Optimal Retention Levels in Dynamic (Re)insurance Markets

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Outline

1. Introduction
2. Setup
3. Dynamic constrained MV problem
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Intro & Motivation

General problem statement:

optimal dynamic insurance strategy for given risk exposure,
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where:

- optimality: maximize expected utility from terminal wealth/dividend payouts, minimize ruin probability...

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○ (re)insurance decision in a MV setting (De Finetti, 1940)
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- setting: sources of randomness, financial market...

my setup:

  ○ Brownian filtration
  ○ financial market
  ○ randomness in claims/market parameters
The model

\( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), with \( \mathbb{F} \hat{=} \mathbb{F}^B \) (\( B \) Brownian motion)

Wealth and shocks:

- Wealth/exposure process: \( X(t) \)
- Proportional wealth shocks: \( dX(t) = -X(t) [\delta(t)dt + \sigma(t) \cdot dB(t)] \)
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Insurance market:

- Exposure: \(w(t)\) covered by insurance, \(v(t) = X(t) - w(t)\) retained.
- Premium rate per unit of insured capital/wealth: \(\pi(t)\)
  \[w(t)\pi(t)dt = \text{premium paid to instantaneously insure exposure } w(t)\]
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Financial market:

- money market account: \(r(t)\) risk-free rate
- risky stocks: \(dS_i(t) = S_i(t)[\mu_i(t)dt + \sigma_i(t) \cdot dB(t)]\) \(i = 1, \ldots, m\)
- \(\delta, \sigma, \mu, \bar{\sigma}\) are \(\mathbb{F}\text{-adapted}\)
Insurance-investment strategies

Insurance/investment strategy: \( u(t) = (v(t), \bar{v}_1(t), \ldots, \bar{v}_m(t))^T \).
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Wealth dynamics:

\[
dX(t) = X(t)(r(t) - \pi(t))dt + \nu(t)[(\pi(t) - \delta(t))dt - \sigma(t) \cdot dB(t)] \\
+ \sum_{i=1}^{m} \nu_i(t)[(\mu_i(t) - r(t))dt + \sigma_i(t) \cdot dB(t)]
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Wealth dynamics:

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Constraints: $v(t) \geq 0$ and...
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- no shorting: $0 \leq \bar{v}_{k+1}(t), \ldots, \bar{v}_m(t)$ $(k = 0, \ldots, m + 1)$;
- ‘safer’ assets $S_1, \ldots, S_k$, ‘riskier’ assets $S_{k+1}, \ldots, S_m$: $\sum_{i=1}^{k} \bar{v}_i(t) \geq v(t)$;
- combinations of the above.
The Optimization Problem

• Constrained dynamic MV problem:

\[
\begin{align*}
\text{min} & \quad V [X(T)] \\
\text{sub} & \quad E [X(T)] = z^* \\
& \quad (X, u) \quad \text{admissible} \\
& \quad u \in C \quad \text{constraints set}
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- Efficient strategy:

\[
\begin{aligned}
u^*(t) &= \begin{cases}
[X(t) - A(t)] \xi_1^*(t) & \text{if } X(t) > A(t) \\
[A(t) - X(t)] \xi_2^*(t) & \text{if } X(t) \leq A(t)
\end{cases}
\end{aligned}
\]

...and mean-variance frontier:

\[
E[X(T)] = f \left( \sqrt{V[X(T)]}; A(0), X(0) \right)
\]
Key BSDEs

$A(\cdot)$ depends on the solutions to the following BSDEs:

\[
\begin{align*}
    dP_{1,2}(t) &= f(t, P_{1,2}, \Lambda_{1,2}, H_{1,2}^*)dt + \Lambda_{1,2}(t) \cdot dB(t) \\
    P_{1,2}(T) &= 1 \\
    P_{1,2}(t) &> 0 \text{ a.s. } t \in [0, T]
\end{align*}
\]

$$
H_{1,2}^*(t, \omega) \doteq \min_{u \in C} H_{1,2}(t, \omega, u, P_{1,2}, \Lambda_{1,2})
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Look to $H_1$ when $X(t) > A(t)$, to $H_2$ when $X(t) \leq A(t)$. 
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Look to \( H_1 \) when \( X(t) > A(t) \), to \( H_2 \) when \( X(t) \leq A(t) \).

\( \xi_{1,2}(\cdot) \) are given by:

\[
\xi_{1,2}(t, \omega, P_{1,2}, \Lambda_{1,2}) \doteq \arg \min_{u \in C} H_{1,2}(t, \omega, u, P_{1,2}, \Lambda_{1,2})
\]
Example: \( u(t) = (v(t), \bar{v}(t))^T \)
Example: $\nu(t) \geq 0$, $\overline{\nu}(t)$ unrestricted
Example: $\nu(t), \overline{\nu}(t) \geq 0$
Conclusion

- De Finetti’s result in continuous-time:
  - Insurance market only:
    \[
    v^*(t) = \frac{\pi(t) - \delta(t)}{\sigma^2(t)} (A(t) - X^*(t))
    \]
Conclusion

- De Finetti’s result in continuous-time:
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    \[ v^*(t) = \frac{\pi(t) - \delta(t)}{\sigma^2(t)} (A(t) - X^*(t)) \]

- De Finetti & Markowitz:
  - efficient frontier
    \[ \rightarrow \text{for fixed } \epsilon \in (0, 1), x^* \geq 0 \text{ choose } z^* \text{ such that } \mathbb{P}(X(T) < x^*) \leq \epsilon \]
  - constant retention levels are inefficient
    \[ \rightarrow \text{market portfolio inefficient} \]
Some references

- Bäuerle N. (2005), Benchmark and mean-variance problems for insurers, MMOR.
- Briys E. (1986), Insurance and consumption: the continuous-time case, JRI.
  - De Finetti B. (1940), Il problema dei pieni, GIIA.
- Gollier C. (1994), Insurance and precautionary capital accumulation in a continuous-time model, JRI.
- Hipp C. (2003), Stochastic control with application in insurance, LNM, Springer.
- Hojgaard B. and M. Taksar (1998), Optimal proportional reinsurance policies for diffusion models with transaction costs, IME.
  - Hu Y. and X.Y. Zhou (2005), Constrained LQ control with random coefficients, and application to portfolio selection, SIAM JCO.
- Mnif M. (2002), Optimal risk control under proportional reinsurance contract: a dynamic programming duality approach, WP.