Limiting tail behaviour of some discrete compound distributions

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Abstract

Asymptotic formulae are derived for some discrete compound distributions which have been found to be useful in insurance claims modelling. These formulae provide insight into the distributional form, and often complement recursive computational algorithms which are cumbersome in the right tail of the distribution. Similar formulae are derived for the tail and the stop-loss premium. Some practical aspects of the use of these formulae are discussed.

Keywords: Ruohonen's distribution, stop-loss premium.

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1. Introduction

Recently, asymptotic formulae have been derived for compound distributions when the number of claims distribution is of a particular type. Sundt (1982), for example, considered the compound geometric distribution, thus generalizing Cramer's asymptotic ruin formula. These results were in turn generalized by Milidiu (1985) (see also Jewell and Milidiu, 1986) to the compound negative binomial, and by Embrechts, Maejima, and Teugels (1985) to a general family of compound distributions in the case of a non-arithmetic claim size distribution. In this paper the claim size distribution will be assumed to be arithmetic.

Relevant background information and notation is provided in section 2, and in section 3 asymptotic formulae are derived for the probability function, the tail, and the stop-los premium of the compound distribution. These results apply with several number of claims distributions which have been found to be of use in fitting insurance claim count data, and some of these examples are given in section 4. Finally, in section 5, some practical aspects of the use of these formulae are discussed, as well as some comments about their accuracy.

2. Background and Notation

The distribution of total claims with probability generating function (pgf)

$$G(z) = \sum_{n=0}^{\infty} g_n z^n = P\{F(z)\} \tag{1}$$

is of interest, where $P(z) = \sum_{n=0}^{\infty} p_n z^n$ and $F(z) = \sum_{n=0}^{\infty} f_n z^n$ are the number of claims and claim size pgf's respectively. Specifically, the limiting behaviour of g_n as $n \to \infty$ will be considered. The notation $a(n) \sim b(n)$, $n \to \infty$, will be taken to mean

$\lim_{n \to \infty} a(n)/b(n) = 1$. The claim size distribution $\{f_n;\ n = 0,1,2,\ldots\}$ is assumed to be such that the greatest common divisor of the set of values of n with $f_n>0$ is 1. If not, a new choice of monetary units will achieve this.

The class of number of claims distributions considered is that for which

$$p_n \sim Cn^{\alpha}\theta^n, \quad n \to \infty, \tag{2}$$

where $C>0$, $-\infty<\alpha<\infty$, and $0<\theta<1$. This class is quite large, and includes the extended truncated negative binomial (Willmot, 1988), the Poisson-negative binomial convolution of Ruohonen (1988), the Poisson-inverse Gaussian (Willmot, 1987), the generalized Poisson-Pascal (Kestemont and Paris, 1985), and Sichel's Poisson mixture by the generalized inverse Gaussian (Sichel, 1971), among others. See Teugels and Willmot (1987) for further examples.

The important special case of the negative binomial is now stated.

Lemma 1 (Milidiu, 1985)

For the compound negative binomial distribution, (1) becomes

$$G(z) = \{1-\beta[F(z)-1]\}^{-\alpha} \tag{3}$$

with $\alpha>0$, $\beta>0$. In this case

$$g_n \sim \frac{n^{\alpha-1}\tau^{-n}}{\Gamma(\alpha)\{\beta\tau F'(\tau)\}^{\alpha}}, \quad n \to \infty, \tag{4}$$

where $\tau>1$ satisfies $F(\tau) = 1+\beta^{-1}$.

The special case with $\alpha = 1$ was found by Sundt (1982).

Lemma 1 will be used later on, as well as the following results. It is assumed here that $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, and $C(z) = \sum_{n=0}^{\infty} c_n z^n$.

Lemma 2 (Bender, 1974)

Suppose that $b_n \sim \phi b_{n+1}$, $n \to \infty$, where $\phi>0$ and $C(z)$ has radius of convergence exceeding ϕ. Then if $A(z) = B(z)C(z)$, it follows that $a_n \sim C(\phi)b_n$, $n \to \infty$.

Lemma 3 (Meir and Moon, 1987)

Suppose that $c_n \sim Cn^{\phi}\theta^{-n}$, $n \to \infty$, where $C>0$, $\phi<-1$, and $\theta>0$. Then if $B(z)$ has radius of convergence exceeding $C(\theta)$ and $A(z) = B\{C(z)\}$, it follows that $a_n \sim CB'\{C(\theta)\}n^{\phi}\theta^{-n}$, $n \to \infty$.

