An Empirical-based Approach for Optimal Reinsurance

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Outline of Today’s Presentation

- Background
- Motivation
- Empirical-based Approach
- Conclusion
Effect of Reinsurance

Policyholders (Insureds)

Premium $\pi_0$

Loss $X$

Insurance Company (Insurer or Cedent)
Effect of Reinsurance

Policyholders (Insureds) $\xrightarrow{\text{Premium } \pi_0} \text{Insurance Company (Insurer or Cedent)}$

Loss $X$

Reinsurance Company (Reinsurer)

$f(X)$: Ceded Claims
- e.g. stop loss:
  $f(X) = (X - d)_+, d \geq 0$
- e.g. quota-share:
  $f(X) = cX, 0 \leq c \leq 1$
Effect of Reinsurance

Policyholders (Insureds)

Premium $\pi_0$

Loss $X$

Insurance Company (Insurer or Cedent)

Reinsurance Company (Reinsurer)

$\Pi(f)$: Reinsurance Premium

e.g. Expectation premium principle:

$$\Pi(f) = (1 + \theta)\mathbb{E}[f(X)]$$

$f(X)$: Ceded Claims

e.g. stop loss:

$$f(X) = (X - d)^+, d \geq 0$$

e.g. quota-share:

$$f(X) = cX, 0 \leq c \leq 1$$
**Effect of Reinsurance**

- **Policyholders (Insureds)**
- **Insurance Company (Insurer or Cedent)**
- **Reinsurance Company (Reinsurer)**

- **Premium** $\pi_0$
- **Loss** $X$

$\Pi(f)$: Reinsurance Premium
- E.g., Expectation premium principle:
  \[ \pi(f) = (1 + \theta)\mathbb{E}[f(X)] \]

$f(X)$: Ceded Claims
- E.g., stop loss:
  \[ f(X) = (X - d)_+ , d \geq 0 \]
- E.g., quota-share:
  \[ f(X) = cX , 0 \leq c \leq 1 \]

- Insurer’s retained risk: $R_f(X) = X - f(X)$
- Insurer’s total risk: $T_f(X) = R_f(X) + \Pi(f) = X - f(X) + \Pi(f)$

- **tradeoff** between the amount of loss retained and the reinsurance premium payable to a reinsurer
A plausible optimal reinsurance model:

\[
\begin{align*}
\min_f \quad & \rho(T_f(X)) = \rho(X - f(X) + \Pi(f(X))) \\
\text{s.t.} \quad & \Pi(f(X)) \leq \pi \\
& 0 \leq f(x) \leq x \quad \text{for all} \quad x \geq 0
\end{align*}
\]

where \( \rho \) is a chosen risk measure, such as variance, VaR and CTE.
A plausible optimal reinsurance model:

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    \text{s.t.} & \quad \Pi(f(X)) \leq \pi \\
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\end{align*}
\]

where \( \rho \) is a chosen risk measure, such as variance, VaR and CTE.

Complexity of solving the above risk measure minimization models:

- if the form of \( f \) is specified: technically tractable.
  - stop-loss \( f(x) = (x - d)_+ \)
  - quota-share \( f(x) = cx \)
- if a general \( f \) is considered: infinite dimensional problem, very challenging to obtain the explicit solutions.
- relies on the premium principle and the risk measure \( \rho \).
Empirical Approach: Motivation

The general reinsurance model can be formulated as:

\[
\begin{aligned}
\min_f & \quad \rho(X, f) \\
\text{s.t.} & \quad 0 \leq f(x) \leq x \text{ for all } x \geq 0, \\
& \quad \Pi(f(X)) \leq \pi.
\end{aligned}
\]

Often difficult to solve (due to infinite dimension and nonlinearity)

In practice, the distribution of the underlying risk \(X\) is estimated from the observed data \(\{x_1, \cdots, x_N\}\).

Empirical-based reinsurance model:
- exploits the observed data directly
Empirical-based Model: Formulation

- Collect samples \( x := (x_1, x_2, \cdots, x_N) \) corresponding to the underlying risk \( X \)

- Introduce decision vector \( f := (f_1, f_2, \cdots, f_N) \) where each \( f_i \) represents the reinsurance indemnification for the loss amount \( x_i, i = 1, 2, \cdots, N \).

- Define empirical-based estimates as:

\[
\rho(X, f) \quad \rightarrow \quad \hat{\rho}(x, f)
\]

\[
0 \leq f(x) \leq x, \quad \text{for all } x \geq 0 \quad \rightarrow \quad 0 \leq f_i \leq x_i, \quad i = 1, 2, \cdots, N,
\]

\[
\Pi(f) \leq \pi \quad \rightarrow \quad \hat{\Pi}(f) \leq \pi
\]
Empirical-based Model: Formulation (Cont’d)

- General reinsurance model

\[
\begin{align*}
\min_f \ & \ \rho(x, f) \\
\text{s.t.} \ & \ 0 \leq f(x) \leq x, \ \text{for all } x \geq 0, \\
& \Pi(f) \leq \pi.
\end{align*}
\]

- Empirical reinsurance model:

\[
\begin{align*}
\min_f \ & \ \widehat{\rho}(x, f) \\
\text{s.t.} \ & \ 0 \leq f_i \leq x_i, \ i = 1, 2, \cdots, N, \\
& \widehat{\Pi}(f) \leq \pi.
\end{align*}
\]

- Many empirical reinsurance model can be cast as Second-Order Conic (SOC) programming:

  - A wide class of optimization problems
  
  - Efficient softwares are available for solving SOC programming: e.g., CVX (Grant and Boyd, 2008)
Consider the following variance minimization model:
\[
\begin{aligned}
\min_f \quad & \text{Var}(T_f) = \text{Var}(X - f(X) + \Pi(f)) \\
\text{s.t.} \quad & 0 \leq f(x) \leq x, \quad \Pi(f) \leq \pi.
\end{aligned}
\]

Empirical version of the goal function:
\[
\hat{\text{Var}}(T_f) = \frac{1}{N-1} \sum_{i=1}^{N} \left[ (x_i - f_i) - (\bar{x} - \bar{f}) \right]^2,
\]
where $\bar{x}$ denotes the mean of $x$, and $\bar{f}$ denotes the mean of $f$.

