Optimal investment strategies and intergenerational risk sharing for target benefit pension plans

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Outline

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Target Benefit Plans

Key features:

- Predefined contribution level
- Sponsor liability limited to contributions
- Target benefit level
- Actual benefits vary

- Collective asset pool
- Members bear risk collectively
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Practical objectives:

- Provide adequate benefits
- Maintain stability
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Given some starting asset value and contribution commitment, how should assets be invested and benefits be paid out to achieve these goals?
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Stochastic optimization in pension literature

- **DB optimization: asset mix and contribution rate**
  

- **DC optimization: asset mix and payout pattern**
  
  Gerrard et al. (2004), He and Liang (2013, 2015), etc.

- **Gollier (2008): asset mix, benefit payout, dividend policy**

- **Cui et al. (2011): asset mix, contribution rate, benefit payout**
Dynamics of financial market

- Risk-free asset $S_0(t)$
  \[ dS_0(t) = r_0 S_0(t)dt, \quad t \geq 0, \]
  where $r_0$ represents the risk-free interest rate.

- Risky asset $S_1(t)$
  \[ dS_1(t) = S_1(t)[\mu dt + \sigma dW(t)], \quad t \geq 0, \]
  where $\mu$ is the appreciation rate of the stock, $\sigma$ is the volatility rate, and $W(t)$ is a standard Brownian motion.
Plan membership

The fundamental elements of demographic model:

- $n(t)$: density of new entrants aged $a$ at time $t$,
- $s(x)$: survival function with $s(a) = 1$ and $a \leq x \leq \omega$.

The density of those who attain age $x$ at time $t$ is

$$n(t - (x - a))s(x), \quad x > a.$$
Dynamics of salary rate for a member who retires at time $t$:

$$dL(t) = L(t) \left( \alpha dt + \eta d\overline{W}(t) \right), \quad t \geq 0,$$

where $\alpha \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. $\overline{W}$ is a standard Brownian motion correlated with $W$, such that $E[W(t)\overline{W}(t)] = \rho t$.

For a retiree age $x$ at time $t$ ($x \geq r$), define assumed salary at retirement ($x - r$ years ago) as

$$\tilde{L}(x, t) = L(t)e^{-\alpha(x-r)}, \quad t \geq 0, \quad x \geq r.$$
The time-age structure of the pension plan
Contribution process

- Individual contribution rate for active member aged $x$ at time $t \geq 0$:

$$C(x, t) = c_0(x)e^{\alpha t}, \quad a \leq x < r.$$ 

- Aggregate contribution rate in respect of all active members at time $t$:

$$C(t) = \int_a^r n(t - x + a)s(x)C(x, t)dx, \quad t \geq 0.$$
Individual pension payment rate at time $t$:

For a new retiree aged $r$:

$$B(r, t) = f(t)L(t)$$

For an existing retiree aged $x > r$:

$$B(x, t) = f(t)\tilde{L}(x, t)e^{\zeta(x-r)}$$

$$= f(t)L(t)e^{-(\alpha-\zeta)(x-r)}$$
Benefit payment process

Individual pension payment rate at time $t$:

- for a new retiree aged $r$:
  \[ B(r, t) = f(t)L(t) \]

- for an existing retiree aged $x > r$:
  \[
  B(x, t) = f(t)\tilde{L}(x, t)e^{\zeta(x-r)} \\
  = f(t)L(t)e^{-(\alpha-\zeta)(x-r)}
  \]
Benefit payment process

- Aggregate pension benefit rate for all retirees at time $t$:

$$B(t) = \int_{r}^{\omega} n(t - x + a)s(x)B(x, t)dx = l(t)f(t)L(t), \quad t \geq 0.$$ 

- The updated aggregate target benefit is $B^* e^{\beta t}$. 
The pension fund dynamic can be described as

\[
\begin{align*}
\frac{dX(t)}{dt} &= \pi(t) \frac{dS_1(t)}{S_1(t)} + (X(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} + (C(t) - B(t)) dt, \\
X(0) &= x_0.
\end{align*}
\]
The objective function

Let $J(t, x, l)$ be the objective function at time $t$ with the fund value and the salary level being $x$ and $l$. It is defined as

$$
J(t, x, l) = E_{\pi, f} \left\{ \int_t^T \left[ (B(s) - B^* e^{\beta s})^2 - \lambda_1 (B(s) - B^* e^{\beta s}) \right] e^{-r_0 s} ds \right. $$

$$
\left. + \lambda_2 (X(T) - x_0 e^{r_0 T})^2 e^{-r_0 T} \right\},
$$

$$
J(T, x, l) = \lambda_2 (X(T) - x_0 e^{r_0 T})^2 e^{-r_0 T}.
$$

The value function is defined as

$$
\phi(t, x, l) := \min_{(\pi, f) \in \Pi} J(t, x, l), \quad t, x, l > 0.
$$

Using variational methods and Itô’s formula, we get the following HJB equation satisfied by the value function $\phi(t, x, l)$:

$$
\min_{\pi, f} \left\{ \phi_t + \left[ r_0 x + (\mu - r_0)\pi + C(t)e^{\alpha t} - fl \cdot l(t) \right] \phi_x + \alpha l \phi_l \\
+ \frac{1}{2} \pi^2 \sigma^2 \phi_{xx} + \frac{1}{2} \eta^2 l^2 \phi_{ll} + \rho \sigma \eta l \pi \phi_{xl} + \left[ (fl \cdot l(t) - B^* e^{\beta t})^2 \\
- \lambda_1 \left( fl \cdot l(t) - B^* e^{\beta t} \right) \right] e^{r_0 t} \right\} = 0.
$$
Solution to the optimization problem

Optimal strategies are

$$\pi^*(t, x, l) = -\frac{\delta}{2\sigma} [2x + Q(t)],$$

$$f^*(t, x, l) = \frac{1}{l \cdot I(t)} \left[ \frac{\lambda_1}{2} + \frac{\lambda_2}{2} (2x + Q(t)) P(t) + B^* e^{\beta t} \right],$$

where $\delta = (\mu - r_0)/\sigma$ is the Sharp Ratio.

The corresponding value function is given by

$$\phi(t, x, l) = \lambda_2 e^{-r_0 t} P(t)[x^2 + xQ(t)] + K(t).$$
\[ P(t) = \begin{cases} 
\frac{1}{\lambda_2(T-t)+1}, & r_0 = \delta^2, \\
\frac{r_0 - \delta^2}{\lambda_2 + (r_0 - \delta^2 - \lambda)e^{-(r_0 - \delta^2)(T-t)}}, & r_0 \neq \delta^2, 
\end{cases} \]

\[ Q(t) = \begin{cases} 
2e^{r_0 t} \left[ \int_t^T C_1(s)e^{(\alpha-r_0)s}ds - B^*(T - t) - x_0 \right], & \beta = r_0, \\
2e^{r_0 t} \left[ \int_t^T C_1(s)e^{(\alpha-r_0)s}ds - B^* \left( \frac{e^{(\beta-r_0)t}e^{(\beta-r_0)s}}{\beta-r_0} \right) - x_0 \right], & \beta \neq r_0, 
\end{cases} \]

\[ K(t) = \lambda_2 \int_t^T e^{-r_0 t} \left\{ P(s)Q(s) \left[ C_1(s)e^{\alpha s} - B^*e^{\beta s} 
- \frac{1}{4} (\delta^2 + \lambda_2 P(s))Q(s) \right] - \frac{\lambda_1^2}{4} \right\} ds. \]
Assumptions for numerical illustrations

- $a = 30$, $r = 65$, $\omega = 100$.
- Force of mortality follows Makeham’s Law.
- $n(t) = 10$ for all $t \geq 0$, implying a stationary population.
- $B^* = 100$, $\beta = 0.025$.
- Cost-of-living adjustment rate $\zeta = 0.02$.
- $r_0 = 0.01$, $\mu = 0.1$, $\sigma = 0.3$, $\Rightarrow \delta = 0.3$.
- $\alpha = 0.03$, $\eta = 0.01$; initial salary rate $L(0) = 1$.
- Correlation coefficient $\rho = 0.1$; $\lambda_1 = 15$, $\lambda_2 = 0.2$.
- $X(0) = 2500$, $c_0 = 0.1$.

See Dickson et al. (2013)
Numerical analysis

Percentiles of $\pi^*(t)/X^*(t)$ and $f^*(t)$

![Graphs showing percentiles of $\pi^*(t)/X^*(t)$ and $f^*(t)$ for $B^* = 100$. The graphs display the 95th, 75th, 50th, 25th, and 5th percentiles over time $t$ from 0 to 20.]
Numerical analysis

Sample paths of $f^*(t)$ and $B(t)$
Numerical analysis

Effects of asset returns

- Bar chart showing the relationship between asset returns and other variables.
- 3D scatter plot illustrating the effects of different parameters on asset returns.

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Numerical analysis

Effects of salary and target benefit growth rates

![3D surfaces representing numerical analysis of salary and target benefit growth rates.]
Medians of $f^*(t)$ for different values of $\lambda_1$ and $\lambda_2$
Conclusion

- We apply the Black-Scholes framework for plan assets, and consider a correlated salary process.

- We consider three key objectives of plan trustees (benefit adequacy, stability and intergenerational equity).

- We derive closed form expressions for optimal investments and payouts.

- The model is useful for identifying combinations of inputs that can meet stakeholders’ stated objectives.


Questions?