It is useful to note that the radius of convergence of $B(z)$ is ϕ in lemma 2, as follows easily from the ratio test. For a result related to lemma 3, see Embrechts and Hawkes (1982).

3. A general asymptotic result

The following shows that the compound distribution $\{g_n\}$ satisfies (2) quite generally if $\{p_n\}$ does.

Theorem

If (1) and (2) hold, and $F(z)$ has radius of convergence exceeding τ where $\tau>1$ satisfies

$$F(\tau) = \theta^{-1}, \tag{5}$$

then

$$g_n \sim \frac{Cn^{\alpha}\tau^{-n}}{\{\theta\tau F'(\tau)\}^{\alpha+1}}, \quad n \to \infty. \tag{6}$$

Proof: First assume that $\alpha \geq 0$ and consider the negative binomial pgf

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \{\frac{1-\theta}{1-\theta z}\}^{\alpha+1}.$$

Then from lemma 1 it follows that

$$a_n \sim \frac{(1-\theta)^{\alpha+1}}{\Gamma(\alpha+1)} n^{\alpha}\theta^{n}, \quad n \to \infty, \tag{7}$$

and that

$$b_n = \sum_{k=1}^{\infty} a_k f_n^{*k} \sim \frac{1}{\Gamma(\alpha+1)} \{\frac{1-\theta}{\theta\tau F'(\tau)}\}^{\alpha+1} n^{\alpha} \tau^{-n}, \quad n \to \infty. \tag{8}$$

In (8), b_n is the coefficient of z^n in $A\{F(z)\}$ and f_n^{*k} the coefficient of z^n in $\{F(z)\}^k$. Then

$$S_n = \left| \frac{g_n}{b_n} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} p_k f_n^{*k}}{\sum_{k=1}^{\infty} a_k f_n^{*k}} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} \right|$$

$$= \frac{\left| \sum_{k=1}^{\infty} a_k f_n^{*k} \{\frac{p_k}{a_k} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}}\} \right|}{\sum_{k=1}^{\infty} a_k f_n^{*k}}$$

$$\leq \frac{\sum_{k=1}^{\infty} a_k f_n^{*k} \left| \frac{p_k}{a_k} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} \right|}{\sum_{k=1}^{\infty} a_k f_n^{*k}}.$$

By (2) and (7), there exists k_0 such that $\left| \frac{p_k}{a_k} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} \right| < \frac{\epsilon}{2}$ for any $\epsilon > 0$ and $k > k_0$. Thus

$$S_n < \frac{\sum_{k=1}^{k_0} a_k f_n^{*k} \left| \frac{p_k}{a_k} - \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} \right| + \frac{\epsilon}{2} \sum_{k=k_0+1}^{\infty} a_k f_n^{*k}}{\sum_{k=1}^{\infty} a_k f_n^{*k}}$$

$$\leq \frac{\epsilon}{2} + \frac{\sum_{k=1}^{k_0} \{p_k + \frac{C\Gamma(\alpha+1)}{(1-\theta)^{\alpha+1}} a_k\}\{f_n^{*k} \tau^n\}}{b_n \tau^n}.$$

But from (5), $\{F(\tau)\}^k = \sum_{n=0}^{\infty} f_n^{*k} \tau^n = \theta^{-k} < \infty$, implying that $f_n^{*k}\tau^n \to 0$ as $n \to \infty$. Also, from (8),

$b_n\tau^n \sim \frac{n^\alpha}{\Gamma(\alpha+1)}\{\frac{1-\theta}{\theta\tau F'(\tau)}\}^{\alpha+1} = Mn^\alpha$, $n \to \infty$. Thus, there exists N such that for $n>N$ one has $b_n\tau^n > M/2$ since $\alpha \geq 0$, and $f_n^{*k}\tau^n < \epsilon_k = \epsilon M/4k_0\{p_k + C\Gamma(\alpha+1)a_k/(1-\theta)^{\alpha+1}\}$ for $k = 1,2,3,\ldots,k_0$. With this choice of N, it follows that for $n>N$ one has

$$S_n < \frac{\epsilon}{2} + \frac{\sum_{k=1}^{k_0}\{\epsilon M/4k_0\}}{M/2} = \epsilon,$$

and the theorem is proved when $\alpha \geq 0$.

