance at a time of distress, that finance may not be forthcoming, or may come only at a very high price. As a result, 'bankruptcy costs' may temper shareholder preference for high debt-equity ratios.

2.10 *The Implication of Second-order Effects*

Because second-order effects act in both directions (some pushing towards more debt financing and some towards more equity financing), the possibility exists that individual companies will have optimal capital structures, despite the irrelevance of the first-order effects.

2.11 *How does Modigliani-Miller Relate to Actuaries*?

2.11.1 Previous sections provide little more than a flavour of the Modigliani-Miller propositions and some of the debate that emerged after they were published. That debate continues to advance. Much recent work has centred on whether changes to capital structure or dividend policy are ways that management can pass more information to providers of capital. They act as signals to draw attention to the firm, especially when management feels that it has good news to tell.

2.11.2 However, even restricting the discussion to these fundamental insights implies a significant change for actuaries. The remaining sections of this part of the paper briefly outline how the Modigliani-Miller propositions affect the work of actuaries in pensions, but this is by no means the limit of their application.

2.11.3 Tepper (1981) and Black (1980) identified that the above types of irrelevance proposition would apply in a more general sense, in particular to the investment policy of defined benefit pension schemes and other institutional funds, such as insurance company funds. Exley, Mehta & Smith (1997) introduced many of these ideas to the U.K. actuarial community.

2.11.4 In order to illustrate how these might work in the case of investment strategy for defined benefit pension schemes, we again return to an idealised economy that lasts for a single period and has no 'imperfections'. We analyse the situation, bearing in mind the key insights of Modigliani & Miller: a focus on the people involved; and a notion that what holds at an aggregate level will also hold at an individual level by virtue of the principle of no-arbitrage.

2.11.5 The defined benefit pension scheme that we consider is funded by the members, who give up a unit of salary in return for a deferred pension of $(1 + i)$, where i is the cost of risk-free debt. There are only two (fully equity financed) companies in the economy, company C and company D, as well as a risk-free bond issuer (the government). Company C runs a defined benefit pension scheme for its employees.

2.11.6 *Scenario 1*

2.11.6.1 In this scenario the economy is in equilibrium, such that investors provide 100 of equity financing to Company C and 100 to the debt issuer. The members of the pension scheme forego 100 of salary that is invested by the pension scheme in Company D, as in Figure 1.

2.11.6.2 At the end of the period, the two companies C and D return $1 + r_C$ and $1 + r_D$, respectively, per unit of investment to their owners; the bond issuer returns $100(1 + i)$ to the investors and the pension fund discharges its liabilities by paying $100(1 + i)$ to the members. Any difference between i and r_D is passed on to Company C (by virtue of contribution holidays, or return of surplus) and on to the investors, as in Figure 2.

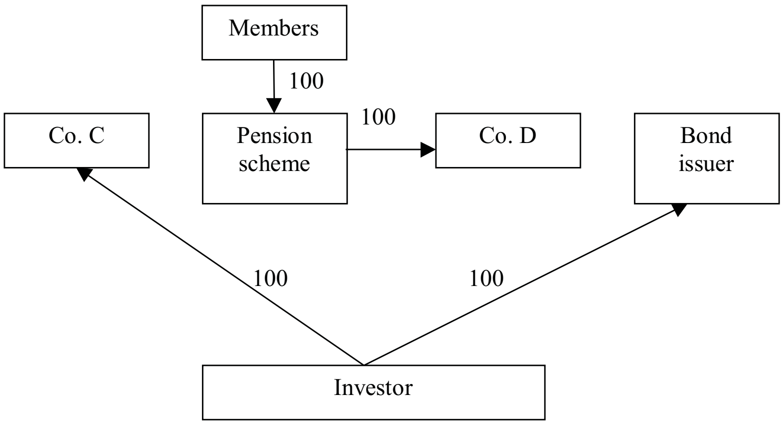

Figure 1. Initial investments

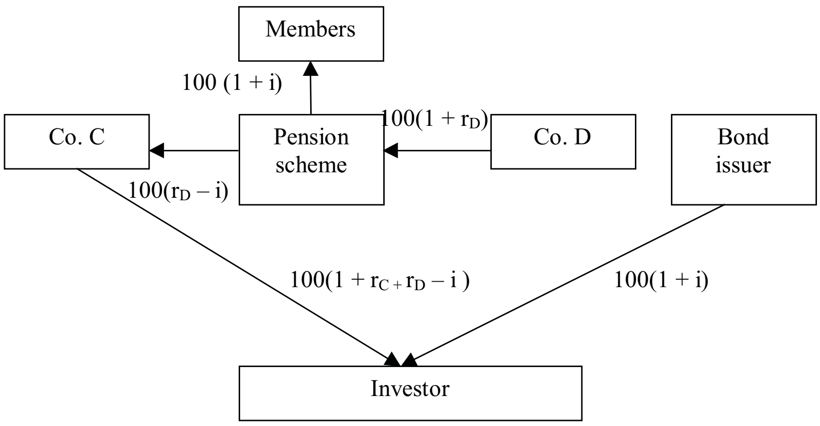

Figure 2. Returns at the end of the period

The aggregate investor receives a total return of $100(2 + r_C + r_D)$ and the members receive $100(1 + i)$.

