Integration by Differentiation

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Abstract

This pedagogical note presents several techniques for evaluating integrals: the exponential shift, the formulas $\sin(x) = \mathrm{Im}\{\exp(ix)\}$ and $\cos(x) = \mathrm{Re}\{\exp(ix)\}$, and the transformation of an integral into a differential equation. These methods are applied to show that a moment sequence need not uniquely determine its distribution and to give a heuristic proof of the inversion formula for the characteristic function.

1. Introduction

Many students find the method of integration by parts tedious. However, if the integrand has a factor which is an exponential function, the integral may be obtained by means of differentiation; most students make fewer mistakes in differentiation than integration. Also, if the sine or cosine functions appear in the integrand, applying Euler's formula

$$e^{ix} = \cos x + i \sin x \tag{1}$$

may lead to simplification. In (1), x is a real number and $i = \sqrt{(-1)}$. This paper gives some interesting applications of these techniques.

2. Exponential Shift

Let D denote the differentiation operator,

$$D = \frac{d}{dx} .$$

By the product rule

$$D[e^{\alpha x}f(x)] = \alpha e^{\alpha x}f(x) + e^{\alpha x}Df(x) = e^{\alpha x}(\alpha + D)f(x). \tag{2}$$

The operator D^{-1} may be called the *indefinite-integral* operator. It follows from (2) that

$$D^{-1}[e^{\alpha x}f(x)] = e^{\alpha x}(\alpha + D)^{-1}f(x); \tag{3}$$

for details see Agnew (1960, Sec. 6.8), Ayres (1952, Chap. 16), Bateman (1918, Sec. 12), Brand (1966, Sec. 36) or Friedman (1969, Sec. 6.1).

If n is a nonnegative integer, then

$$\begin{aligned} D^{-1}(e^{-\alpha x}x^n) &= e^{-\alpha x}(-\alpha + D)^{-1}x^n \\ &= e^{-\alpha x} \frac{1}{(-\alpha)\,(1 - \frac{D}{\alpha})} x^n \\ &= -e^{-\alpha x} \sum_{j=0}^{\infty} \frac{D^j x^n}{\alpha^{j+1}} \\ &= -e^{-\alpha x}(x^n/\alpha + nx^{n-1}/\alpha^2 + \ldots + n!/\alpha^{n+1}). \end{aligned}$$

Evaluating the integral

$$\int_a^b e^{-\alpha x} x^n \, dx$$

by the formula

$$\int_a^b g(x)dx = D^{-1}g(x) \Big|_a^b \tag{4}$$

yields the same answer as integration by parts. Thus, we have the following relation between the incomplete gamma and the Poisson

$$\int_0^x \frac{\beta^n y^{n-1} e^{-\beta y}}{(n-1)!}\,dy = 1 - \sum_{j=0}^{n-1} \frac{e^{-\beta x}(\beta x)^j}{j!},$$

a special case of which is formula (11.3.25) of Bowers, *et al.* (1986). Another interesting proof of the last formula can be found in Chao (1982).

Note that, if α is a number with positive real part, then

$$\int_0^\infty e^{-\alpha x} x^n\,dx = \frac{n!}{\alpha^{n+1}}. \tag{5}$$

3. Moment Problem

We shall show that, for each nonnegative integer m, the integral

$$\int_0^\infty u^m \exp(-u^{\frac{1}{4}}) \sin(u^{\frac{1}{4}})\,du \tag{6}$$

is zero (Widder, 1946, p. 126; Feller, 1966, p. 224; Neuts, 1973, p. 252). Thus, the family of probability density functions

$$f_\lambda(u) = [1 + \lambda \sin(u^{1/4})] \exp(-u^{1/4})/24, \qquad u \geq 0, \quad \lambda \in [-1, 1],$$

constitutes an example which shows that a moment sequence need not uniquely determine its distribution. This example was given by T.J. Stieltjes in 1894.