Now assume that the theorem holds for $\alpha \geq \alpha_0$, and we demonstrate that it holds for $\alpha \geq \alpha_0 - 1$, and then the theorem follows by induction since it holds for $\alpha \geq 0$. If $\alpha \geq \alpha_0 - 1$, one has from (1) that

$$\{zG'(z)\} = \{H(z)\}\{zF'(z)P'(1)\} \tag{9}$$

where

$$H(z) = \sum_{n=0}^{\infty} h_n z^n = \frac{P'\{F(z)\}}{P'(1)} = \sum_{n=0}^{\infty} \frac{(n+1)p_{n+1}}{P'(1)}\{F(z)\}^n. \tag{10}$$

(We remark that (2) implies that the radius of convergence of $P(z)$ is $\theta^{-1}>1$, implying that $P'(1)<\infty$. While not necessary, this definition leads to $H(z)$ being a pgf.) From (2),

$$\frac{(n+1)p_{n+1}}{P'(1)} \sim \frac{C\theta n^{\alpha+1}\theta^n}{P'(1)}, \quad n \to \infty.$$

Since $\alpha+1 \geq \alpha_0$, it follows from the theorem that

$$h_n \sim \frac{C\theta n^{\alpha+1}\tau^{-n}}{P'(1)\{\theta\tau F'(\tau)\}^{\alpha+2}}, \quad n \to \infty. \tag{11}$$

Clearly, $zF'(z)P'(1)$ has radius of convergence greater than τ, and from (9), (11), and lemma 2 one has

$$ng_n \sim \tau F'(\tau)P'(1)h_n, \quad n \to \infty.$$

In other words, (4) holds for $\alpha \geq \alpha_0 - 1$, and the theorem is proved.

It is apparent from the proof of the theorem that it may be generalized in various ways, but in the present form it covers most situations of practical interest.

Furthermore, it follows directly from the theorem that the tail of the distribution $\{g_n\}$ satisfies

$$q_n = \sum_{k=n+1}^{\infty} g_k \sim \frac{Cn^{\alpha}\tau^{-n}}{(\tau-1)\{\theta\tau F'(\tau)\}^{\alpha+1}}, \quad n \to \infty, \tag{12}$$

and the stop-loss premium satisfies

$$\sum_{k=n+1}^{\infty} \{k-(n+1)\}g_k = \sum_{k=n+1}^{\infty} q_k \sim \frac{Cn^{\alpha}\tau^{-n}}{(\tau-1)^2\{\theta\tau F'(\tau)\}^{\alpha+1}}, \quad n \to \infty. \tag{13}$$

4. Some examples

Several number of claims distributions of interest in insurance satisfy the results of the previous section.

Example 1 - The extended truncated negative binomial

A generalization of the negative binomial distribution is that with pgf

$$P(z) = p_0+(1-p_0)[\frac{\{1-\beta(z-1)\}^{-\alpha} - (1+\beta)^{-\alpha}}{1-(1+\beta)^{-\alpha}}], \quad |z|<1+\beta^{-1}, \tag{14}$$

where $0 \le p_0 < 1$, $\beta > 0$, and $\alpha > -1$, $\alpha \ne 0$ (cf. Willmot, 1988). The modified geometric (Gossiaux and Lemaire, 1981) is another special case which has been found to be of use in claim count modelling. As with the negative binomial, one has easily that

$$p_n \sim \frac{\alpha(1-p_0)n^{\alpha-1}}{\Gamma(\alpha+1)\{(1+\beta)^{\alpha}-1\}}(\frac{\beta}{1+\beta})^n, \quad n \to \infty, \tag{15}$$

which is of the form (2). Hence, the compound distribution $\{g_n\}$ with pgf $G(z) = P\{F(z)\}$ satisfies

$$g_n \sim \frac{\alpha(1-p_0)\{\beta\tau F'(\tau)\}^{-\alpha}}{\Gamma(\alpha+1)\{1-(1+\beta)^{-\alpha}\}} n^{\alpha-1}\tau^{-n}, \quad n \to \infty, \tag{16}$$

where $\tau>1$ satisfies $F(\tau) = 1+\beta^{-1}$, as follows from the theorem.

Example 2 - Ruohonen's distribution

Ruohonen (1988) has suggested the use of the convolution of a Poisson and a negative binomial, with pgf

$$P(z) = e^{\lambda(z-1)}\{1-\beta(z-1)\}^{-\alpha}, \quad |z|<1+\beta^{-1}. \tag{17}$$

Since the Poisson has infinite radius of convergence, it follows from lemmas 1 and 2 that

$$p_n \sim \frac{e^{\lambda/\beta} n^{\alpha-1}}{\Gamma(\alpha)(1+\beta)^{\alpha}}\left(\frac{\beta}{1+\beta}\right)^n, \quad n \to \infty. \tag{18}$$

Hence, the compound distribution $\{g_n\}$ with pgf $G(z) = P\{F(z)\}$ satisfies

$$g_n \sim \frac{e^{\lambda/\beta} n^{\alpha-1}\tau^{-n}}{\Gamma(\alpha)\{\beta\tau F'(\tau)\}^{\alpha}}, \quad n \to \infty, \tag{19}$$

where $\tau>1$ satisfies $F(\tau) = 1+\beta^{-1}$, as follows from the theorem.