2.11.7 *Scenario 2*

2.11.7.1 If we suppose that we change Scenario 1 such that the pension fund no longer invests in Company D, but rather invests in the bond issuer,

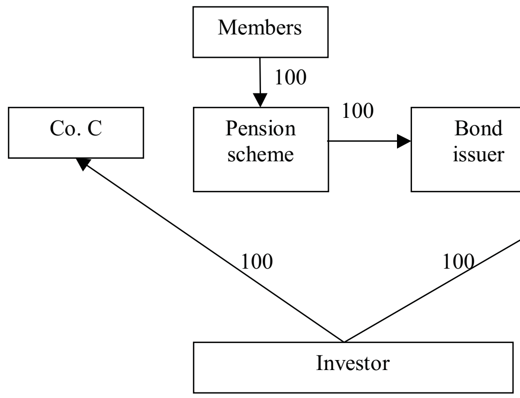

Figure 3. Scenario 2 investments

and we suppose that the aggregate investor then invests 100 in Company D and 100 in Company C, then we have Figure 3.

2.11.7.2 At the end of the period the bond issuer returns $100(1+i)$ to the pension fund, which, in turn, passes this on to the members. As before, the members receive their defined benefit and the aggregate investor receives the same return as before: $100(2+r_C+r_D)$. All the people in this scenario receive exactly the same rates of return as in Scenario 1. All the companies and the bond issuer are provided with the same level of capital. This is shown in Figure 4.

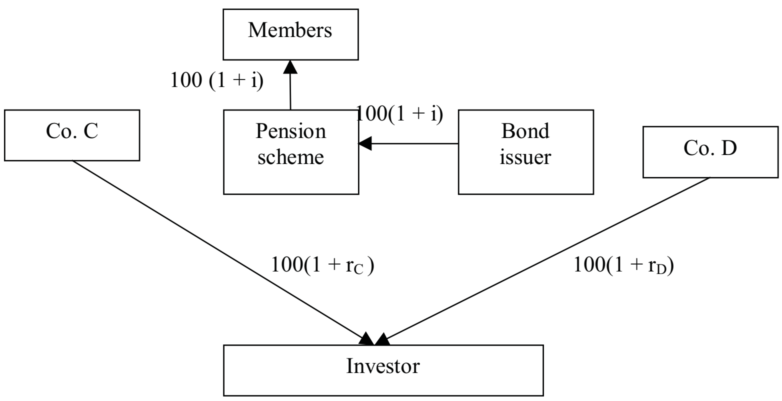

Figure 4. Returns at the end of the period

2.12 *So What?*

2.12.1 The insight is that, as far as the people are concerned, the investment strategy of the pension scheme is irrelevant, at least under the assumptions made in this simple model. There is no argument with the fact that if r_D is expected to be higher than i with a high degree of confidence, then the pension fund is likely to be better off in Scenario 1 than in Scenario 2. It is just that a better-off pension fund is not of any use to anyone (as a first order observation).

2.12.2 The aggregate investor has the same level of risk and expected return, and the pension fund members receive the same defined benefits. As in the case of the capital structuring irrelevancy proposition, what holds for the aggregate investor will also hold for individual investors. Individuals can all rearrange their own holdings so that they are faced with the same level of risk and expected return without affecting the aggregate position of the economy.

2.12.3 In many ways, the irrelevancy of pension fund investment strategy with regard to affecting economic value is obvious; pension funds are not (to first order) economic entities. They do not consume, and so cannot change their patterns of consumption to provide capital investment; they cannot produce widgets, and so cannot provide the labour required for economic production.

2.13 *The Impact for Actuaries*

2.13.1 The impact of this realisation for actuaries is on at least two fronts.

2.13.2 First, in valuation, the irrelevance of the investment strategy implies that the value of the scheme is (largely) unrelated to how it is invested, and much more to do with how equivalent promises of future benefits are valued. Using similar reasoning to that outlined by Modigliani & Miller, employees and shareholders can undo any investment strategy repackaging in the scheme. In economic terms, the pension scheme is a corporate bond (with some complicated options), and should be valued as such (Bodie & Merton, 2000).

2.13.3 Second, the irrelevancy of the investment strategy, to first order, implies that the focus of advice for investment actuaries to pension schemes should be on second-order effects. A focus on second-order effects implies a closer examination of the many groups of people affected by pension scheme investment, rather than scheme-centric approaches.

2.13.4 For valuation actuaries, financial economics is all about determining what combination of market instruments gives the best approximation (in terms of risk and amount) to the benefits promised within the scheme. Once the portfolio of market instruments has been determined, the actual valuation is a simple process of adding up the market prices of the instruments. However, finding the best approximation is decidedly non-trivial in most cases. This is because pension schemes involve a series of implicit options (the sponsor can wind up the scheme or close it to further accrual, and the scheme members can withdraw from the scheme) without obvious market equivalents. Furthermore, the benefits derive their value from fairly complex relationships with salary and price inflation, and are contingent on demographic elements that are far from deterministic.

