

# ENHANCING the UTILITY of COMPLEX TABLES IN ACTUARIAL TEACHING

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## ABSTRACT

The presentation studies the following utilities of tables. Tables i) provide speedier access to information, ii) function as a sort of *Alexa*, offering diagnostic assistance to readers and problem solvers, iii) dynamically facilitate problem creation, iv) provide an organizational technique for lengthy problem solutions by providing convenient pauses, roadside stops on the road to solution, v) are an easier alternative to complex abstract formula, vi) provide a metric for textbook assessment and vii) facilitate problem writing. The presentation also applies the prime implicant Boolean form (PIBF) to tables to mechanically expose underlying *intuition*. We believe these techniques can enhance the teaching, problem solving, and assessment experience.

### 1. INTRODUCTION– *Tables As Facilitating Access*

This paper is about use of the *table* as a vehicle of pedagogic delivery. *Tables* are familiar to most instructors. In this paper we summarize and make explicit half a dozen uses of tables. We begin in this section by showing, using a simple example from the K-12 educational literature, that the fundamental utility of tables is not provision of new knowledge but facilitation of access.

In Section 2, we show how mechanical tools from the theory of Boolean Algebras can be used to provide *intuition* to complex tables. In subsequent sections we show: i) *Tables* as an Alexa, helping and coaching students, ii) *Tables* as a creative problem tool. iii) *Tables* as road stops on a complex highway facilitating writing lengthy solutions to complex problems, iv) *Tables* as an easier alternative to complex formula, v) *Tables* as a textbook evaluation tool, and vi) *Tables* as an instructional tool, We hope this review of *Tables* from several points of view enriches the reader's understanding of this tool and will encourage them to go further in their application.

The following beautiful contrast is provided by Robinson and Kierra (1995. *Visual argument: Graphic organizers are superior to outlines in improving learning from text. Journal of Educational Psychology*, 87, 455-467), and is derived from material teaching schizophrenics. Consider four tiles, A,B,C,D. Figure 1 presents pictures of these tiles. Using an *outline bullet method*, the tiles are analyzed in Table 2 in three dimensions: shade, texture, and border.

**Figure 1:** *Four tiles, labeled A, B, C, D*



\*NOTE: The grid texture of Tile B may not appear in certain viewers.

**Figure 2:** Analysis of tiles by dimensions of shade, texture, and border

A:	Shade:	Light
	Texture:	Solid
	Border:	Single
B:	Shade:	Dark
	Texture:	Grid
	Border:	Single
C:	Shade:	Dark
	Texture:	Solid
	Border:	Thick
D:	Shade:	Dark
	Texture:	Solid
	Border:	Solid

Contrastively, Table 1, presents the *same information* in tabular form. The point of this study is that there is no difference in the *information* contained in Figure 2 versus Table 1; however, the speed of *access to the information* is greater in the Table. For example, it is instantaneous to see in Table 1 that A is the only light tile while B is the only tile with a grid texture.

**Table 1:** Tabular Presentation of Information on the Four Tiles

Tile	A	B	C	D
Shade of Grey	Light	Dark	Dark	Dark
Texture	Solid	Grid	Solid	Solid
Border	Single Line	Single Line	Thick	Thick

In summary: The purpose of *Tables* is not presenting new information, but rather, presenting information in a more accessible manner. The simultaneity of rows and columns allows multiple perspectives. Such multiple perspectives are not readily available in the outline version.

## 2. The Prime Implicant Normal Form (PINF) Provides *Intuition* To Complex Tables

In this section we study how tools from Boolean algebra can heighten the utility of tables. The example (which we supplement with the PINF approach) is taken from a standard Electrical Engineering (EE) textbook (P. Meenen and R. Adhami, 2005, **Fundamental Concepts in Electrical and Computer Engineering**, Universal Publishers). Section 7.7, “Logic Word Problems” presents the following question: *How do you program a front door electronic lock to a business store?* A response using three dimensions and Boolean Normal form is presented in Table 2.