Denote expression (6) by I. By a change of variable and equation (1),

$$I = 4\int_0^\infty e^{-x} x^{4m+3} \sin(x)dx$$

$$= 4\,\mathrm{Im}\left\{\int_0^\infty e^{-(1-i)x} x^{4m+3}\,dx\right\}.$$

It follows from (5) that

$$\begin{aligned} I &= 4 \,\mathrm{Im} \left\{ \frac{(4m+3)!}{(1-i)^{4(m+1)}} \right\} \\ &= 4 \,\mathrm{Im} \left\{ \frac{(4m+3)!}{(-4)^{m+1}} \right\} \\ &= 0. \end{aligned}$$

Remarks

(i) It follows from the Weierstrass approximation theorem that a distribution which is concentrated on some finite interval is uniquely determined by its moments. It had been proved by F. Hausdorff (Feller, 1966, p. 223, Theorem 2; 1971, p. 225, Theorem 1) that a sequence of numbers $\mu_0, \mu_1, \mu_2, \ldots$ represents the moments of some probability distribution concentrated on [0, 1] if and only if $\mu_0 = 1$ and for m, n = 0, 1, 2, ...

$$(-1)^{m-n} \Delta^{m-n} \mu_n \geq 0 .$$

(ii) Feller (1971, p. 227) shows that the log-normal distribution is not determined by its moments.

4. Inversion Formula

As pointed out by Kotlarski (1975), most books in probability theory no longer use moment generating functions, but instead use characteristic functions. A rigorous proof of the inversion formula for the characteristic function involves mathematics at a level more advanced than that normally assumed in an introductory probability or risk theory course. Books such as Hoel, Port and Stone (1971) and Neuts (1973) simply state the inversion formula without proof. We now give a heuristic proof of the inversion formula.

Let f be a continuous probability density function with characteristic function ϕ,

$$\phi(t) = \int_{-\infty}^{\infty} e^{ity} f(y)\, dy. \tag{7}$$

If ϕ is absolutely integrable, then the integral

$$\int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt$$

exists and the following inversion formula holds:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt. \tag{8}$$

Substituting (7) into (8) yields

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left(\int_{-\infty}^{\infty} e^{ity} f(y)dy\right)dt. \tag{9}$$

Interchanging the order of integration in (9), we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-it(x-y)}dt\right) f(y)dy.$$

To verify the inversion formula (8) we shall show that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itw}\, dt \tag{10}$$

is the Dirac delta-function.

Define

$$h(w) = \int_{-\infty}^{\infty} \frac{e^{-itw}}{-it}\, dt;$$

then the derivative of $\frac{h(w)}{2\pi}$ is (10). Let

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx ;$$

then

$$h(w) = \begin{cases} J/i & w > 0 \\ -J/i & w < 0 \end{cases}$$

We shall prove that expression (10) is the Dirac delta-function by showing that the step function $h(w)$ has a jump of height 2π at $w = 0$, i.e.,

$$J = \pi i. \tag{11}$$

Equation (11) can be proved by contour integration (Friedman, 1969, p. 18); however, the following method (Agnew, 1960, p. 362; Apostol, 1974, p. 285; Parzen, 1960, p. 411) is more elementary.

Since the cosine is an even function and the sine an odd function, by equation (1)

$$\begin{aligned} J &= i \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \\ &= 2i \int_{0}^{\infty} \frac{\sin x}{x}\, dx . \end{aligned}$$

For $\alpha \geq 0$, define

$$w(\alpha) = \int_{0}^{\infty} \frac{\sin x}{x}\, e^{-\alpha x}\, dx .$$

For $\alpha > 0$,

$$\begin{aligned} w'(\alpha) &= - \int_{0}^{\infty} \sin x\, e^{-\alpha x}\, dx \\ &= - \operatorname{Im} \left\{ \int_{0}^{\infty} e^{ix}\, e^{-\alpha x}\, dx \right\} \end{aligned}$$

$$= -\mathrm{Im}\{(\alpha - i)^{-1}\}$$

$$= -(1 + \alpha^2)^{-1}, \tag{12}$$

which implies that

$$w(\alpha) = k - \arctan(\alpha).$$

Since $w(\alpha) \to 0$ as $\alpha \to \infty$, the constant k is $\pi/2$. Hence,

$$J = 2i\, w(0)$$

$$= 2i\, w(0+)$$

$$= 2i\, (\pi/2 - \arctan(0+))$$

$$= \pi i,$$

proving equation (11). The step $w(0) = w(0+)$ can be justified by the theory of uniform convergence; cf. Apostol (1974, p. 286).

Remarks

(i) A rigorous proof of the inversion formula can be found in Rudin (1974, Chapter 9).