2.13.5 For investment actuaries, financial economics implies that attention needs to be given to determining what the important second-order effects are, and how and for whom they come into play. Financial economics suggests that conventional tools, such as asset/liability models, need to be radically altered, particularly in terms of output.

2.13.6 The profession has already started to rise to the challenge of building these alternative models, see Chapman *et al.* (2001). In addition, some large schemes in the U.K. have already explicitly accepted the theory and made their own decisions as to what the best way of responding to the perceived second-order effects is.

2.13.7 The fundamental idea behind the new types of model is to list the stakeholders who are, or might be, affected by the asset allocation decision within the scheme. The most clear-cut stakeholders are the scheme members and shareholders of the sponsoring company. However, beyond these, it is worth considering the different classes of scheme members (employee, deferred pensioners, pensioners), the management of the company, the advisors to the scheme (including investment managers, as well as consultants), the Inland Revenue, the debt holders of the company (particularly via the credit rating agencies), the financial services regulators, the trustees of the scheme and others. In public sector schemes, the stakeholders will be different, but will include the local taxpayer, the national taxpayer and the participating employers.

2.13.8 When second-order effects are considered, many of the same issues (taxation, agency effects, bankruptcy costs) come through, as in the analysis of capital structuring. Equities are typically taxed at a lower rate in the hands of investors than are bonds. This makes it more valuable for investors for companies to use their (tax exempt) pension funds to invest in bonds rather than in equities, which the investors can hold themselves. Moreover, not all the tax applied to equities can be reclaimed by pension schemes. That also makes it tax inefficient for pension funds to hold equities. Consideration of taxation would, therefore, seem to support heavy bond investment by pension schemes. Bond investment (and a commensurately higher contribution rate into the scheme by the company) will help enforce financial discipline within the company, and so, it can be argued, will help defray agency costs. Management may well see value in diversifying the investment risk within the company in a very obvious way, by using the pension scheme to hold assets in other companies. Shareholders will typically not find value in having the pension scheme of one part of their portfolio undertaking diversification actions.

2.13.9 Scheme members may also value a high-risk/high expected return strategy, especially if the scheme sponsor's covenant is good. Any surpluses generated in the scheme may be returned to them in the form of benefit improvements, but, in the case of deficit, their benefits are underpinned by the 'guaranteed' part of the defined benefit.

2.13.10 On the other hand, the enforced discipline of a high bond investment strategy and a funding plan for the scheme that requires 100% funding levels may also exacerbate any financial distress or the costs of avoiding financial distress.

2.13.11 In local authority pension schemes, different considerations may apply. Local taxpayers are not necessarily shareholders, and so may benefit from equity investment within pension schemes by virtue of lower recommended contribution rates (passed on as lower local tax rates or improved local services), based on higher expected returns from equity. However, the national taxpayer essentially stands security should the equity market perform really badly and jeopardise benefit security.

2.13.12 If investment strategy is going to add value, advice about asset allocation needs to be given in an integrated manner, and the value of the strategies needs to be evaluated in (approximate) market values, rather than in probabilities of funding levels falling below (arbitrary) 'critical' levels or other common asset/liability model outputs.

2.13.13 One of the most marked differences that actuaries notice in applying financial economics is just how heavily the worst-case scenarios affect market prices. Economic historians are fond of quoting statistics about how rarely equities have underperformed bonds over 'long' periods. Even leaving aside some serious doubts about the quality of the statistics and the appropriateness of the data in many of these studies, it is also clear that the market has persistently priced equities as being far from risk free, even over long periods of time. The cost of removing the risk is high. This is quite a different discipline from one where risks that have low probability are sometimes glossed over. The models required to value high impact/low probability risks are often quite different from the models that have been designed to analyse the first order effects (that turn out to be irrelevant).

2.13.14 As with the reaction to Modigliani & Miller and their propositions about capital structuring and dividend policy, the notion that pension investment strategy is irrelevant, or that second-order effects mean that the strategy should be all bond or all equity, is uncomfortable. When we look at the investment decisions of pension schemes today, they do not (despite popular misconception) all have the same strategy. Strategies range from 100% bonds to 100% equities.

2.13.15 It is difficult to accept that nearly all of them have adopted strategies that are destroying value. It is more comforting (to those who run and those who advise schemes and companies) to suppose that each scheme will have an optimal strategy that is different and that depends on the specifics on the scheme sponsor and scheme members. It is also comforting (in the sense of being able to have confidence in the market) to suppose that the market has already enforced any optimal solutions on the schemes — after all, the same level of market information is needed to establish the theoretical optima.

2.13.16 Are pension funds trustees and the market ahead of the theory, and has the theory yet to find all the second-order effects that make what we observe optimal? Is there large scale irrationality, fed by imprecise reporting, that has led to systematic value destruction? If it is the latter, then some careful communication is required to redress the situation without further destruction of value.