**Table 2:** *Solution to the problem of programming a front door electronic lock to a business store*

Response of Electronic System	Issue Number 1	Issue # 2	Issue #3
Open	During Business hours	Weekday	Valid Code Provided
Open	During Business hours	Weekday	Code invalid
Open	During Business hours	Weekend	Valid Code Provided
Close	During Business hours	Weekend	Code invalid
Open	Not During Business hours	Weekday	Valid Code Provided
Close	Not During Business hours	Weekday	Code invalid
Open	Not During Business hours	Weekend	Valid Code Provided
Close	Not During Business hours	Weekend	Code invalid

The Table presents *information* but no *intuition*. It provides *information* since a complete requirements listing of all cases is provided. But there is no immediate intuition of why this solution is correct. The Boolean Prime Implicant Normal Form (PINF) can provide *intuition*. The PINF for the above table is presented in Figure 3.

**Figure 3:** *Prime Implicant Normal Form For the Boolean Normal Form of Table 2*

$\text{Open door lock} = \text{Valid Code Provided} \} \bigvee \{ \text{During Business hours \& Weekdays} \}$
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We first show that the PINF presented in Figure 3 provides intuition. We then explain the underlying characteristics of the PINF. Finally, we mention several well-known methods for obtaining a PINF.

Intuition: The PINF is intuitive. If a *valid code* is provided you want the door to open independent of whether it is a *weekday* or during *business hours*. Similarly you do want the door open if it is a *weekday* provided it is during *business hours*; you don't want the door open say at midnight when crime is likely. This is also intuitive. One reason the PINF is intuitive is that the PINF has only 3 Boolean variables: *Valid code*, *Business hours*, *Weekdays*, while Table 2 written in normal form has a conjunction of eight sets each of three variables resulting in a presence of 24 Boolean variables. However, the explanation of the definition of prime implicant will clarify further the intuitiveness of the PINF.

What is a prime implicant? A prime implicant has 2 definitional attributes. First the prime implicant *implies* the desired outcome. For example, if a *valid code* is submitted *then* the door lock will be open independent of the other parameters. Second and more importantly, a prime implicant has a minimal number of conjuncts. For example, *weekday* by itself does *not* imply that the door should be open. Contrastively, *weekday* and *business hours* do imply that the door should be open. The minimality of *weekday* and *business hours* is seen by the fact that if it is a *weekday* during *business hours* the lock is open whether or not a *valid code* has been presented.

The PINF is simply the disjunction of all *prime implicants*. As just seen each *prime implicant* has a minimal number of conjuncts and it is this minimality that provides *intuition* by not bringing in extraneous parameters.

PINF Methods: Based on the above, every table, particularly complex ones can be supplemented, perhaps even replaced by a PINF. The methods for producing PINF from a disjunctive normal form are well known and routinely taught in undergraduate logic courses; they are presented in both in books on logic and electrical engineering. Two popular methods for producing PINFS are i) Karnaugh maps (which work well if there are 4 or less parameters), ii) Quine-rule reduction methods (which work especially well on computers).

### 3. Tables as Alexa, Diagnostic Hinting For Students

Tables, when properly constructed, can serve as a diagnostic template facilitating helping students, similar to say Alexa in software. We illustrate using Question 4.6 in the LTAM SOA questions presented in Figure 4.

**Figure 4:** Question 4.6 from SOA LTAM questions

For a 3-year term insurance of 1000 on (70), you are given:

- (i)  $q_{70+k}^{\text{SULT}}$  is the mortality rate from the standard ultimate life table (SULT) for  $k=0,1,2$
- (ii)  $q_{70+k}$  is the mortality used to price this insurance
- (iii)  $q_{70+k} = (0.95)^k q_{70+k}^{\text{SULT}}$
- (iv)  $i = 5\%$

Calculate the single net premium.

Table 3 presents the solution to this problem using an *Alexa* type approach that provides hints to candidates that have mastered the template. The idea that the headers *suggest* hints to students, similar to the hints an instructor might offer during office-hours, should be obvious. Some elaborative comments will now be made on how these *Alexa* tables can be used.

