On the Application of the Wilkie Model to the TSX Price Index

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The Wilkie Model is the first stochastic investment model of long term returns of multiple assets designed for actuarial application. This model was first published by Wilkie (1986).
Introduction

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Many applications of Wilkie’s hierarchy structural modeling method were done late on, such as Mulvey and Thorlacius (1996), Sharp (1992), and Tomson (1996).
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Many applications of Wilkie’s hierarchy structural modeling method were done late on, such as Mulvey and Thorlacius (1996), Sharp (1992), and Tomson (1996).

Critics and adjustment to the Wilkie Model are also very active, such as Huber (1995), Chang and Wang (1998), and Whitten and Thomas (1999).
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Validation of the Wilkie Model

Outline

1. Introduction

2. Validation of the Wilkie Model
   - Evaluation of Other Models
   - Further Evaluation

3. Discussion of Dividend Yield

4. Discussion of New Model Structure

5. Conclusion

6. Bibliography
Cascade structure of Wilkie’s Model

\[ T_t = D_t + \frac{D_t}{Y_t} \quad \& \quad P_t = \frac{D_t}{Y_t} \]

\[ R_t = \log\left(\frac{T_t}{P_{t-1}}\right), \quad \text{where } R_t \text{ is the TSX Total Return yield} \]
Formulas of Four Components

- Model of Retail Price Index $Q(t)$:
  \[
  \nabla \log Q(t) = QMU + QA(\nabla \log Q(t-1) - QMU) + QSD \times QZ(t)
  \]

- Model of dividend yield $Y(t)$
  \[
  \log Y(t) = YW \times \nabla \log Q(t) + YN(t)
  \]
  \[
  YN(t) = \log YMU + YA(YN(t-1) - \log YMU) + YE(t)
  \]
Formulas of Four Components

- **Model of dividend index** $D(t)$:

\[
\nabla \log D(t) = DW\left(\frac{DD}{1 - (1 - DD)B}\right) \nabla \log Q(t) + DX \times \nabla \log Q(t) \\
+ DMU + DY \times YE(t - 1) + DE(t) + DB \times DE(t - 1)
\]

where: \(\frac{DD}{1 - (1 - DD)B}\) is exponential weighted average

- **Model of Consols yield** $C(t)$:

\[
C(t) = CW\left(\frac{CD}{1 - (1 - CD)B}\right) \nabla \log Q(t) + CN(t),
\]

\[
\log CN(t) = \log CMU + CA \times (\log CN(t - 1) - \log CMU) \\
+ CY \times YE(t) + CSD \times CZ(t)
\]
Log-likelihood of the Wilkie Model for TSX Yield

\[
\sum_{t} \log f(R_t|\mathcal{F}_{t-1}) = \sum_{t} \log(D_t + D_t Y_t) - \sum_{t} \log(D_t) \\
+ \sum_{t} \log f(\log(Y_t)|\mathcal{F}_{t-1}) + \sum_{t} \log f(\nabla \log(D_t)|\mathcal{F}_{t-1})
\]

where:

\[
\sum_{t} \log f(\log(Y_t)|\mathcal{F}_{t-1}) : \text{Log-likelihood of logarithmic dividend yield}
\]

\[
\sum_{t} \log f(\nabla \log(D_t)|\mathcal{F}_{t-1}) : \text{Log-likelihood of logarithmic dividend force}
\]
Prediction with the Wilkie Model

Figure: Prediction for 10 years based on Canadian data 1956–1995
Independent log-normal model for short term data: \( T_t = T_0 \cdot e^{\mu t + \sigma W(t)} \), where \( W(t) \) is a standard Brownian motion.

Figure: ILN model for monthly data of TSX
ARIMA–GARCH Model

- ARMA(1) model: $R_t - a(R_{t-1} - \mu) = \mu + \sigma \epsilon_t + \theta \sigma \epsilon_{t-1}$
- GARCH(1,1) model: $R_t = \mu + \sigma \epsilon_t, \quad \sigma_t^2 = a_0 + a_1 (R_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2$.
- Fitted ARMA(1,1)+GARCH(1,1) for monthly Canadian data and ARIMA(0,0,0) $\times$ (0,0,1)$^5$ for quarterly data.