3. An Option Pricing Primer

3.1 *Overview of Option Pricing*

This final section is intended to give a basic introduction to the insights provided by Black-Scholes-Merton (BSM) on the valuation of options. However, before reviewing their analysis, we will begin by explaining why the valuation of options is so important to actuaries. We will conclude the section with a brief review of some of the limitations of their model, with an eye towards the future application of these ideas.

3.2 *What is an Option?*

3.2.1 For the sake of completeness, let us begin with a definition:

— An *option* is an agreement which confers the right (but not the obligation) to exchange one asset for another at an agreed rate on an agreed date.

3.2.2 Options can be seen in many places — in financial markets, in markets for real assets, as well as in everyday life. Consider some examples of financial options:

— an option to buy 1000 BP shares at 500p before end September 2003; and
— an option to deposit £1m for 3 months at an interest rate of 5% p.a. at end December 2002.

3.2.3 These two sorts of options are traded on financial exchanges every day. You can read the price of these sorts of options in the financial newspapers. These kinds of options are bought and sold by financial institutions as part of their day-to-day investment business. They usually appear on the asset side of a life company's balance sheet.

3.2.4 Now, consider three other options that might appear on the other side of a life company balance sheet:

— an option to exchange the cash benefits of a savings contract for a pension at a rate of £10 fund/£1 pension;

— an option to exchange an 'asset share' for a guaranteed amount on a pre-specified date; and
— an option to surrender a term assurance contract and renew at a lower premium level.

3.2.5 These options all fall within the definition set out above. These options have all been provided (i.e. written) by U.K. life companies as part of their business over the past few decades. The first is an annuity option of the type that has caused U.K. life offices so many problems in recent years. In this case, when the option is exercised, the asset given by the option holder is his rights specified in terms (typically) of a with-profits fund value. The asset received is a pension at a pre-specified rate. The second option forms part of a plain vanilla with-profits endowment contract that provides a guarantee in the form of sum assured and reversionary bonus. The final example concerns the implicit option that a holder of a term assurance contract is granted — to surrender and to take out a replacement contract if mortality rates move in the policyholder's favour.

3.2.6 You cannot read the prices of these options in the financial newspapers. Transactions and 'open interest' are not reported. Yet, it is important to appreciate that large exposures do exist on the books of life assurance companies. Life assurers write huge volumes of options exposure every day. They are in the business of selling options embedded in a wide range of business:
— with-profits (sum assured/reversionary bonus);
— guaranteed annuities;
— equity-linked bonds with guarantees;
— high-income bonds with geared capital risk; and
— term assurance contracts, etc.

3.2.7 What is astonishing, given the scale of these exposures, is that options know-how is still viewed by some actuaries as remote from their everyday work. In reality, understanding, pricing, and managing options exposures is fundamental to actuarial work. Option pricing theory offers a set of tools for dealing with options.

3.2.8 Before we review the basic idea behind option pricing theory, let us say something about models in general. It is important to remember that:
— a model is a cut-down, simplified version of reality that captures essential features and aids understanding.

By implication, you can see that a good model does not need to capture all real-world complexity in order to be useful. In our view, the best models are parsimonious (simple), transparent (easy to understand) and evolve (you can add more complexity later, if it is really necessary). One of the things that is so remarkable about the BSM model is that practitioners would tell us that it does *all* of these things. Moreover, no practitioner (that we have ever met) believes that the model's assumptions hold in reality.

3.3 *Key Insights of Black-Scholes-Merton*

3.3.1 Black & Scholes published their celebrated paper in 1973. It is interesting to note that other researchers derived apparently very similar results, so someone might easily ask: "What is all the fuss about? Why is Black, Scholes and Merton's work now viewed as being so special?"

3.3.2 In order to appreciate their work, we will present a simple pricing example based on the work of three other researchers who contributed to explaining and simplifying the work and to helping practitioners put the BSM model into practice. Cox, Ross & Rubenstein (1979) developed analytical tools in 'discrete time' by considering the binomial development of an asset price and the implications for the prices of options based on the asset. Let us use this framework as an introduction to pricing.

3.3.3 Consider a share whose price today is $S = 50$. The price of the asset after a period of time is signified by S^*. The end-of-period price can take on two possible values; S^* is either 100 or 25. A call option (which confers the right, but not the obligation, to buy the asset at the end of the period at a pre-specified 'strike' price) on the stock is available, with strike price $K = 50$. Let us assume that it is possible to borrow and lend at a 25% interest rate. Can you find the price of the call option C, given only this information? We know that, if the end-period share price is 100, then an option to buy at 50 will be worth 50. By contrast, if the share price ends at 25, an option to buy at 50 is worthless. The call option will be worth nothing when the asset price ends up below the strike price of the call. Figure 5 shows the two possible paths for the stock price and the final values of the option. We know everything except the initial value of the call.