**Table 3:** *Alexa* type table providing a solution to Question 4.6, SOA LTAM Questions

In which age period did the person die	70 – 71	71 - 72	72 - 73	After 73	Total
Express this as absolute time from 70	0- 1	1 – 2	2 - 3	3+	
When is payoff made upon death	1	2	3	3+	
What is payoff	1000	1000	1000	0	
What is discount of 1 to time 0	0.952380952	0.907029478	0.863837599	Varies	
What is the discounted payoff	952.3809524	907.0294785	863.8375985	0	
Which endpoint of year is used in $q_x$	0	1	2	Varies	
What is $q_x$ in SULT	0.010413	0.01167	0.013081	Varies	
What is the improvement factor	100%	95%	90.25%	Varies	
What is $q_x$ for this insurance	0.010413	0.0110865	0.011805603	Varies	
What is $p_x$	0.989587	0.9889135	0.988194398	Varies	
What is the probability of death in the period	0.010413	0.010971056	0.011553151	0.967062793	100%
Calculate the weighted discounted payoff	9.917142857	9.951071452	9.98004607		29.85

Of course, for any one problem, you can make a table that gives you the right answer. That is not the goal. The goal is to provide a table template that facilitates the student receiving hints from the row headers to solve a wide variety of problems. It is in this sense that the table functions as an *Alexa*; it provides a set of hints that is applicable to solving many similar problems.

The table has various additional subtleties. One blatant example is inclusion of death after 73. This affords the student the ability to see a collection of mutually exclusive events whose total probability is 1 as shown in the check-total in the next to last row.

#### 4. **Problem Writing:** *The Table as a Dynamic Problem Creator*

Consider by way of example the nine money growth functions in the SOA FM syllabus (simple interest, compound interest, nominal interest, simple discount, compound discount, nominal discount, constant force, force, general accumulation functions). Consider the two problems presented in Figure 4.

**Figure 5:** *Two problems on SOA FM Money growth functions with solutions and commentary*

**Problem 1:** Compute using money growth method X, the present value (PV) (fair price) of an increasing annuity immediate paying 1000 and 2000 at  $t=1$  and  $t=2$  respectively

**Solution 1:** Let  $v_X(t)$  be the present value at  $t=0$  of a payment of 1 at time  $t$  under money-growth method X. Then

$$(1) \quad PV = 1000 v_X(1) + 2000 v_X(2)$$

**Problem 2:** Compute using money growth method X, the current value at  $t=1$  (CV(1)) of an increasing annuity immediate paying 1000 and 2000 at  $t=2$  and  $t=3$  respectively

**Solution 2:** Here is the subtlety. Let  $v_X(t)$  be defined as in Problem 1. Then if X is compound money growth then the solution is

$$(2) \quad CV(1) = 1000 v_X(1) + 2000 v_X(2).$$

However if X represents say a random accumulation function (say quadratic or square root) then the solution as given by (1) is not correct while the following gives the correct solution

$$(3) \quad v_X(1) CV(1) = 1000 v_X(2) + 2000 v_X(3)$$

The point that Figure 5 makes is that for certain problems like Problem 1, there is no *distinction* between money growth methods. That is the same functional format, as given in (1), solves all equations. Contrastively, problem 2 *differentiates* problems: For some money growth methods *either* (2) or (3) will solve the problem while for other money growth methods only (3) will solve

the problem. Readers who have taught FM might recognize this difference as an *error* that weaker students make in applying (2) to solve problems like problem 2.

Using the language of tables, we can summarize as follows:

Under problem 1, the nine money growth methods are a *list*, while under problem 2, the nine money growth methods become a table since the need for absolute time versus relative time *differentiates* the money growth method. This means, that if you were to create a table whose rows were money growth methods you would need at least two columns to *differentiate* the rows: One column might give the money growth formula while the second row might indicate if absolute or relative time is needed or sufficient in solutions.

In this precise sense, Problem 2 in Figure 5 points to a need to *create* a table rather than just suffice with a list. These problems which *differentiate* lists are important pedagogically. Thus, we have another important aspect of tables.

**5. Roadside Stops: The Table as a Roadside Stop, providing convenient pauses**

Consider by way of example the problem of graphing  $P(x) = x^4 - 2x^3 - x^2 + 2x$ . Calculus books are quick to tell instructors and students about *how* to graph: the first derivative test, the second derivative test etc. But I have not found any books discussing solution organization. Solution organization of lengthy solutions is a critical skill, needed as much as other critical skills by mathematicians and actuaries. It is sometimes just more convenient to ask students during assessment for snapshots: e.g. *Graph the region of  $P(x)$  which is increasing but concave downward.*

I have found tables useful as an organizational solution tool. Think of the solution as a road or highway. Every highway needs road stops, places where you can rest, get a snack, fill up on gas, or just pause during a long trip. Skillful use of tables allows a problem solver to organize a lengthy solution with decorative road stops; each table naturally suggests a place to pause and take a breather.

