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Bounding And Asymptotic Behavior Of Ruin Probabilities In Collective Risk Theory: Final Report III*

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Abstract

The results obtained during the last one-third of the research project and the project at whole are presented. The report only contains brief descriptions of the results which are placed in addendums and which are in reality copies of papers and a chapter of a book submitted for publication. Some theoretical and applied problems induced by the research are discussed.

Key words: RUIN PROBABILITY, PROBABILITY METRIC, GEOMETRIC SUM, CONTINUITY ESTIMATE, CLAIMS COST INFLATION, TWO-SIDED BOUNDS

1 Introduction

This report highlights the results obtained during the research "Bounding And Asymptotic Behavior Of Ruin Probabilities In Collective Risk Theory" supported by The Society of Actuaries Committee on Knowledge Extension Research (CKER).

Sections 2 through 4 of the report describe new results obtained during the last one-third of the research. These sections should be viewed as a continuation of precedings reports [1] and [2].

Section 5 is a summary of the results obtained during the whole research.

Section 6 dwells on a discussion of further investigations inspired by the research.

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2 Applications Of Geometric Sums

The first group of the results is associated with applications of various estimates obtained by the researcher for random geometric sums. These results can be found in the addendum to the report which is actually Chapter 6 of the book [3], written owing to the support of CKER. Some of them have already been described in the preceding reports [1] and [2], while the others are new and will be described below, in this section.

The basic object of these results is the Sparre Andersen risk process $(X(t))_{t\geq 0}$, described in Section 2 of the first intermediate report [1], and the corresponding probability $\Psi(x)$ of ruin (given the initial capital x), introduced in Section 3 of the same report [1].

We now describe the contents of Chapter 6 of [3].

Section 6.1 from Chapter 6 of the book [3] contains necessary notations and setups.

Section 6.2 is devoted to the following new setup (appeared owing to our joint work with Professor A. Nagaev). Let us measure an insurer's risk by the value $\Psi(x)$ of the ruin probability. Assume that the values of this probability laying below a prescribed level Ψ^* are only tolerable. This means that the insurer accepts at most Ψ^* as an *acceptable risk level*. Let us define the *minimal admissible initial capital*

$$x^* = \inf\{x : \Psi(x) \leq \Psi^*\}. \tag{2.1}$$

Suppose the insurer wants to attract as many clients as possible keeping the relative safety loading at the lowest possible level. Formally, we consider the case, where all parameters governing the risk process (i.e. the distribution function of inter-occurrence times, the distribution function of claim sizes, and the premium rate) can vary in such a manner that the relative safety loading ρ tends to 0. Then the minimal admissible initial capital x^* varies too. Evidently, $x^* \to \infty$ as $\rho \to 0$. The problem consists in disclosing the limiting behavior of x^* and its bounding.

For this, we used the results developed in Chapter 3 of the book [3]. Using the generalized Renyi theorem, we proved that, in the case of the classical risk model, the limiting shape of the underlying ruin probability is exponential (see Lemma 2.1, [3], pp. 176-177). We also obtained the accuracy bounds for this limiting result (see Lemma 2.2, [3], p. 177). The desired asymptotic result as well as the bounds for the minimal admissible initial capital is in Theorem 2.1 (see [3], p. 178, and formulas (2.17) and (2.18)).

These results were generalized to the case of the S. Andersen model in Lemma 2.3 and Theorem 2.2 (see [3], pp. 178-179), where we could not obtain explicit quantitative bounds of x^* and this can be regarded as a problem to be solved in the sequel.

Section 6.3 contains the following applications of the results of Chapter 4 of [3], which were described in [1] (Section 3), [2] (Section 1), and [4]:

(1) Two-sided bounds of $\Psi(x)$ under the Crámer condition;

(2) Two-sided bounds of $\Psi(x)$ for the case of large claims;

(3) A generalization of upper bounds of $\Psi(x)$ proposed by G. Willmot (see [4] for further references);

(4) Numerical results showing the advantage and quality of the proposed bounds.

Section 6.4 from Chapter 6 of [3] is devoted to the continuity problem of ruin probabilities discussed in the intermediate report [2] (Sections 1 to 4).

3 Bounds Of Ruin Probabilities Under Uncertain Claim Sizes Distributions

In practice, we often encounter the situation where claim sizes distribution $B(u)$ is unknown. Applying the technique evolved in [3], let us consider such a problem.

Let us assume that, for a classical risk process, the expectation μ, variance σ^2, and the range $[0, b]$ for the claim severity function are only known. It is necessary to give reasonable bounds for the probability $\Psi(x)$ of ruin under these minimal restrictions.