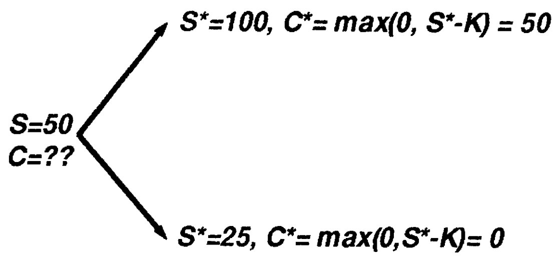

Figure 5. Values

Table 6. Call value

	Cost today	Value at option expiry $S^* = 25$	$S^* = 100$
Sell 3 calls	$3C$	0	−150
Buy 2 shares	−100	50	200
Borrow	40	−50	−50
Total	?	0	0

$3C - 100 + 40 = 0$
The current value of the call must be $C = 20$

3.3.4 In order to discover the price of the call, consider the following portfolio:
(1) sell 3 calls at (unknown) price C each;
(2) buy 2 shares at 50 each; and
(3) borrow 40 at 25% per period, to be paid off at the end of the period.

3.3.5 Given the information that we already have, we can calculate the value of this portfolio for the two possible states for the share price. As Table 6 shows, if the stock price falls to 25, the 3 calls are worthless, the 2 shares are worth 50 in total and the outstanding loan stands at −50 (with interest). The total portfolio is worthless. Likewise, if the stock price has risen to 100, the 3 calls (worth 50 each to the holder of the option) result in a cash outflow of −150, the shares are worth 200 and the loan −50. Again the portfolio is worthless.

3.3.6 If the portfolio is worthless in all possible future states of the world, then the law of one price tells us that it must have zero cost today. Why pay something for an asset that is worthless in all future states? With some very simple algebra, it can be seen that the value of the call must be 20. It is also now straightforward to write down the portfolio that will replicate the option's payoffs in both possible future states, which comprises 2/3 share (cost = $(2^*50)/3$) and a loan of 40/3 (cost = −40/3). The total cost of this replicating portfolio is $100/3 - 40/3 = 20$. Notice that, in this case, the replicating portfolio is a 'geared' holding in the underlying equity.

3.3.7 Now, just suppose that the call price $C = 21$. Table 7 shows the cash flows that would arise if a trader were to sell 3 call options and buy the replicating portfolio. As before, the portfolio that the trader holds is worthless in both future states of the world. However, in this case, selling 3 calls will raise 63 (not 60) while the cost of the replicating portfolio is unchanged. This produced a profit of 3 today for the trader with no risk. This is a pure arbitrage profit.

Table 7. Arbitrage profit

		Value at option expiry	
	Cost today	$S^* = 25$	$S^* = 100$
Sell 3 calls	63	0	−150
Buy 2 shares	−100	50	200
Borrow	40	−50	−50
Total	+3	0	0

3.3.8 Now, suppose that the call price $C = 19$. In this situation we would expect the trader simply to reverse his positions — buying options, selling shares and lending — to produce an arbitrage profit of 3.

Table 8. Alternative arbitrage profit

		Value at option expiry	
	Cost today	$S^* = 25$	$S^* = 100$
Buy 3 calls	−57	0	150
Sell 2 shares	100	−50	−200
Lend	−40	50	50
Total	+3	0	0

3.4 *Expectation Pricing*

3.4.1 We now know the correct price for the call option $C = 20$. In our simple model, *any* other price will allow traders to generate risk-free arbitrage profits. The call must trade at 20. It is a little surprising that we have achieved this without any knowledge of the probability of the two states for the share price. Let us suppose that we do know the 'true' probability assigned by investors.

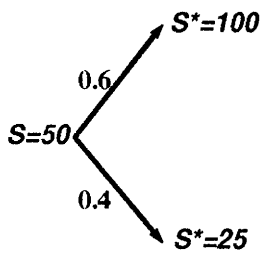

Figure 6. Probabilities

3.4.2.1 If the probability p of an up move to 100 is 0.6, and the complimentary probability of a down move to 25 is 0.4, then the expected price $E_P(S^*) = 0.6(100) + 0.4(25) = 70$. The expected return on the share is, $E_P(S^*)/S - 1 = 70/50 - 1 = 40\%$. Now, consider the expected value of the call option at the end of the period.

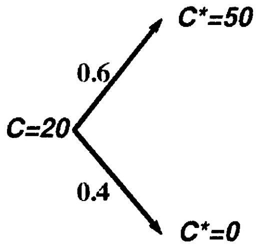

Figure 7. Expected call values

The expected call price is the probability-weighted expectation across the two possible states, so $E_P(C^*) = 0.60(50) + 0.40(0) = 30$. The expected call return is $E_P(C^*)/C - 1 = 50\%$.

3.4.2.2 Note that this can also be calculated by applying the expected returns that we have derived on the share and cash to the two parts of the replicating portfolio.

$$\text{Expected call return} = \frac{\left(\frac{100}{3}\right)1.40 - \left(\frac{40}{3}\right)1.25}{20} - 1 = 50\%. \quad \text{(Equation 1)}$$

The expected return on the option is higher than the share, because the replicating portfolio is a geared investment in the share.

