Risk Premiums and Their Applications

Jeffrey S. Pai
I. H. ASPER SCHOOL OF BUSINESS
University of Manitoba

Guohong Sun
Manulife Financial
Abstract:

In this paper we discuss some properties of the $n$th stop-loss order and their application in risk premium principles. We give a necessary condition and a sufficient condition for $n$th stop-loss orders. They are convenient tools to construct risk pairs with $n$th stop-loss orders.

The maintenance properties of $n$th stop-loss orders under the operation of compound, in the situation where counting variables $N_1$ and $N_2$ are not identical, are proved. The necessary condition for $n$th stop-loss orders is applied to the valuation of risk premium principles.

We show that exponential premium principles can differentiate between losses more finely than the net premium principles under some conditions.

Key Words:

$n$th stop-loss transform, $n$th stop-loss order.
1 Introduction

For an insurance company, each contract of insurance brings a risk with it. A claim may occur some time in the future and the amount of the claim is a nonnegative random variable which is called a risk. One of the main tasks of actuaries is to compare the attractiveness of different risks. This helps them to determine insurance premiums and to decide on the reinsurance needed. Another task of actuaries is to calculate the risk premiums. The basis of insurance is the hypothesis that claims can be compensated by fixed payments called premiums. Premiums are calculated by a premium calculation principle. The partial orders on a family of risks are called risk orders. The theory of risk orders is a useful mathematical tool for comparing risks and risk premium principles.

From Bowers (1997), we know if the decision maker has decided on the fixed amount to be paid for insurance, also the expected claims is a fixed value, the stop-loss insurance will maximize the expected utility of the decision maker. Consequently, we concern more with the feature of the stop-loss insurance. The properties of \( n \)th stop-loss orders provide much more information for studying the stop-loss insurance, since the 1st stop-loss transforms are the stop-loss premiums.

This paper is based upon the works of Goovaerts et al. (1990) and Cheng and Pai (1999a). Many kinds of partial orders were discussed in Goovaerts et al. (1990).
The $n$th stop-loss order is one of them. In Cheng and Pai (1999a), the concept of stop-loss transforms was generalized to the $n$th stop-loss transforms. The maintenance properties of the $n$th stop-loss order under the individual risk model and the collective risk model were developed. In this paper, we first discuss the properties of the $n$th stop-loss order, later apply them to risk premium principles and ruin probabilities.

This paper is organized as follows. In Section 2, we introduce some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a). In Section 3, we continue the study by Cheng and Pai (1999a) on $n$th stop-loss orders. We give a necessary condition and a sufficient condition for $n$th stop-loss orders. They are convenient tools to construct risk pairs having $n$th stop-loss orders. The maintenance properties of $n$th stop-loss orders under the operation of compound, in the situation where counting variables $N_1$ and $N_2$ are not identical, are to be proved. In Section 4, the necessary condition for $n$th stop-loss orders will be applied in the valuation of risk premium principles. We will prove that exponential premium principles can differentiate between losses more finely than the net premium principles under some conditions.
2 \textit{nth Stop-Loss transform and Order}

This article deals with risks to be insured, which are defined as non-negative random variables. Here we cite some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a).

\textbf{Definition 1 (\textit{nth Stop-Loss Transform})} Suppose loss random variable $X$ is nonnegative with its distribution function being $F(x)$, its survival function being $\bar{F}(x) = 1 - F(x)$, and $E[X^n] < \infty$. Let

\[
\Pi^{(n)}(u) = E[(X - u)_{+}^n], \quad u \geq 0, \quad n = 1, 2, \cdots, \tag{1}
\]

where

\[
(X - u)_{+} = \begin{cases} 
0, & \text{for } x \leq u, \\
x - u, & \text{for } x > u,
\end{cases}
\]

\[
\Pi^{(0)}(u) = \bar{F}(u) = 1 - F(u). \tag{2}
\]

As a function of $u$, $\Pi^{(n)}(u), \ n = 1, 2, \cdots,$ will have domain $[0, \infty)$. We call function $\Pi^{(n)}(u)$ the \textit{nth stop-loss transform} of $X$. 
Definition 2. (*nth Stop-Loss Order*) We say that $X$ is less than $Y$ in the meaning of the $n$th stop-loss order, denoted by $X <_{sl(n)} Y$, if

$$E[X^k] \cdot E[Y^k], \quad k = 1, 2, \ldots, n - 1,$$

(3)

and

$$\Pi_X^{(n)}(u) \cdot \Pi_Y^{(n)}(u), \quad \text{for all } u \geq 0.$$

(4)

When $n = 0$, the formula (3) disappears and formula (4) becomes

$$F_X(u) \cdot F_Y(u), \quad \text{for all } u \geq 0.$$

When $n = 1$, the formula (3) is trivial and formula (4) becomes

$$\int_u^\infty F_X(x)dx \cdot \int_u^\infty F_Y(x)dx, \quad \text{for all } u \geq 0.$$

Definition 3. (Weak *nth Stop-Loss Order*) Let

- $= \{ H(x), \ x \geq 0 : H(x) \geq 0 \text{ monotonous decreasing and } \lim_{x \to \infty} H(x) = 0 \}.$

Suppose $H(x), G(x) \in -$. We say that $H(x)$ is less than $G(x)$ in the meaning of weak $n$th stop-loss order, denoted by $H <_{wsl(n)} G$, if

$$\Pi_H^{(n)}(u) \cdot \Pi_G^{(n)}(u), \quad \text{for all } u \geq 0.$$

(5)
The following results are important and will be used in this paper.

