Optimal Consumption Strategy in the Presence of Default Risk: Discrete-Time Case

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Regime-Switching Model

- Market situation may change $\Rightarrow$ distribution of asset’s return will change over time

- Regime-Switching model: market environment may switch among different regimes in a Markovian manner $\Rightarrow$ distribution of asset’s return will change over time in a Markovian manner
Regime-Switching Model


Model

Discrete-time setting: investor can decide the level of consumption, \( c_n \) at time \( n = 0, 1, 2, \ldots, T \)

After consumption, all the remaining money will be invested in a risky asset

The random return of the risky asset in different time periods will depend on the state of a time-homogeneous Markov chain \( \{\xi_n\}_{0 \leq n \leq T} \) with state space \( \mathcal{M} = \{1, 2, \ldots, M\} \) and transition probability matrix \( P = (p_{ij}) \)
Absorption State — Default Risk

Assume that state $M$ of the Markov Chain is an absorbing state:

\[
p_{Mj} = 0 \quad j = 1, 2, \ldots, M - 1, \\
p_{MM} = 1.
\]

Default occurs at time $n$ if $\xi_n = M$. In this case, the investor can only receive a fraction, $\delta$, of the amount that he/she should have received.

The recovery rate $\delta$ is a random variable, valued in $[0, 1]$. 
\{W_n\}_{0 \leq n \leq T}: \text{wealth process of the investor}

\begin{align*}
W_{n+1} &= \begin{cases} 
(W_n - c_n)R^\xi_n (1_{\{\xi_n+1 \neq M\}} + \delta 1_{\{\xi_n+1 = M\}}) & \text{if } \xi_n \neq M, \\
W_n - c_n & \text{if } \xi_n = M,
\end{cases} \\
& \quad n = 0, 1, \ldots, T - 1, \text{ where } 1\{\ldots\} \text{ is the indicator function.}
\end{align*}

$R_i^m$ is the return of the risky asset in the time period $[n, n + 1]$, given that the Markov chain is at regime $i$ at time $n$. 
Assumptions

1. The random returns $R^i_0, R^i_1, \ldots, R^i_{T-1}$ are i.i.d. with distribution $F_i$; they are strictly positive and integrable

2. $R^i_n$ is independent of $R^j_m$, for all $m \neq n$

3. The Markov chain $\{\xi\}$ is stochastically independent to the random returns in the following sense:

$$
\mathbb{P}(\xi_{n+1} = i_{n+1}, R^i_n \in B \mid \xi_0 = i_0, \ldots, \xi_n = i_n) = p_{i_n i_{n+1}} \mathbb{P}(R^i_n \in B)
$$

for all $i_0, \ldots, i_n, i_{n+1} \in S, B \in \mathcal{B}(\mathbb{R})$ and $n = 0, 1, \ldots, T - 1$
Assumptions

4. $0 \leq c_n \leq W_n$ (Budget constraint)

5. The recovery rate $\delta$ is stochastically independent of all other random variables
Given that the initial wealth is $W_0$ and the initial regime is $i_0 \in \mathcal{M}^* := \mathcal{M} \setminus \{M\}$, the objective of the investor is to

$$\max_{\{c_0, \ldots, c_T\}} \mathbb{E}_0 \left[ \sum_{n=0}^{T} \frac{1}{\gamma} (c_n)^\gamma \right]$$

over all admissible consumption strategies. Here $0 < \gamma < 1$.

**Admissible consumption strategy**: a feedback law $c_n = c_n(\xi_n, W_n)$ satisfying the budget constraint

**Optimal Consumption Strategy**: $\hat{C} = \{\hat{c}_0, \ldots, \hat{c}_T\}$
Definition 1 For \( n = 0, 1, \ldots, T \), the value function \( V_n(\xi_n, W_n) \) is defined as

\[
V_n(\xi_n, W_n) = \max_{\{c_n, c_{n+1}, \ldots, c_T\}} \mathbb{E}_n \left[ \sum_{k=n}^{T} \frac{1}{\gamma} (c_k)^\gamma \right].
\]

Bellman’s Equation:

\[
\begin{cases}
V_n(\xi_n, W_n) = \max_{0 \leq c_n \leq W_n} \mathbb{E}_n [U(c_n) + V_{n+1}(\xi_{n+1}, W_{n+1})] \\
V_T(\xi_T, W_T) = \frac{1}{\gamma} W_T^\gamma \\
\end{cases}
\]