3.4.3 Suppose that we had set out to work out the option price by discounting the expected payoff on the option. This does not seem an unreasonable approach to the problem. We might choose to discount using the risk-free rate or, perhaps, the equity expected rate of return. Let us see what the results look like. Discounting the expected option payoff of 30 at the risk-free rate gives a present value of 24. We already know that this is incorrect as an estimate of the option price. For sure, if we invest 24 at the risk-free rate we will have 30 at the end of the period. However, if we were to trade at this price, we would offer some other investor the opportunity of making a risk-free profit. We would not be able to sell at this price while other investors are prepared to offer the option at 20, and we would be unlikely to buy at 24 whilst the option (or replicating portfolio) costs 20.

$$C = E_P(C^*)/(1 + r) = 30/1.25 = 24 \qquad (\times \text{ incorrect})$$

✓3.4.4 Alternatively, suppose that we had discounted the expected call price at the expected equity rate of return of 40%. In this case the present value is 21.4, which we know is still incorrect. When we use the probabilities set out above, the correct discount rate is 50%, the rate of return on the (geared) replicating portfolio. Notice that, if we had used our knowledge of the composition of the replicating portfolio, we could have derived the correct discount factor for the option payoff.

$$C = E_P(C^*)/(1 + r_e) = 30/1.40 = 21.4 \qquad (\times \text{ incorrect})$$

$$C = E_P(C^*)/(1 + r_C) = 30/1.50 = 20 \qquad (\checkmark \text{ correct})$$

3.5 *'Risk-Neutral' Pricing*

3.5.1 'Risk-neutral' pricing is a neat trick which lies at the foundation of the Black-Scholes option pricing model and the large body of research that has followed since its publication. The concepts behind risk-neutral pricing can be difficult to grasp, and, as a consequence, there has been a fair amount of debate among actuaries and other practitioners about the basic idea and its practical application. Let us attempt to give some intuition for the basic idea behind risk-neutral pricing (and Black, Scholes and Merton's important work).

3.5.2 Remember, we calculated the option price without having to worry about probabilities at all. In other words, given the share price today, together with its possible states next period and the risk-free rate, the option price fell directly out of the analysis. We can observe that *any set of probabilities* could have been used in the calculation of the correct option price, so long as they are consistent with the possible share price states and the risk-free rate. This peculiar observation results because, as we saw above, the option pay-offs could be perfectly replicated by an appropriate portfolio of the underlying asset and the risk-free asset. The risk-neutral pricing technique simply recognises this feature of the problem and exploits it in a way that simplifies the calculations.

3.5.3 In the simple example above, we saw that, by observing the composition of the replicating portfolio, we could calculate the appropriate discount rate to apply to the expected option cash flow. However, when we move to a multiple-period framework (or eventually to 'continuous' time) this becomes a much more complex exercise. This presents a problem; we know that the price of the option, like any other asset, can be expressed as the present value of a set of cash flows, but finding the appropriate discount rate is usually very difficult. The crucial contribution of BSM was to show that we could circumvent the problem of finding the appropriate discount rate by changing the probabilities used to calculate the expected option cash flow.

3.5.4 We can choose to work with any probability we like. Someone might ask: "Is there a set of probabilities that is best to work with?" The answer to this question is: "Yes, pick a probability that allows you to discount all payoffs at the risk-free interest rate." In our example, let us solve for this probability q, assuming that the expected share price grows in line with the risk-free rate 25%:

$$50(1.25) = q(100) + (1 - q)25$$

$$q = 0.5.$$

3.5.5 You can see that q is 0.5. This is a 'pseudo-probability' or 'risk-neutral probability'. It is one of many probability measures that we could choose to use. It is the simplest and most convenient to work with, as it allows us to use the risk-free interest rate as our discount rate. It is known as the 'risk-neutral' probability measure, since it is consistent with a world where all assets have an expected return equal to the risk-free interest rate. Note that this does not imply that options priced using risk-neutral pricing have a 'real-world' expected return equal to the risk-free rate of return — as we saw above, the option can be replicated with a combination of the underlying asset and the risk-free asset, and its expected return will be consistent with this. The risk-neutral technique is an artificial construction where we have already nailed down prices and the interest rate. It is a clever trick.

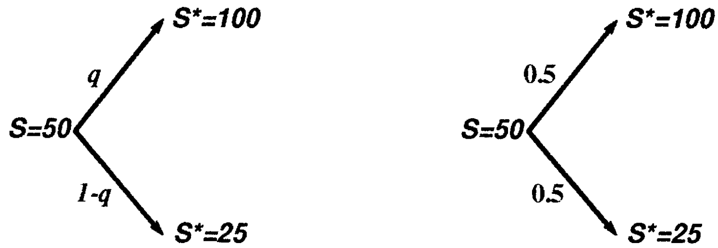

Figure 8.

