A Cautionary Note on Pricing Longevity Index Swaps
(Joint work with Johnny S.H. Li)

Rui Zhou

Department of Statistics and Actuarial Science
University of Waterloo

44th Actuarial Research Conference 2009
Objectives

- Pricing QxX index swap
- Examining the parameter risk and model risk in the pricing
- Determining the effect of the uncertainty on the pricing

Outline

- Mortality derivatives
- QxX index Swap
- Parameter risk
- Model risk
- Conclusion
Mortality Derivatives

What are mortality derivatives?

- Financial contracts that have payoffs tied to the level of a certain longevity or mortality index
- Examples: survivor bond, survivor swap, . . .

How to price mortality derivatives?

- Mortality model
- Wang’s Transform, Q measure, . . .
A two-factor stochastic mortality model (Cairns, Blake and Dowd (2006))

Mathematical Specification:

$$\ln \frac{q_{x,t}}{1 - q_{x,t}} = A_1(t) + A_2(t)x. \quad (1)$$

- $x \rightarrow$ age
- $t \rightarrow$ time
- $q_{x,t} \rightarrow$ realized single-year death probability
- $\{A_1(t)\}$ and $\{A_2(t)\} \rightarrow$ discrete-time stochastic processes
A two-factor stochastic mortality model (con’t)

Stochastic Mortality: \[ \ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \]

\[ D(t+1) = A(t+1) - A(t) \]
\[ = \mu + CZ(t+1) \] (2)

- \( A(t) = (A_1(t), A_2(t))' \)
- \( \mu \rightarrow \text{constant } 2 \times 1 \text{ vector} \)
- \( C \rightarrow \text{constant } 2 \times 2 \text{ upper triangular matrix} \)
- \( Z(t) \rightarrow 2\text{-dim standard normal random variable} \)
Model fitting

Data

- $q_{x,t}, \quad x = 65, 66, \ldots, 109, \quad t = 1971, 1972, \ldots, 2005$

Model fitting

\[
\ln \frac{q_{x,t}}{1 - q_{x,t}} = A_1(t) + A_2(t)x \quad D(t + 1) = \mu + CZ(t + 1)
\]

- First step: Estimate $A(t)$ by least square method
- Second step: Estimate $\mu$ and $C$ through maximum likelihood estimation
A Cautionary Note on Pricing Longevity Index Swaps

Mortality Derivatives

Mortality model

Forecasting

Steps

\[
\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \quad D(t+1) = \mu + CZ(t+1)
\]

- Simulate a set of $Z$
- Obtain corresponding $D(2005 + k), \ k = 1, 2, \ldots, 10$
- $A(2005 + k) = A(2005) + \sum_{n=1}^{k} D(2005 + n), \ k = 1, 2, \ldots, 10$
- Calculate $q_{x,2005+k}$
Pricing in Risk-adjusted world

Real-world probability measure (P measure)

\[ D(t + 1) = \mu + CZ(t + 1) \]  

(3)

Risk-adjusted probability measure (Q measure)

\[ D(t + 1) = \mu + C(\tilde{Z}(t + 1) - \lambda) \]
\[ = \tilde{\mu} + C\tilde{Z}(t + 1), \]  

(4)

where \( \lambda \) is the market price of risk and \( \tilde{\mu} = \mu - C\lambda \).
QxX Index

“allows market participants to measure, manage and trade exposure to longevity and mortality risks in a standardized, transparent, and real-time manner"

- Launched by Goldman Sacs in 2007
- Based on a reference pool consisting of a set of lives underwritten by AVS Underwriting LLC
- The index value is the number of lives in the reference pool
- Published monthly, providing “real-time” mortality information
Payment structure of QxX index swap

\[ X \left( \frac{S_{k-1}}{S_0} \cdot \frac{\sigma}{12} \right) \]

\[ X \left( \frac{S_{k-1} - S_k}{S_0} \right) \]