\( n = 0, 1, \ldots, T - 1 \)
Define some symbols recursively:

\[ M^{(i)} = \{ \mathbb{E}[R^i]^\gamma \} \frac{1}{1-\gamma}, \quad i \in \mathcal{M}^*, \]
\[ L_0^{(i)} = 0, \quad i \in \mathcal{M}, \]
\[ L_n^{(i)} = M^{(i)} K_n^{(i)} 1_{\{i \neq M\}} + n 1_{\{i = M\}}, \quad i \in \mathcal{M}, n = 1, 2, \ldots, T, \]
\[ K_1^{(i)} = [1 - p_{iM} + p_{iM} \mathbb{E}(\delta^\gamma)] \frac{1}{1-\gamma}, \quad i \in \mathcal{M}^*, \]
\[ K_n^{(i)} = \left\{ \frac{M-1}{\sum_{j=1}^{M-1} p_{ij} (1 + L_{n-1}^{(j)})^{1-\gamma} + p_{iM} \mathbb{E}(\delta^\gamma)(1 + L_{n-1}^{(M)})^{1-\gamma}} \right\}^{\frac{1}{1-\gamma}}, \]
\[ i \in \mathcal{M}^*, n = 2, \ldots, T. \]

Note that \( K_{(M)} \)'s are not defined. \( M^{(i)} \) is well-defined since we have assumed that \( R^i \) is integrable.
Theorem 1  For $n = 0, 1, \ldots, T$, the value functions are given by

$$V_{T-n}(i, w) = \frac{1}{\gamma} w^\gamma (1 + L_n^{(i)})^{1-\gamma},$$

and the optimal consumption strategy $\hat{C}$ is given by

$$\hat{c}_{T-n}(i, w) = \frac{w}{(1 + L_n^{(i)})}.$$
From Theorem 1, we see that if we are now at time $T - n$, and at regime $i$, then we should consume a fraction of our wealth which is equal to

$$\frac{1}{1 + L_i^{(n)}}.$$ 

Thus our optimal consumption strategy depends heavily on the current regime and the remaining investment time through the function $L$. 
Proposition 1  (a) For fixed $i \in \mathcal{M}$, $L_n^{(i)}$ is increasing in $n$:

$$0 = L_0^{(i)} \leq L_1^{(i)} \leq \ldots \leq L_T^{(i)}.$$ 

(b) For fixed $i \in \mathcal{M}^*$, $K_n^{(i)}$ is increasing in $n$:

$$0 \leq K_1^{(i)} \leq K_2^{(i)} \leq \ldots \leq K_T^{(i)}.$$
The monotonicity of $L$ implies at the same regime, we should consume a larger fraction of our wealth when we are closer to the maturity.

This strategy is quite reasonable. If we are closer to the maturity, a short-term fluctuation in the return of the risky asset will bring a loss to us that we may not have enough time to cover. Therefore, we should consume more and invest less.
Next, we may guess that at any time period, say $T - n$, if we are at a “better” regime, then we should consume less and invest more.

Need two ingredients:

1. A criterion to compare the distributions of the returns in different regimes $\implies$ **second order stochastic dominance**

2. Market has to “regular” enough $\implies$ **stochastically monotone transition matrix**
Definition 2 Suppose that \( X \) and \( Y \) are two random variables satisfying

\[
\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]
\]

for any increasing and concave function \( g \) such that the expectations exist, then we say \( X \) is dominated by \( Y \) in the sense of second order stochastic dominance and it is denoted by \( X \leq_{SSD} Y \).
Definition 3 Suppose $P = (p_{ij})$ is an $m \times m$ stochastic matrix. It is called stochastically monotone if

$$\sum_{l=k}^{m} p_{il} \leq \sum_{l=k}^{m} p_{jl}$$

for all $1 \leq i < j \leq m$ and $k = 1, 2, \ldots, m$. 
Suppose $P$ is a $M \times M$ matrix. Let $e_k = (1, \ldots, 1, 0, \ldots, 0)'$ (i.e. first $k$ coordinates are 1, the rest are 0) for $k = 1, 2, \ldots, M$. Let $\mathcal{D}_M = \{(x_1, \ldots, x_M)' \in \mathbb{R}^M \mid x_1 \geq \cdots \geq x_M\}$ and $P_D = \{y \in \mathcal{D}_M \mid Py \in \mathcal{D}_M\}$.