Besides a solution presenting a final labeled graph of  $P(x)$ , the following four tables are a complete solution and also provide convenient road stops along the road of solution

**Table 4a: Functions, Derivatives, and Roots**

Function & Derivatives	Formula	Roots
$P(x)$	$x^4 - 2x^3 - x^2 + 2x$	-1, 0, 1, 2
$P'(x)$	$4x^3 - 6x^2 - 2x + 2$	-.62, 0.5, 1.62
$P''(x)$	$12x^2 - 12x - 2$	-0.1455, 1.1455

**Table 4b:** Signs of  $P'$  and  $P''$  in Six Regions

Region	$(-\infty, -0.62)$	$(-0.62, -0.1455)$	$(-0.1455, 0.5)$	$(0.5, 1.1455)$	$(1.1455, 1.62)$	$(1.62, \infty)$
Sign of $P'$	-	+	+	-	-	+
Sign of $P''$	+	+	-	-	+	+

**Table 4c:** Calculus Results: Graph Components Using Signs of  $P'$  and  $P''$  \*

Sign of $P'$	Sign of $P''$	Description of Graph Component	Illustrative example using clock notation
+	+	Increasing, concave upward (holding water)	Position 6 to position 3
+	-	Increasing, concave downward (Spilling water)	Position of 9 to position 12
-	+	Decreasing concave upward (Holding water)	Position 9 to position 6
-	-	Decreasing concave downward (spilling water)	Position 12 to position 3

\*The imagery of holding and spilling water for concave upward and downward was first suggested by G. Thomas a noted calculus book author. The use of clock positions to illustrate graphical components should be self-explanatory: Thus tracing position 9 to position 6 from left to right gives you a decreasing curve that is concave upwards

**Table 4d:** Application of Calculus Theory (Table 4c) To the Function  $P$  (Table 4b)

Region	$(-\infty, -0.62)$	$(-0.62, -0.1455)$	$(-0.1455, 0.5)$	$(0.5, 1.1455)$	$(1.1455, 1.62)$	$(1.62, \infty)$
Sign of $P'$	-	+	+	-	-	+
Sign of $P''$	+	+	-	-	+	+
Graphical component	Position 9 to 6	Position 6 to 3	Position 9 to 12	Position 12 to 3	Position 9 to 6	Position 6 to 3

These four tables, possibly interspersed with accompanying narrative can compactly take up a page, provide complete detail, distinguish calculus theory and application, and yet, give the solution reader some breathing moments to catch up with the flow of thought. The idea of *table* as *road stops* is very useful in organizing lengthy solutions; while this example only needed 4 tables, some problems can naturally use up to a dozen tables. The tables just make the trip to solution more enjoyable.

## 6. The Table as an Easier Alternative to Complex Formula

Figure 6 presents The following very charming example which comes from the SOA published questions and solutions to LTAM.

**Figure 6:** *Problem 2.2 of LTAM SOA Questions*

Scientists are searching for a vaccine for a disease. You are given:

- (i) 100,000 lives age  $x$  are exposed to the disease.
- (ii) Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1.
- (iii) The probability that the vaccine will be available is 0.2.
- (iv) For each life during year 1,  $q_x = 0.02$ .
- (v) For each life during year 2,  $q_{x+1} = 0.01$  if the vaccine is given;  $q_{x+1} = 0.02$  if the vaccine is not given.

Calculate the standard deviation of the number of survivors at the end of year 2.

The problem is challenging since it *appears* to be a mixture problem but is not. Unlike a *smoker – non-smoker* problem where a cohort consists of two mutually exclusive *groups*, this problem deals with two possible *worlds* the world with a vaccine versus the world without a *vaccine*. Hence, different techniques must be employed.

A *formula* approach to a solution uses the conditional variance formula:

$$\text{Var}(S) = \text{Var}[E(S|A)] + E([\text{Var}(S|A)]).$$

My experience is that most students, including good ones, have difficulty with this formula. The SOA presents a *table* approach to the solution reproduced in Table 5.