A. Steenackers and M.J. Goovaerts (see [14]) have considered a similar setup. Their approach was based on the bounds resulting from the stochastic ordering technique. These bounds are not good in the range of interest and, in addition, they are expressed in the form which does not fit immediate numerical calculations because it does not contain accuracy control (see formulas (1) and (2) in [10]).

We proposed another approach based on the geometric sum technique developed in Chapters 3 and 4 of the book [3] (see Lemma 1 in [10]). This resulted in the upper bound (3) and lower bound (6) for the unknown ruin probability (see [10]). Both bounds are expressed in the terms which can be easily calculated given μ, σ^2 and b.

The numerical results, collected in Section 3 of [10], show that in the range where the probability of ruin is less than 0.2 (approximately) our bounds are substantially better than those proposed in [14]. In addition, the adjustment coefficients for our bounds are rather close to the corresponding coefficient of the well-known De-Vylder approximation which proved to have a good accuracy.

This rather simple example shows that the method elaborated can successfully be applied in the case where the complete information of the risk process is not available.

4 Non-Traditional Risk Models

Let us consider the risk model

$$X(t) = x + S(t) - \sum_{i \le N(t)} Z_i,$$

where x is the initial capital of an insurance company, $S(t)$ denotes the premium accumulated during $[0, t]$, $N(t)$ is the number of claims occurred during $[0, t]$, and Z_i is the ith claim size.

Let us assume that the ith inter-occurrence time is equal to θ_i and, during this time, $S(t)$ increases with rate c_i. We assume that sequences $\{Z_i\}$ and $\{\theta_i\}$ are independent, sequence $\{Z_i\}$ consists of i.i.d.r.v.'s having the common exponential distribution

$$\mathbf{P}(Z_i \leq u) = 1 - \exp(-u)$$

and sequence $\{\theta_i\}$ consists of independent exponentially distributed r.v.'s,

$$\mathbf{P}(\theta_i \leq u) = 1 - \exp(-\delta_i u).$$

Note that θ_i have different distributions, in general.

Let $\Psi_n(x)$ be a probability of ruin until the nth claim occurs and

$$\Psi(x) = \lim_{n\to\infty} \Psi_n(x)$$

be the probability of ruin during $[0, \infty)$.

In paper [13], we derive recurrence equations (see Theorems 1 and 2, p. 5) allowing us to find $\Psi_n(x)$ using simple computations.

At the first glance, the model described above is very restrictive. However, this is not the case as the following remarks show.

Remark 1. If we put $c_i = 0$, $2 \leq i \leq k$ $(k \geq 2)$, then this is equivalent to the case, where the first claim size has the Erlang distribution of order k that is it has the density $u^k e^{-u}/(k-1)!$ (see Corollary 1 in [13] and Example 2 below). Playing with δ_i and c_i (in particular, assuming that they are random), we can consider the cases when inter-occurrence intervals are dependent and have distributions different from the Erlang ones. Moreover, it is well-known that any distribution function of a nonnegative r.v. can be approximated by a mixture of Erlang distributions. This gives us the possibility to approximate the ruin probability when the claim sizes have an arbitrary distribution. Therefore, the proposed model is sufficiently general. Of course, the mentioned possibilities require more complicated calculations but we anticipate that these difficulties can be overcome.

Remark 2. Theorems 1 and 2 do not require the positiveness of safety loading $(c_i > \delta_i)$ during each inter-occurrence time. It is possible to admit that $c_i < \delta_i$ for some i. Such a situation can be of interest in insurance, modeling burst claims. The effect of negative safety loadings is demonstrated below in Example 2.

Remark 3. The proposed model gives the possibility to consider the following insurance model proposed by J. Galambos (see [13]). Let n clients of an insurance company belongs to the same risk group. Then the time to the accident for these clients can be regarded as i.i.d.r.v.'s. The actual claims arrive at the insurance company in increasing order which means that occurrence times form the order statistics of the mentined sampling. The proposed approach solves this problem (see [13] and Example 1 below).

After [13] has been prepared, numerical experiments were done. In all cases, the programs worked extremely fast and this confirms the efficiency of the approach.

Example 1. Order statistics.

Let us consider the model proposed by J. Galambos, see Corollary 2 in [13]. Let $n = 100$, $c = 3$, and $\beta = 0.004$. Then, for the first twenty five claims the relations $c_i < \delta_i$ hold. The following Table 1 contains the values of ruin probability $\Psi_n(x)$ depending on the initial capital x.

Table 1

inital capital x	ruin probability $\Psi_n(x)$
0.0	0.9729340
1.0	0.9325102
2.0	0.8840706
3.0	0.8288801
4.0	0.7684787
5.0	0.7045540
6.0	0.6388195
7.0	0.5729099
8.0	0.5082983
9.0	0.4462379
10.0	0.3877298

Example 2.