**Figure:** Forecast of 96–05 for quarterly Canadian data
RSLN2 Model

- Log–Normal Regime Switching (RSLN) model: $R_t|\rho_t \sim N(\mu_{\rho_t}, \sigma^2_{\rho_t})$, where $\rho$ represents the regime, $P_{i,j}$ represents transition probabilities between regimes.

**Figure:** Forecast of 96–05 for 10–year Canadian data
Residual Analysis

The residuals of ARIMA-GARCH model are white noise. The following are the residuals of ILN, the Wilkie Model, and YE and DE values.
Comparison of Various Models

Table: Comparison of Fitted Models for Quarterly TSX Yield 1956-2005

<table>
<thead>
<tr>
<th>Statistics</th>
<th>ILN</th>
<th>ARMA(0,0)x(0,1)^5</th>
<th>RSLN</th>
<th>the Wilkie Model</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Parameters</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>logL</td>
<td>218.43</td>
<td>220.44</td>
<td>224.28</td>
<td>212.58</td>
</tr>
<tr>
<td>LRT p-value</td>
<td>0.0197</td>
<td>0.104</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>
Comparison of Various Models

**Table:** Comparison of Simulation of Monthly TSX Yield over 10 Years

<table>
<thead>
<tr>
<th>Statistics</th>
<th>ILN</th>
<th>ARMA(1,1)</th>
<th>ARMA–GARCH(1,1)</th>
<th>RSLN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimun value</td>
<td>-0.1986</td>
<td>-0.2153</td>
<td>-0.3558</td>
<td>-0.3463</td>
</tr>
<tr>
<td>2.5 percentile</td>
<td>-0.0806</td>
<td>-0.0803</td>
<td>-0.0817</td>
<td>-0.0958</td>
</tr>
<tr>
<td>5 percentile</td>
<td>-0.0663</td>
<td>-0.0661</td>
<td>-0.0663</td>
<td>-0.0677</td>
</tr>
<tr>
<td>Pr(crash)</td>
<td>0</td>
<td>0</td>
<td>0.0008</td>
<td>0.0088</td>
</tr>
</tbody>
</table>

Prob(crash) is based on the event in October 1987 when the TSE index crashed with the historical low yield value of -0.2552. For monthly data, \[ Pr(Crash) = Pr\left\{ \min_{1 \leq t \leq 120} Y_t \leq -0.2552 \right\} \]
Comparison of Various Models

<table>
<thead>
<tr>
<th>Statistics</th>
<th>ARMA(0,0)x(0,1)</th>
<th>RSLN</th>
<th>the Wilkie Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.0231</td>
<td>0.0229</td>
<td>0.0145</td>
</tr>
<tr>
<td>std.dv</td>
<td>0.0811</td>
<td>0.0807</td>
<td>0.2556</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.0082</td>
<td>3.6518</td>
<td>3.8208</td>
</tr>
<tr>
<td>Minimum value</td>
<td>-0.3700</td>
<td>-0.4215</td>
<td>-0.6826</td>
</tr>
<tr>
<td>2.5 percentile</td>
<td>-0.1356</td>
<td>-0.1586</td>
<td>-0.3709</td>
</tr>
<tr>
<td>5 percentile</td>
<td>-0.1099</td>
<td>-0.1258</td>
<td>-0.3220</td>
</tr>
<tr>
<td>10 percentile</td>
<td>-0.0806</td>
<td>-0.0864</td>
<td>-0.2590</td>
</tr>
</tbody>
</table>
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Discussion of Dividend Yield

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Discussion of Dividend Yield

Calculation of Dividend Yield

The calculation under discrete assumption of dividend yield is as follows:

\[ D_t = (T'_t - T'_{t-1}) \frac{P_{t-1}}{T'_{t-1}} - (P_t - P_{t-1}) \]

where \( T'_t \): TSX Total Return Index, \( P_t \): TSX Price Index

Dividend is assumed to be discrete at the end of each period

\[ Y_t = \frac{D_t}{P_t} = \frac{(T'_t - T'_{t-1})(P_{t-1})}{T'_{t-1}P_t} - \frac{P_t - P_{t-1}}{P_t} \]

The calculation under continuous assumption is as follows:

\[ Y'_{t-1} = \log(\frac{T'_t}{T'_t}) - \log(\frac{P_t}{P_{t-1}}) \]

\[ D_t = P_{t-1} \times (e^{Y'_{t-1}} - 1) \]

\[ Y_t = \log(\frac{D_t}{P_t} + 1) \]
Prediction of US Market Dividend Yield

The following plots are based on the data series for US dividend yield $Y(t)$ of 1927 – 2006. The first 70 years data are used to modeling and the last 10 years data are used to compare the prediction.
High AR Coefficient for Dividend Yield Model