**Table 5:** *Alternative Table Solution to Problem 2.2 in LTAM SOA problems.*

Availability of Vaccine=A	Pr (A)	$2p = p$	$E(S=\text{Survivors})$ from $n=100,000$ : $E(S) = n p$	$Var(S) = n p (1-p)$	$E(S^2) = E(S)^2 + Var(S)^*$
A is True	20%	$(1-2\%)(1-1\%)$	97,020	2,891	9,412,883,291
A is False	80%	$(1-2\%)(1-2\%)$	96,040	3,803	9,223,685,403
Weighted Average			96,236		9,261,524,981
$Var(S) = E(S^2) - E(S)^2$					$157,285 = 397^2$

\*NOTE: For binomial distribution, the second moment is preferably computed as the sum of variance and expectation squared since there is no nice formula for the binomial 2<sup>nd</sup> moment.

## 7. Textbook Assessment

Routinely when I review a textbook, one metric I use, is the table-figure density. Take this paper for example. The first 10 pages have 9 tables and 6 figures giving a table to page ratio of 1.5. Such a ratio is one of many metrics that can indicate that a book is good. A high ratio indicates that the material is dense requiring tables to summarize and navigate the complexity. Contrastively, if the density is low, either the book is shallow, not presenting complex material requiring tables, or, the book is rich but ignores the need to summarize for the reader using tables.

## 8. Problem Writing

Frequently a table will underly an instructional unit. Preferably the table structure should be made explicit during instruction. Students are said to have mastery of the table if they can do all the following:

- Subtables: *Compare and Contrast Specified Rows by Specific Column Attributes*
- Single Column Subtable: *Compare rows by the attribute of a single column*
- Row specific, Open Ended: *Give the students two row names and ask them to compare*
- Fully Open ended: *Ask the students to explain key attributes (columns) as pros or cons*
- Explicit Table fill in: *Give the student a snapshot and ask them to fill in with yes or no*

To illustrate these techniques we have selected the topic of *interest rate models* from the Quantitative Finance syllabus. This *topic* may be *taught* using a table structure whose rows are various interest rate models and whose columns represent attributes of the interest rate models. Because of the complexity of this table, about half a dozen models and about half a dozen to a dozen important attributes, the SOA exams have utilized all the techniques in the above bulleted list. Figures 7A through 7D and Table 6E illustrate these techniques are instructive to show the varied issues of mastery in an instructional unit with an underlying table

**Figure 7A: S19-Q5, QFIQF: Subtable: Compare /Contrast Specified Rows by specified columns**

Compare and contrast the following properties under the Hull White and Black-Karasinski models.

- (i) Arbitrage vs. non-arbitrage model
- (ii) Distribution of  $r$
- (iii) Possible values of  $r$
- (iv) Analytical Bond Price

**Figure 7B: F13-10, QFIADV: Single Column Comparison of Rows**

You are considering the following models to price a European swaption, in a country where negative interest rates are possible.

- (i) Vasicek Model
- (ii) Cox, Ingersoll and Ross Model

**Figure 7C:S18-2 QFIADV: Row specific, column open ended**

Describe the advantages of the two-factor G2++ model versus the one factor Model in this situation.

Describe the advantages of using the two-factor G2++ model over the CIR2++ model when the Brownian motions within each model are correlated.

**Figure 7D: F13-2 QFIADV: Fully open ended – Description or recall of column attributes**

Describe the advantages of working with affine term structure

**Table 6E: S15-Q11 QFIADV: Explicit Table: Fill in Cells below with Yes, No, NA**

Model Feature or implications	Model A	Model B	Model C	Model D
The model is Time homogeneous				
The model has an Affine term structure				
Short rate of model always positive				
Short rate of model is normally distributed				
Model has a Closed form formula for zero coupon bond				
Closed form formula for European options on a zero-coupon bond				

## 9. Conclusion

This paper has presented several known and new uses of tables that can significantly enhance the pedagogic experience. We have covered the *table* as providing speedier access, the *table* as *Alexa*, the *table* as a roadside stop, the *table* as an easier alternative to complex formulae, the *table* as a textbook assessment vehicle, and the *table* as a vehicle for writing problems.

It is hoped that these ideas will prove useful to other instructors in their pedagogic endeavors.