We can check the equity price by discounting the expected share price (under the pseudo-probabilities) at the risk-free rate:

$$S = E_Q(S^*)/(1 + r) = 62.5/1.25 = 50. \qquad (\checkmark \text{ correct})$$

Now let us price the call option by discounting the expected option price (under the pseudo-probabilities) at the risk-free rate:

$$C = E_Q(C^*)/(1 + r) = 25/1.25 = 20. \qquad (\checkmark \text{ correct})$$

3.5.6 Again, we have the correct result for the option price. Remember that finding the correct discount factor to use with the true, real-world probabilities required analysing the composition of the replicating portfolio. Since — for a call option — it is geared, the correct discount rate was higher than the equity expected return. When we pick a set of probabilities that ensures that the expected return on the share is the same as the risk-free rate, this also ensures that the expected rate of return on the option must be the same. Look at equation 1 in ¶3.4.2. You can see that, if the expected return on the share is set equal to the risk-free rate, then the expected return on the option will also be the risk-free rate.

3.5.7 We could now use the pseudo-probabilities to value *any* other claim that is contingent on the share price at the end of the period. They provide a neat short cut to the correct answer every time. It is important to appreciate that, although we have modified the probability associated with each state, all states are still possible. It is the relation between the possible states and the current share price that determines the price of a contingent claim.

3.6 *Beyond the Binomial Branch*

3.6.1 Is everything that we have seen so far trivial, or does it arm us with genuine insights that can be applied to real world problems? Certainly, someone might argue that a single binomial branch does not really carry us very far. However, they should appreciate that Black, Scholes and Merton applied the basic concepts presented above in a way that has provided practitioners with a practical tool for analysing, valuing and hedging the option exposures that are so widespread.

3.6.2 One way to arrive at BSM's famous result is to extend the binomial tree presented in Figure 9. More steps can be added. Time can be

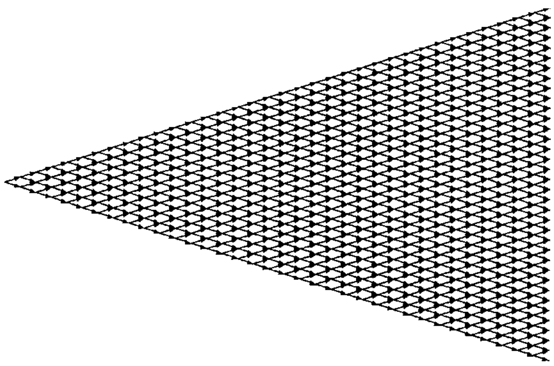

Figure 9. The binomial tree

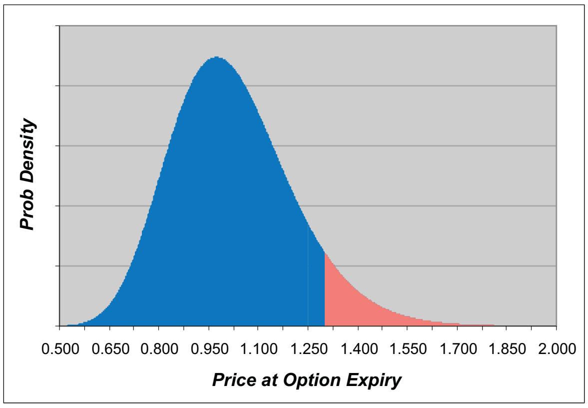

Figure 10. Probability density function

segmented into smaller pieces. The states for the share price can be chosen so as to be consistent with investors' views on possible future prices. Figure 9 shows a tree with a mere 35 steps. If the step size is reduced, the binomial distribution becomes closer to the normal distribution and (in the limit) the BSM result emerges. Just like in the binomial branch, you can think of the BSM price as the present value of the expected option payoff discounted at the risk-free rate, but where the mean of the share price distribution is fixed in line with the risk-free rate. It is what emerges when the binomial model is extended to its limit.

3.7 *Some Comments*

3.7.1 The BSM model — and extensions — reveals the portfolio that will replicate the option's payoffs in all states of the world. BSM showed that, if time is divided into small enough increments, a replicating portfolio can be found and then adjusted to match any contingent claim. In order to comment on the usefulness of the model, it is worth highlighting some of the key assumptions:

— Price changes are continuous (prices do not 'gap'), and the process that generates price changes is known.
— Markets do not contain frictions, re-balancing is continuous and costless.

3.7.2 Now, any practitioner will tell you that these assumptions do *not* hold in the real world. As Figure 11 shows, although price changes conform very approximately to the log-Normal distribution, there are frequent 'gaps' where prices move by 5% or even 10% in a day.