Table: Fitted AR Coefficients of $YN(t)$ in Dividend Yield Model for US Market

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarterly Data AR1</td>
<td>0.77</td>
<td>0.93</td>
</tr>
<tr>
<td>Yearly Data AR1</td>
<td>0.82</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Table: AR Coefficients of $YN(t)$ for Canadian Data

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarterly Data AR1</td>
<td>0.63</td>
<td>0.91</td>
</tr>
<tr>
<td>Yearly Data AR1</td>
<td>0.85</td>
<td>0.97</td>
</tr>
</tbody>
</table>

Comments: From the above tables, we can see that the time series models for the dividend yield is becoming not very stationary in recent years.
The Q–Q plot of dividend yield based on monthly S&P 1926–2005 shows that the percentiles of year 11 (y coordinate) are smaller than those of year 1 (x coordinate), which is verified by paired–rank test and regression test.
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Discussion of Dividend Yield

Dividend for Price

- Based on Variance Bound Test:

\[ p_t = E_t(\beta d_{t+1} + \beta^2 d_{t+2} + \ldots + \beta^{n+1} d_{t+n+1} + \beta^{n+1} p_{t+n+1}) \]

let \( p_t^* = \sum_{n=1}^{\infty} \beta^n d_{t+n} \)

therefore \( V(p_t) \leq V(p_t^*) \)

- Based on the test of monthly S&P from Jan. 1926 to Dec. 2006 with some selected yield rate, the results contradict the above claim.

- We can see that the behaviors of dividend and share price are differently, which make the dividend yield less predictable.
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This new structure is constructed based on stepwise regression models on different components, their lagged values, and some rollover values. In the structure, we discard dividend yield and use share yield directly in the modeling.
New Structure

This structure is constructed based on the analysis of residuals of Univariate ARIMA Models.

- Retail Price Index
- Share Dividend Force
- Consols Yield
- Return Index Yield
New Structure

- This structure is constructed based on the Vector Autoregressive modeling for the four economic components, with the covariance matrix for the residuals of the model of each component.
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Discussion of New Model Structure

Simulation of Year TSX yield with VAR Model of New Structure

<table>
<thead>
<tr>
<th>Statistics of VAR model</th>
<th>Statistics of RSLN2 model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistic</td>
<td>Value</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0898</td>
</tr>
<tr>
<td>Minimum value</td>
<td>-0.7535</td>
</tr>
<tr>
<td>St. dev.</td>
<td>0.1972</td>
</tr>
<tr>
<td>2.5 percentile</td>
<td>-0.2984</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.0252</td>
</tr>
<tr>
<td>5 percentile</td>
<td>-0.2351</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.0202</td>
</tr>
<tr>
<td>10 percentile</td>
<td>-0.1622</td>
</tr>
</tbody>
</table>

Comment: The variance of VAR model is between those of the Wilkie Model and RSLN2, while the tails of RSLN2 is thicker than that of VAR model.
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- In this case both VAR models and the Wilkie Model create larger variance than the univariate models. Actually in the Wilkie Model, its variance is so wide that almost everything could happen.
Conclusion

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- It is better to include the series of TSX yield to predict the movement of itself. Otherwise, neither dividend yield nor dividend could predict TSX yield accurately.
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- The monthly data and the quarterly data could have different results and interpretation for modeling. In general, the continuous assumption and discrete assumption are suitable for monthly data and quarterly data respectively.
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- The monthly data and the quarterly data could have different results and interpretation for modeling. In general, the continuous assumption and discrete assumption are suitable for monthly data and quarterly data respectively.

- Even a historically stationary model of dividend yield may not be predictable for future period, because of changing parameters for ARIMA models.
Future Works

- Updating models with other methods like cointegrated VAR models, and state space method for changing model parameters.
- Specifying any measurement for the degree of stationarity of time series for different period, and testing the ergodicity of time series.
- Improving RSLN models with smoothed regimes for multiple variables, and providing better methods for testing RSLN effects.
- Updating models by taking more considerations of realities as constraints of the models.
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Questions?

Thanks!