(ii) There is an interesting way to derive (12). By equation (3)

$$D^{-1}(e^{-\alpha x} \sin x) = e^{-\alpha x} (-\alpha + D)^{-1} \sin x.$$

Since

$$D^2 \sin(bx) = -b^2 \sin(bx),$$

we have

$$(\alpha + D^2)^{-1} \sin(bx) = (\alpha - b^2)^{-1} \sin(bx),$$

provided $\alpha \neq b^2$ (Ayres, 1952, p. 99; Brand, 1966, p. 145). Thus,

$$\frac{1}{-\alpha + D} \sin x = \frac{\alpha + D}{-\alpha^2 + D^2} \sin x$$

$$= \frac{\alpha + D}{-\alpha^2 - 1^2} \sin x$$

$$= -(\alpha \sin x + \cos x) / (1 + \alpha^2).$$

By formula (4)

$$\int_0^\infty e^{-\alpha x} \sin x \, dx = - \frac{e^{-\alpha x} (\alpha \sin x + \cos x)}{1 + \alpha^2} \Big|_0^\infty$$

$$= (1 + \alpha^2)^{-1}.$$

(iii) A main result in Chapter 12 of Bowers *et al.* (1986) is the formula

$$\int_0^\infty e^{ru} [-\psi'(u)] \, du = \frac{\theta}{1 + \theta} \frac{M_X(r) - 1}{1 + (1 + \theta)p_1 r - M_X(r)}. \tag{13}$$

Denote the right-hand side in (13) by g(r). Integrating by parts yields

$$\int_0^\infty e^{ru} \psi(u) \, du = \frac{-\psi(0) + g(r)}{r}. \tag{14}$$

We wish to invert (14) and obtain an expression for the probability of eventual ruin $\psi(u)$.

Compare (14) with (7) and consider $x = u$, $r = it$, $f = \psi$ and $\phi(t) = [-\psi(0) + g(it)]/(it)$. It follows from the inversion formula (8) that

$$\psi(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{-\psi(0) + g(r)}{r} e^{-ru} \, dr. \tag{15}$$

Observe that the point $r = \theta$ is a removable singularity for the integrand in (15) since

$$\lim_{r \to 0} g(r) = \frac{1}{1 + \theta} = \psi(0).$$

Let R denote the adjustment coefficient. For each real number a, $a < R$, it can be shown that

$$\psi(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{-\psi(0) + g(r)}{r} e^{-ru} \, dr. \tag{16}$$

For u > 0 and b > 0,

$$\int_{b-i\infty}^{b+i\infty} \frac{e^{-ru}}{r} \, dr = 0.$$

Hence, (16) can be simplified and we have the following integral expression for the probability of eventual ruin $\psi(u)$, $u > 0$,

$$\psi(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{g(r)}{r} e^{-ru} \, dr, \qquad 0 < c < R,$$

which is identical to formula (118) of Cramér (1955).

5. The Poisson Process

Equation (3) can be applied to solve the system of differential-difference equations which arises in the derivation of the Poisson process (Hogg and Craig, 1978, Sec. 3.2; Parzen, 1960, Sec. 6.5; Ross, 1983, Sec. 2.1): For $n = 1, 2, 3, \ldots$,

$$(D + \lambda)\, p_n(x) = \lambda\, p_{n-1}(x), \tag{17}$$

$$p_n(0) = 0, \tag{18}$$

and

$$p_0(x) = e^{-\lambda x}. \tag{19}$$

By (17) and (19)

$$\begin{aligned} p_n(x) &= \lambda^n (D + \lambda)^{-n} p_0(x) \\ &= \lambda^n (D + \lambda)^{-n} e^{-\lambda x}. \end{aligned}$$

Applying (3) repeatedly yields

$$\begin{aligned} p_n(x) &= \lambda^n e^{-\lambda x} (D - \lambda + \lambda)^{-n} 1 \\ &= \lambda^n e^{-\lambda x} D^{-n} 1. \end{aligned}$$

By (18)

$$p_n(x) = \lambda^n e^{-\lambda x} x^n / n!$$

$$= (\lambda x)^n e^{-\lambda x} / n! .$$

6. The Laguerre Polynomials

The Laguerre orthogonal polynomials have been applied to approximate the distribution of aggregate claims (Bowers, 1966; Gerber, 1980, Sec. 4.4) and the ruin function (Pfenninger, 1974; Seal, 1975; Taylor, 1977). Apart from a normalization factor, a Laguerre polynomial of degree n is

$$x^{-\alpha} e^{x} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}) . \tag{20}$$

It is not immediately clear that (20) is a polynomial. However, by repeated applications of equation (2), (20) becomes

$$x^{-\alpha} e^{x} e^{-x} (D - 1)^n x^{n+\alpha}$$

$$= x^{-\alpha} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} D^j x^{n+\alpha} ,$$

which is obviously a polynomial of degree n.

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