Empirical version of the constraints:
\[
0 \leq f_i \leq x_i, \quad i = 1, 2, \cdots, N, \quad \text{and} \quad \hat{\Pi}(f) \leq \pi.
\]

Empirical variance minimization model:
\[
\begin{aligned}
\min_{f \in \mathbb{R}^N} \quad & \sum_{i=1}^{N} \left[ (x_i - f_i) - (\bar{x} - \bar{f}) \right]^2 \\
\text{s.t.} \quad & 0 \leq f_i \leq x_i, \quad i = 1, 2, \cdots, N. \\
& \hat{\Pi}(f) \leq \pi.
\end{aligned}
\]
P1. Expectation principle: $\Pi(f) = (1 + \theta)E[f]$ with $\theta > 0$.

$$\hat{\Pi}(f) \leq \pi \iff (1 + \theta)\bar{f} \leq \pi,$$

P2. Standard deviation principle: $\Pi(f) = E[f] + \beta \sqrt{\text{Var}[f]}$, where $\beta > 0$.

$$\hat{\Pi}(f) \leq \pi \iff \bar{f} + \frac{\beta}{\sqrt{N-1}} \left[ \sum_{i=1}^{N} (f_i - \bar{f})^2 \right]^{1/2} \leq \pi.$$
Empirical Model: CTE minimization

- Theoretical CTE minimization model:

\[
\begin{align*}
\min_f & \quad \text{CTE}_\alpha(T_f) \equiv \text{CTE}_\alpha \left( X - f(X) + \Pi[f(X)] \right) \\
\text{s.t.} & \quad 0 \leq f(x) \leq x, \quad \Pi[f(X)] \leq \pi,
\end{align*}
\]

- Technical model:

\[
\begin{align*}
\min_{(\xi, f)} & \quad G_\alpha(\xi, f) \equiv \xi + \frac{1}{\alpha} \mathbb{E} \left[ \left( X - f(X) + \Pi(f(X)) - \xi \right)^+ \right] \\
\text{s.t.} & \quad 0 \leq f(x) \leq x, \quad \Pi(f(X)) \leq \pi.
\end{align*}
\]

- We developed the following fact:

\((\xi^*, f^*) \in \arg \min_{(\xi, f)} G_\alpha(\xi, f)\) if and only if

\[f^* \in \arg \min_f \text{CTE}_\alpha(T_f), \quad \xi^* \in \arg \min_\xi G_\alpha(\xi, f^*).\]

- We proved that stop-loss is the optimal solution given that \(\Pi\) is the expectation principle.
Empirical model:

\[
\begin{align*}
\min_{(\xi, f)} \ G_\alpha(\xi, f) & \equiv \xi + \frac{1}{\alpha N} \sum_{i=1}^{N} \left[ \left( x_i - f_i + \hat{\Pi}(f) - \xi \right)^+ \right], \\
\text{s.t.} \quad 0 \leq f_i \leq x_i, \ i = 1, 2, \cdots, N, \ \hat{\Pi}(f) \leq \pi.
\end{align*}
\]

Empirical CTE minimization model:

\[
\begin{align*}
\min_{(\xi, f, z)} \ \xi + \frac{1}{\alpha N} \sum_{i=1}^{N} z_i, \\
\text{s.t.} \quad 0 \leq f_i \leq x_i, \ \hat{\Pi}(f) \leq \pi, \\
\quad z_i \geq 0, \ z_i \geq \hat{\Pi}(f) - f_i - \xi + x_i, \\
\quad i = 1, 2, \cdots, N.
\end{align*}
\]

The above empirical model can be cast as Second-order conic programming for as many as ten reinsurance premium principles.
Example: CTE minimization

- Use samples from an exponential loss distribution with mean 1000

- Reinsurance premium principle:
  - expectation principle with loading factor $\theta = 0.2$
  - standard deviation principle with loading factor $\beta = 0.2$

- Consider different levels of the reinsurance premium budget

- The solutions $f^*$ are illustrated by their scatter plots against sample $x$
1) $\pi = 80$

2) $\pi = 200$

3) $\pi = 400$

4) $\pi = 600$
5) \( \pi = 800 \)

6) \( \pi = 1000 \)

7) \( \pi = 1500 \)

8) \( \pi = 2000 \)
1) $\pi = 50$

2) $\pi = 80$

3) $\pi = 100$

4) $\pi = 120$

5) $\pi = 150$

6) $\pi = 200$
7) \( \pi = 400 \)

8) \( \pi = 600 \)

9) \( \pi = 800 \)

10) \( \pi = 1000 \)

11) \( \pi = 1500 \)

12) \( \pi = 1500 \)
Conclusion: Pros & Cons

Pros:

- empirical data based
- finite dimensional reinsurance models
- flexibility of the goal function and the reinsurance premium principle
- empirical solutions are consistent with the theoretical solutions
  - e.g., Variance and CTE minimization with expectation premium principle

Cons:

- empirical-based model will turn out to be a large scale programming when the sample size is extremely large
  - issues such as computational time and requirement for a substantially large computer’s memory will arise
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