(i) Let us take $c_1 = c_3 = \ldots = c_n = c > 0$ and $c_2 = 0$. It follows, from Corollary 1 in [13] that this corresponds to the case where the density of the first claim size is xe^{-x} and the density of other claim sizes is e^{-x}. If we choose δ and c such that $\delta < c < 2\delta$, then the safety loading is negative during the first inter-occurrence time, and it is positive during all other intervals. The total number of claims is equal to $n - 1$. The column (i) of Table 2 corresponds to such a case where $n = 100$, $c = 3.0$, $c_2 = 0$.

(ii) Now, let again $n = 100$, but $c_i = 3.0$ and $\delta_i = 2.0$ *for all* $i = 1, 2, \ldots, 100$. In this case (marked as (ii) in Table 2) the values of ruin probabilities are essentially different from the preceding case. This means that the negative safety loading (even appearing during one interval) affects heavily the results.

(iii) At last (see column (iii) in Table 2), take $n = \infty$, $c_i = 3.0$ and $\delta_i = 2.0$ for $i \geq 1$. The results are pretty close to those of the case $n = 100$ which demonstrates a fast speed of convergence of $\Psi_n(x)$ to $\Psi(x)$

Table 2

inital capital x	ruin probability $\Psi_n(x)$ (i)	(ii)	(iii)
0.0	0.7999543	0.6665945	0.6666667
1.0	0.6428743	0.4775759	0.4776875
2.0	0.4862354	0.3421417	0.3422781
3.0	0.3577831	0.2451030	0.2452530
4.0	0.2597806	0.1755765	0.1757314
5.0	0.1873684	0.1257633	0.1259171
6.0	0.1346803	0.0900754	0.0902235
7.0	0.0966355	0.0645084	0.0646480
8.0	0.0692706	0.0461931	0.0463223
9.0	0.0496271	0.0330736	0.0331914
10.0	0.0355413	0.0236766	0.0237827

5 The Results Obtained During The Research

5.1 The following publications were prepared during the research:

- **Book** – see [3] (accepted in Kluwer Acad. Publ., Dordrecht, to appear in September 1997);
- **Papers** – [4] (submitted to *NAAJ*), [7] (accepted in *J. Math. Sci.*, 1997), [9] (published in *Fundamental'naya i Prikladnaya Mathematika*, in Russian), [10] (submitted to *Scand. Act. J.*), [13] (submitted to *Theory Prob. Appl.*, in Russian);
- **Preprints** – [4], [5] and [6] appeared as preprints of Laboratory of Actuarial Mathematics, The University of Copenhagen; [9] will appear as a preprint of Bulgarian Actuarial Society in July 1997;
- **Presentations at conferences** – see abstracts [8] and [11] of two talks at the *International Seminar on Stability Problems for Stochastic Models*, Debrecen, Hungary, January 1997; these abstracts appeared in *Theory Prob. Appl.*, **42**, No 2, in Russian.

5.2 In these publications, the following general mathematical results were obtained:

- Two-sided bounds of geometric sums under the Cramér condition and in the presence of heavy-tailed summands.
- Metric estimates of geometric sums.

- Continuity estimates of geometric sums.
- Generalyzed Renyi's limit theorem for geometric sums.
- Specific links between risk theory and branching processes.

5.3 Concerning the risk theory, these results were transformed into:

- Two-sided bounds of ruin probabilities in the cases where claim sizes satisfy the Cramér condition and have heavy tails.
- Continuity estimates of the probability of ruin.
- Estimates and asymptotic formulas for the initial capital securing a prescribed risk level.
- Recurrent formulas for calculating the ruin probability in non-traditional risk models.
- The analysis of a risk model under inflatory conditions.
- Bounds for ruin probabilities in the case where the mean, variance, and the range of claim sizes are only known.

5.4 Many computer programs supporting numerical calculations were written.

6 Outline Of Further Research

The following directions of further investigations follow from the present research:

- Elaborating methods giving upper bounds of the probability of ruin having both an acceptable relative accuracy in the range of practical importance and correct asymptotic behavior when the initial capital tends to infinity.
- Generalizing the proposed approaches to more complicated risk models (say, having the Cox occurrence process, dependent claims, etc.)
- Estimates of the initial capital securing a prescribed risk level for more general risk models.
- Obtaining new continuity estimates taking into account the shape of the tail of ruin probability.
- Investigation of the controlled risk process.
- Elaborating specific simulation methods for estimation ruin probabilities.

- Elaborating effective numerical procedures to calculate or estimate ruin probabilities.

We do not discuss these topics in details as these details should appear during the research.

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