Example 3 - Poisson-Pascal extension

The distribution with pgf

$$P(z) = e^{-\mu\{[1-\beta(z-1)]^{r}-1\}}, \quad |z|<1+\beta^{-1} \tag{20}$$

where $\mu>0$, $\beta>0$, and $0<r<1$, has been found to provide an extremely good fit to claim count data by Kestemont and Paris (1985). Willmot (1987) discussed the many attractive properties of its special case $r = 0.5$, the Poisson-inverse Gaussian distribution.

The pgf (20) may be re-expressed in compound Poisson form as $P(z) = \exp\{\lambda[K(z)-1]\}$ where $\lambda = \mu\{(1+\beta)^{r}-1\}$ and $K(z)$ is given by (14) with $p_0 = 0$ and $\alpha = -r$. Thus, the coefficient k_n of z^n in $K(z)$ satisfies (15), again with $p_0 = 0$ and $\alpha = -r$. Since $R(z) = \exp\{\lambda(z-1)\}$ has infinite radius of convergence, it follows from lemma 3 that $p_n \sim R'\{[1-(1+\beta)^{-r}]^{-1}\}k_n$, $n \to \infty$. In other words,

$$p_n \sim \frac{r\mu e^{\mu}(1+\beta)^r}{\Gamma(1-r)} n^{-r-1}\left(\frac{\beta}{1+\beta}\right)^n, \quad n \to \infty, \tag{21}$$

and so the theorem yields, for the distribution $\{g_n\}$ with pgf $G(z) = P\{F(z)\}$,

$$g_n \sim \frac{r\mu e^{\mu}\{\beta\tau F'(\tau)\}^r}{\Gamma(1-r)} n^{-r-1}\tau^{-n}, \quad n \to \infty \tag{22}$$

where $\tau > 1$ satisfies $F(\tau) = 1+\beta^{-1}$.

Example 4 - Sichel's distribution

Another generalization of the Poisson-inverse Gaussian distribution is the Poisson mixture by the generalized inverse Gaussian distribution, with pgf

$$P(z) = \frac{K_\alpha\{\mu[1-\beta(z-1)]^{1/2}\}}{K_\alpha(\mu)}\{1-\beta(z-1)\}^{-\alpha/2}, \quad |z| < 1+\beta^{-1}, \tag{23}$$

where $\mu > 0$, $\beta > 0$, $-\infty < \alpha < \infty$, and $K_\alpha(x)$ is the modified Bessel function of the third kind with index α (cf. Jorgensen, 1982). The Poisson-inverse Gaussian is the special case $\alpha = -1/2$. One has

$$p_n \sim \frac{2^{\alpha-1}}{\mu^\alpha(1+\beta)^\alpha K_\alpha(\mu)} n^{\alpha-1}\left(\frac{\beta}{1+\beta}\right)^n, \quad n \to \infty, \tag{24}$$

(cf. Teugels and Willmot, 1987, with misprint), and so the theorem yields, for the distribution $\{g_n\}$ with pgf $G(z) = P\{F(z)\}$,

$$g_n \sim \frac{2^{\alpha-1}}{\{\mu\beta\tau F'(\tau)\}^\alpha K_\alpha(\mu)} n^{\alpha-1}\tau^{-n}, \quad n \to \infty \tag{25}$$

where $\tau > 1$ satisfies $F(\tau) = 1+\beta^{-1}$.

Other examples may be found in Teugels and Willmot (1987).

5. Practical considerations

The asymptotic results of section 3 give estimates for a large family of compound distributions, their tail probabilities, and their stop-loss premiums.