**Lemma 1** The following statements are equivalent:

1. $P$ is stochastically monotone
2. $P_D = \mathcal{D}_M$
3. $e_k \in P_D$ for all $k = 1, 2, \ldots, M$
Proposition 2 Suppose that the transition probability matrix $P$ is stochastically monotone and

$$R^1 \geq_{SSD} R^2 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}.$$ 

Assume further that

$$M^{(i)}_1 K^{(i)}_1 \geq 1 \quad \forall i \in \mathcal{M}^*.$$ 

Then we have for $n = 1, 2, \ldots, T$

$$L^{(1)}_n \geq L^{(2)}_n \geq \cdots \geq L^{(M-1)}_n \geq L^{(M)}_n,$$

as well as

$$K^{(1)}_n \geq K^{(2)}_n \geq \cdots \geq K^{(M-1)}_n.$$
Meaning of $R^1 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}$

Preference of investor: increasing and concave utility function

+ 
Return of the risky asset in regime $i$: $R^i$

+ 
Definition of SSD order

⇓

The $M - 1$ regimes are ranked according to their favorability to the risk-averse investor:
regime 1 is the most favorable, regime $M - 1$ is the most unfavorable
Meaning of $P$ being stochastically monotone:

For $1 \leq i < j \leq M - 1$ (regime $i$ is more favorable to regime $j$)

- $\sum_{l=k}^{M} p_{il}$ is the probability of switching to the worst $m - k + 1$ regimes from regime $i$

- $\sum_{l=k}^{M} p_{jl}$ is the probability of switching to the worst $m - k + 1$ regimes from regime $j$

Intuitively, if the market is “regular” enough, we should have

$$ \sum_{l=k}^{M} p_{il} \leq \sum_{l=k}^{M} p_{jl} $$

for all possible $k$. This precisely means that $P$ is stochastically monotone.
Meaning of $M^{(i)} K_{1}^{(i)} \geq 1 \quad \forall i \in \mathcal{M}^*$:

If $1$ is invested today (regime $i$), then $M^{(i)} K_{1}^{(i)}$ is the expected utility of the amount one period later, allowing for default risk.

$M^{(i)} K_{1}^{(i)} \geq 1 \quad \forall i \in \mathcal{M}^*$ means that the risk-averse investor would prefer the risky asset to a risk-free asset (risk-free interest rate is zero) in any regimes.
Corollary 1 Suppose that the transition probability matrix $P$ is stochastically monotone and

$$R^1 \geq_{SSD} R^2 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}.$$ 

Assume further that

$$M^{(i)} K_1^{(i)} \geq 1 \quad \forall i \in \mathcal{M}^*.$$ 

Then for $w > 0$ and $n = 0, 1, \ldots, T,$

$$c_n(1, w) \leq c_n(2, w) \leq \cdots \leq c_n(M, w).$$
Effect of Recovery Rate

**Proposition 3** Suppose that $\delta_1$ and $\delta_2$ are two $[0, 1]$-valued random variables that are independent of the Markov chain $\{\xi\}$ and all the random returns. If

\[ \mathbb{E}[\delta_1^\gamma] \leq \mathbb{E}[\delta_2^\gamma], \]

then

\[ c_n(i, w; \delta_1) \geq c_n(i, w; \delta_2). \]
Example

- \( \delta_1 \sim U(0, 1) \rightarrow \mathbb{E}(\delta_1^\gamma) = 1/(1 + \gamma) \)

- \( \delta_2 \equiv 1/2 \rightarrow \mathbb{E}(\delta_2^\gamma) = 1/2^\gamma \)

It is not difficult to show that

\[
\frac{1}{1 + \gamma} \leq \frac{1}{2^\gamma}
\]

for \( 0 < \gamma < 1 \), i.e.

\[ \mathbb{E}(\delta_1^\gamma) \leq \mathbb{E}(\delta_2^\gamma), \]

hence

\[ c_n(i, w; \delta_1) \geq c_n(i, w; \delta_2). \]
THE END
THANK YOU