A Bias Reduction Technique for Monte Carlo Pricing of Early Exercise Options

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Joint work with Tyson Whitehead and Matt Davison of UWO.
Overview

• Biased Monte Carlo Estimators.
• Bias Evaluation and Removal.
American-style Derivatives

• An American-style option that pays the holder $P_w$ upon exercise at time $w$ has time-$t$ value $(t \leq T)$ given by

$$B_t = \sup_{\tau} \mathbb{E} \left[ e^{-r(\tau-t)} P_\tau \bigg| \mathcal{F}_t \right].$$

• Working backwards in time from the option maturity, the option can be priced using a recursive scheme.

• Drawbacks
  • Methods break down in high dimensions.
  • No information on path properties.
Monte Carlo


• MC methods typically generate estimators that are biased (but consistent).

• It is common to use both a high- and low-biased estimator. Here we discuss only high-biased estimators.

• In a stochastic tree, replace the exact values

\[ H_w = \mathbb{E} \left[ e^{-r\Delta T} B_{w+1} \mid \mathcal{F}_w \right] \quad \text{and} \quad B_w = \max(P_w, H_w). \]

with the estimators

\[ \tilde{H}^i_w = \frac{1}{M} \sum_{j=1}^{M} e^{-r\Delta T} \tilde{B}^{i,j}_{w+1} \quad \text{and} \quad \tilde{B}^i_w = \max(P^i_w, \tilde{H}^i_w). \]
## Estimator Error

<table>
<thead>
<tr>
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<th>Held:</th>
<th>Exercised:</th>
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<tbody>
<tr>
<td></td>
<td>$\tilde{H}<em>{w+1} &gt; P</em>{w+1}$</td>
<td>$\tilde{H}<em>{w+1} &lt; P</em>{w+1}$</td>
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<tr>
<td>Should Hold:</td>
<td>$\tilde{H}<em>{w+1} - H</em>{w+1}$</td>
<td>$P_{w+1} - H_{w+1}$</td>
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<tr>
<td>$H_{w+1} &gt; P_{w+1}$</td>
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<tr>
<td>Should Exercise:</td>
<td>$\tilde{H}<em>{w+1} - P</em>{w+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$H_{w+1} &lt; P_{w+1}$</td>
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- The error is

\[
X_{w+1} = \mathbb{1}_{H_{w+1} > P_{w+1}} \left( \tilde{H}_{w+1} - H_{w+1} \right) + \\
\mathbb{1}_{H_{w+1} < P_{w+1}} \mathbb{1}_{\tilde{H}_{w+1} > P_{w+1}} \left( \tilde{H}_{w+1} - P_{w+1} \right) + \\
\mathbb{1}_{H_{w+1} > P_{w+1}} \mathbb{1}_{\tilde{H}_{w+1} < P_{w+1}} \left( P_{w+1} - H_{w+1} \right)
\]
Bias

• Let $\tilde{Y}_{w+1} = \tilde{H}_{w+1} - P_{w+1}$ and $Y_{w+1} = H_{w+1} - P_{w+1}$.

• The bias is

\[
E \left[ e^{-r\Delta T} X_{w+1} \mid F_w \right] = e^{-r\Delta T} E \left[ \mathbb{I}_{Y_{w+1} > 0} \left( \tilde{Y}_{w+1} - Y_{w+1} \right) + \mathbb{I}_{Y_{w+1} < 0} \mathbb{I}_{\tilde{Y}_{w+1} > 0} \tilde{Y}_{w+1} - \mathbb{I}_{Y_{w+1} > 0} \mathbb{I}_{\tilde{Y}_{w+1} < 0} Y_{w+1} \bigg| F_w \right]
\]

Note:

1. $\left( \tilde{H}_{w+1}, H_{w+1}, P_{w+1} \right)$ are functions of $S_{w+1}$.

2. The bias is an integral over the joint density of $S_{w+1}$.

3. Can express this as an integral over the joint density of $\left( \tilde{Y}_{w+1}, Y_{w+1} \right)$.

4. Evaluation?
Bias

Reminder:

\[
\hat{H}^i_{w+1} = \frac{1}{M} \sum_{j=1}^{M} e^{-r\Delta T} \hat{B}_{w+2}^{i,j} \quad \text{and} \quad H_{w+1} = \mathbb{E}\left[ e^{-r\Delta T} B_{w+2} \middle| \mathcal{F}_{w+1} \right]
\]

Assumptions:

1. \( \mathbb{E}\left[ \hat{H}_{w+1} \middle| \mathcal{F}_{w+1} \right] = H_{w+1} \).

2. By the CLT \( \hat{H}_{w+1} - P_{w+1} \sim N\left( H_{w+1} - P_{w+1}, \frac{\Sigma_{w+1}}{M} \right) \)

Implication

- Bias can be approximated by an integral over the joint density of \( \left( \tilde{Y}_{w+1}, \Sigma_{w+1} \right) \).
Bias and Correction

- Bias can be expressed as

$$E \left[ e^{-r\Delta T} X \right] \approx e^{-r\Delta T} \int_{\infty}^{\infty} \int_{0}^{\infty} |\tilde{y}| \Phi \left( \frac{-|\tilde{y}|}{\sigma/\sqrt{M}} \right) f_{\tilde{Y},\Sigma}(\tilde{y}, \sigma) d\sigma d\tilde{y}$$

- Thus the bias-corrected estimator for the hold value is

$$\tilde{H}_{i}^w = \frac{1}{M} \sum_{j=1}^{M} e^{-r\Delta T} \left( \tilde{B}_{i,j}^w - \left| \tilde{H}_{i,j}^w - P_{i,j}^w \right| \Phi \left( \frac{-\left| \tilde{H}_{i,j}^w - P_{i,j}^w \right|}{\sqrt{\Sigma_{i,j}^w/M}} \right) \right)$$
Example: Setup

- 5 underlying stocks \( S = (S^1, S^2, S^3, S^4, S^5)' \)
- Price a 3-year American-style max-max call struck at $100 that, upon exercise at \( \tau \), pays the holder

\[
\max([S^1_\tau - 100]^+, [S^2_\tau - 100]^+, [S^3_\tau - 100]^+, [S^4_\tau - 100]^+, [S^5_\tau - 100]^+).
\]

- In the discretized version, \( \tau \in \{1, 2, 3\} \).
- 100 bootstrap samples drawn to estimate the bias using the bootstrap (Broadie and Glasserman 1995).
- Stochastic mesh technique used.

Broadie and Glasserman
Example: Results

- Comparisons for a fixed standard deviation ($\approx 0.01$) and computational time.
Summary

• Like Guinness, reducing bias is good for you.
  and
• It is very cheap.
Example: Computational Notes

- Stochastic mesh technique used.
- Runtime is $O(M^2)$ where $M$ is the number of mesh points (simulations).
- Standard deviation of the estimator is $O(1/\sqrt{M})$.
- Can reduce the standard deviation of the estimator by repeated, independent valuations. This does NOT reduce the bias.
- Computations were done using many resources, including SHARCNet, involving four different architectures and many Linux versions and execution environments.
- Total computational time of 282 CPU days, performed over a two to four week period.
- 641,350 independent runs.
- Produced approximately 500 MB of output.