From a numerical standpoint, the results are simple to apply, only requiring the value of τ from (5). Normally, this value must be obtained numerically, and the inequality

$$1 < \tau < e^{((m_1^2+2m_2(1-\theta)/\theta)^{1/2}-m_1)/m_2} \tag{26}$$

where $m_i = \sum_{n=0}^{\infty} n^i f_n$; $i = 1,2$, is of use in locating the root τ. In particular, a successive bisection routine using (26) or a Newton-Raphson procedure using the right side of (26) as starting value (eg. Burden and Faires, 1985) is satisfactory. To derive (26), note that from (5)

$$\theta^{-1} = \sum_{n=0}^{\infty} f_n \tau^n = \sum_{n=0}^{\infty} f_n e^{n \log \tau} = \sum_{n=0}^{\infty} f_n \sum_{k=0}^{\infty} \frac{(\log \tau)^k}{k!} n^k$$

$$= \sum_{k=0}^{\infty} \frac{(\log \tau)^k}{k!} \sum_{n=0}^{\infty} n^k f_n > 1 + m_1 \log \tau + m_2 (\log \tau)^2/2.$$

In other words, $m_2(\log \tau)^2 + 2m_1(\log \tau) - 2(1-\theta)/\theta < 0$. Since $m_2 > 0$, the roots of this quadratic in $\log \tau$ bound the actual value, and exponentiating the positive root yields (26).

In practice, the usefulness of an asymptotic formula depends on whether the probabilities are still significantly larger than zero when the asymptotics become accurate. One would expect that since the asymptotic result depends only on the asymptotic behaviour of $\{p_n\}$ and not its exact form, the accuracy should depend to a great extent on the choice of $\{p_n\}$. In particular, if (2) is not accurate, then one could hardly expect (4) to be accurate. The second important factor influencing the accuracy of (4) is the distributional form of $\{f_n\}$. If the distribution is multimodal, then this behaviour may be reproduced by $\{g_n\}$ (see Douglas, 1980, for further discussion), and it may take longer for $\{g_n\}$ to settle down to its asymptotic form.

As a result, it is difficult to assess the accuracy without evaluating the exact value of g_n, yet this is often primarily the reason why one would wish to use the asymptotic formula in the first place. This, fortunately, is not always the case. For example, the exact probabilities of the compound negative binomial may be computed recursively using the algorithm of Panjer (1981), yet this formula becomes quite cumbersome computationally in the extreme right tail, since a large number of previously calculated values may appear in the computation. An obvious use of the asymptotic result in this case would be to compare the asymptotic value to the value from the recursion, and then to use only the former when it is sufficiently accurate. Several successive values should be checked to assess this, however, since the asymptotic value could be close to the recursive value and then the two could diverge again. A similar approach may be used with the extended Poisson-Pascal distribution of example 2. A combination of recursive and asymptotic approaches have been found to be quite useful in connection with the evaluation of some queue length distributions by Tijms (1986).

The accuracy of the asymptotic results may be checked numerically for various choices of $\{p_n\}$ and $\{f_n\}$. Milidiu (1985) suggests that for the compound negative binomial the accuracy of (4) decreases as α increases. Further numerical investigations tend to reinforce this conclusion, and also suggest that the accuracy increases as β increases. In particular, in the compound geometric case with $\alpha = 1$, (4) is normally extremely accurate for quite small values of n, even for β as small as 0.5.

As an example, consider the claim size distribution obtained by applying the compound Poisson approximation to the individual risk model using Gerber's example (Gerber, 1979, pp. 48-55). This yields the claim size distribution

n	f_n
1	.042857
2	.250000
3	.307143
4	.257143
5	.142857

With $\beta = 1$, (4) is extremely accurate even for relatively small values of n, as may be seen from the following table.

Compound Geometric		
n	exact g_n	asymptotic g_n
5	.058276	.051118
10	.017615	.018069
15	.006356	.006387
20	.002258	.002257
25	.000798	.000798
30	.000282	.000282
35	.000100	.000100
40	.000035	.000035

Thus, (4) is of use in obtaining quick estimates of g_n, a fact which when combined with the recursive approach to the evaluation of the compound negative binomial yields a fast, accurate computational scheme.

A similar approach may be used to study the accuracy of (6) for other choices of $\{p_n\}$ and $\{f_n\}$. For example, for the Poisson-Pascal extension of example 3, both (21) and (22) appear to be most accurate when r is very close to 0.5 (the Poisson-inverse Gaussian distribution), particularly for large β, and not very accurate elsewhere. For example, using Gerber's $\{f_n\}$ as above and the compound Poisson-inverse Gaussian ($r = 0.5$) with $\beta = 0.5$ and $\mu = 1$, one obtains the following results.

Compound Poisson-Inverse Gaussian

n	exact g_n	asymptotic g_n
5	.028659	.031664
10	.001954	.002226
15	.000228	.000241
20	.000031	.000031
25	.000004	.000004

6. Conclusions

The use of asymptotic formulae such as (6) appears to have much potential for many compound distributions of practical interest, both as approximations in their own right and in conjunction with other computational schemes such as recursive approaches. However, with any particular distributional assumptions, much numerical work is needed to assess the accuracy, as well as that of related formulae such as (12) and (13).

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