- \( X \rightarrow \) nominal amount
- \( S_k \rightarrow \) index value in the \( k \)th month
- \( \sigma \rightarrow \) fixed spread
- Goldman Sachs: \( \sigma = 500 \) basis points for 10-year swap
A Cautionary Note on Pricing Longevity Index Swaps

QxX index swap

Pricing a 10-year QxX index swap

10-year QxX index swap price

- QxX index swap is priced by determining the “fair” spread $\sigma$
  
  \[
  \text{Market value of future payments from fixed payer} = \text{Market value of future payments from fixed receiver}
  \]

- We need to know the market price of risk $\lambda$. In our analysis,
  - Not enough data to estimate $\lambda$ for QxX index swaps
  - Use the estimated market price of risk from BNP/EIB longevity bond
10-year QxX index swap price (Con’t)

Estimates of $\sigma$ (in basis points) under different choices of $\lambda = (\lambda_1, \lambda_2)$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>627</td>
</tr>
<tr>
<td>0</td>
<td>0.316</td>
<td>619</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>622</td>
</tr>
</tbody>
</table>

Why $\sigma \neq 500$ bps?

- No access to the actual QxX index reference pool
- Lack of market data for the swap
- Existence of parameter risk and model risk
Parameter risk under Bayesian Method

- \( D(t) \sim \text{MVN}(\mu, V) \), where \( V = C' C \).
- Treat \( \mu \) and \( C \) as random variables

\[ D(t) \mid \mu, V \sim \text{MVN}(\mu, V) \tag{5} \]

- Use a non-informative prior distribution

\[ \pi(\mu, V) \propto |V|^{-3/2} \tag{6} \]

- Marginal posterior distribution

\[ V^{-1} \mid D \sim \text{Wishart}(n - 1, n^{-1} \hat{V}^{-1}), \tag{7} \]
\[ \mu \mid D \sim \text{MVN}(\hat{\mu}, n^{-1} \hat{V}), \]
Estimated marginal posterior density functions for the model parameters

**Figure:** Simulated marginal posterior parameter distributions. (We denote the \( i \)th element in \( \mu \) by \( \mu_i \) and the \((j, k)\)th element in \( V \) by \( V_{j,k} \).)
Simulated predictive distribution of $\sigma$, $\lambda = (0.375, 0)$
### 95% Confidence Interval for $\sigma$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>With parameter risk</th>
<th>Without parameter risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>(560,693)</td>
<td>(574,680)</td>
</tr>
<tr>
<td>0</td>
<td>0.316</td>
<td>(553,685)</td>
<td>(567,673)</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>(557,686)</td>
<td>(571,675)</td>
</tr>
</tbody>
</table>

**Table**: 95% confidence intervals for $\sigma$ (in basis points) under different choices of $\lambda_1$ and $\lambda_2$. 
A Cautionary Note on Pricing Longevity Index Swaps

Model risk in pricing

Figure: Estimated values of $A_1(t)$ and $A_2(t)$, 1971–2005.
A Cautionary Note on Pricing Longevity Index Swaps

Model risk

Reason for the reverse trend

What causes the reverse trend?

Crude mortality curves

\[ \ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \]
What causes the reverse trend?

Life expectancies at age 65

\[ \ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t) x \]
Three possible scenarios
How does the change affect QxX index swap price?

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>627</td>
<td>674</td>
<td>566</td>
</tr>
<tr>
<td>0</td>
<td>0.316</td>
<td>619</td>
<td>683</td>
<td>553</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>622</td>
<td>678</td>
<td>558</td>
</tr>
</tbody>
</table>

Table: Swap spread (in basis points) under three different scenarios.
Conclusion

- The swap spread computed from our pricing framework is fairly close to the spread currently offered by Goldman Sachs
- The pricing is still very experimental
  - Parameter risk and model risk are significant in the pricing
  - No sufficient market price data to estimate market prices of risk
  - No clear conclusion on how mortality rates may evolve in the future