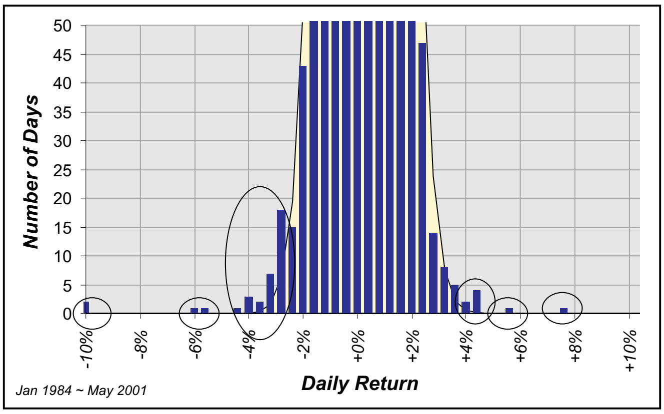

Figure 11. FTSE index prices changes

3.7.3 Of course, transacting in real-world markets does incur costs. Frequent transactions in most financial instruments will be very costly. Does this violation of the model's assumptions mean that it is not a practical tool? Not at all. Like any model, it was never meant to be a perfect representation of the real world. In order to judge its usefulness, we must compare the insights it provides with other alternatives. Judged in those terms, the BSM model and all that has followed should be viewed as a success.

3.7.4 Now consider Figures 12 to 14. They show the analysis of a 10-year savings product, where assets are invested in equities and a money-back guarantee is provided. In Figure 12 we simulate the performance of a hedge

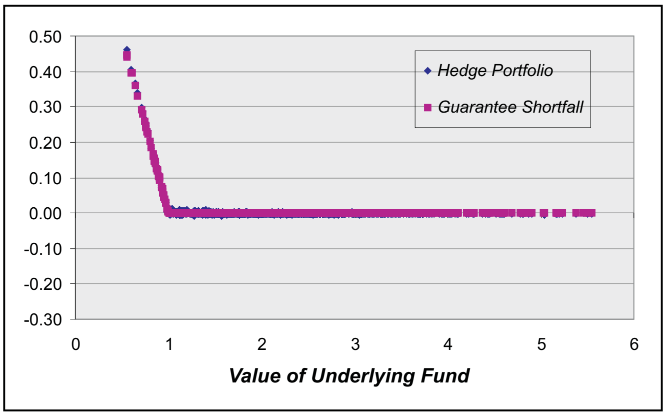

Figure 12. Daily hedge adjustments

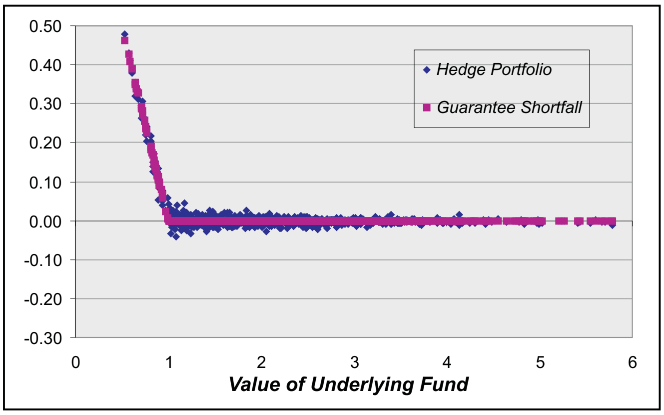

Figure 13. Monthly hedge adjustments

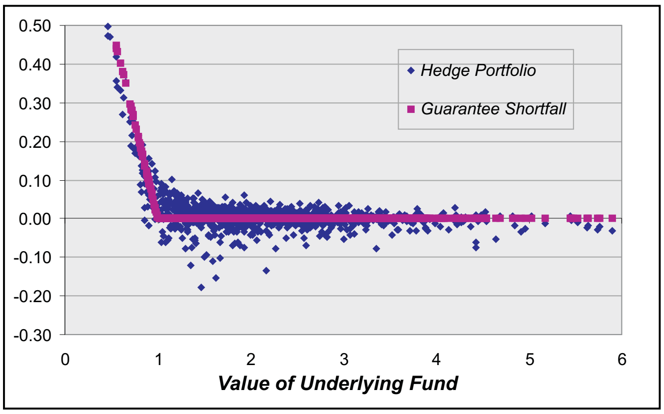

Figure 14. Annual hedge adjustments

portfolio (using Monte-Carlo simulation), where the hedge is adjusted on a daily basis. There is a good match between the hedge and the liability incurred by the provider of the product.

3.7.5 Figures 13 and 14 show the same analysis, but where the hedge is adjusted only monthly and annually, respectively. You can see that the match is no longer perfect, but the hedge helps to limit exposure, even with an annual adjustment. We may not believe in all of the BSM model's assumptions, but it still looks like a useful tool for the actuary.

3.8 *Summary*

3.8.1 Life companies own option exposures as part of both assets and liabilities. The options sold by life assurers mean that they now carry huge option exposures on the liability side of their balance sheets that dwarf anything held in their asset portfolios. They are in the derivatives business.

3.8.2 The BSM model (and the huge literature that has followed) provides a powerful framework for pricing and managing these exposures. The theory developed by Black, Scholes and Merton reveals the portfolio that will replicate the payoffs on a contingent claim in all possible states of the world. Their work provided practitioners and researchers with another key insight — that the probabilities used to price options could be modified in a way that avoids much of the potential complexity of contingent claims pricing.

3.8.3 BSM is only a model. The challenge for actuaries — like their counterparts in the banking industry — is to put the model and the many enhancements and variations that have followed it into everyday use.

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