**Theorem 1.**

\[
\frac{d}{du} \left[ \Pi_X^{(n)}(u) \right] = -n \Pi_X^{(n-1)}(u),
\]

or

\[
\Pi_X^{(n)}(u) = n \int_u^\infty \Pi_X^{(n-1)}(x) dx.
\]

(see Cheng and Pai (1999a), Theorem 6)

**Theorem 2.** Let \( n = 0, 1, 2, \cdots \) and \( m > n \). Suppose risk \( X <_{sl(n)} Y \). Then

\( X <_{sl(m)} Y \).

(see Goovaerts et al. (1990), Theorem 4.2.2)

**Theorem 3.** Suppose \( u(x) \) is a utility function having \( n - 1 \) continuous derivatives of alternating sign:

\[
(-1)^{(k-1)} u^{(k)}(x) \geq 0, \ k = 1, 2, \cdots, n - 1,
\]

\[
(-1)^{(n-1)} u^{(n)}(x) \geq 0, \text{ and non-decreasing in } x.
\]
Let \( U_n = \{ u(x) : u(x) \) satisfies (8) and (9) \}, w(x) = -u(-x), \) and \( W_n = \{ w(x) : w^{(k)}(x) = (-1)^{(k+1)} u^{(k)}(-x) \geq 0 \} \). Then \( X <_{sl(n)} Y \), if and only if

\[
E[u(-X)] \geq E[u(-Y)], \text{ for all } u \in U_n,
\]

if and only if

\[
E[w(X)] \cdot E[w(Y)], \text{ for all } w \in W_n.
\]

(see Cheng and Pai (1999a), Theorem 10)

**Theorem 4.** The \( n \)th stop-loss order is maintained under the summation of independent random variables. That is, if

\[
X_i <_{sl(n)} Y_i, \quad i = 1, 2, \ldots, k,
\]

where \( k \) is a positive integer, then

\[
\sum_{i=1}^{k} X_i <_{sl(n)} \sum_{i=1}^{k} Y_i, \quad n = 0, 1, 2, \ldots.
\]

(see Cheng and Pai (1999a), Theorem 15 )
3 Properties of \( n \)th Stop-Loss Orders

From Theorem 3, we can see that the \( n \)th stop-loss order can be characterized as the common preferences of a group of decision makers with increasingly regular utility functions \( u(x) \in U_n \). We will continue the work of Goovaerts et al. (1990) and Cheng and Pai (1999a), to give more features of the \( n \)th stop-loss order.

Theorem 5 will be used to compare the differences of the net premium principle and the exponential premium principle in Section 4.

**Theorem 5. (Necessary Condition)** Suppose \( X, Y \) are not identically distributed risks. If \( X <_{wsl(n)} Y \) and \( E[X^{n+i}] < \infty \), then

\[
E[X^{n+k}] < E[Y^{n+k}], \quad k = 1, 2, \ldots, i.
\]

**Proof**

If \( E[Y^{n+i}] = \infty \), the result is obvious. If \( E[Y^{n+i}] < \infty \), we first show that for \( k = 1 \), we have \( E[X^{n+1}] < E[Y^{n+1}] \). Indeed, let

\[
g(u) = \Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u).
\]

From Definition 3 and Theorem 1, we have: For all \( u > 0 \), \( g(u) \cdot 0 \), and

\[
g'(u) = \frac{d}{du} [\Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u)] = -(n + 1)\Pi_X^{(n)}(u) - \Pi_Y^{(n)}(u) \geq 0.
\]

Further more, there exists \( u_o \geq 0 \), such that

\[
g'(u_o) = -(n + 1)[\Pi_X^{(n)}(u_o) - \Pi_Y^{(n)}(u_o)] > 0.
\]
(Otherwise we will have \( F_X(u) = G_Y(u) \) differentiating \( g(u) \) \( n \) times.)

So the following inequality must be true:

\[
g(0) = \Pi_X^{(n+1)}(0) - \Pi_Y^{(n+1)}(0) = E[X^{n+1}] - E[Y^{n+1}] < 0.\]

Applying the same method and the fact that \( \Pi_X^{(n+j)}(u) \cdot \Pi_Y^{(n+j)}(u) \) for \( j = 1, 2, \cdots \) and for all \( u > 0 \), we obtain the relation

\[
E[X^{n+k}] < E[Y^{n+k}], \quad k = 2, \cdots, i. \]

A sufficient condition for the \( n \)th stop-loss order is given by Theorem 4.2.3 of Goovaerts (1990): \( n+1 \) sign changes in density functions implies the \( n \)th stop-loss order. Here we give another sufficient condition: \( n \) sign changes in distribution functions implies the \( n \)th stop-loss order.

**Theorem 6. (Sufficient Condition)** Suppose that for two risks \( X \) and \( Y \) there is a partition of \([0, \infty)\) into \( n+1 \) consecutive non-empty intervals (closed intervals containing only one point are acceptable) \( I_0, I_1, \cdots, I_n \) such that

\[
(-1)^{n+1-j} \{F_X(t) - F_Y(t)\} \cdot 0 \text{ on } I_j.
\]

and the first \( n \) moments satisfy \( E[X^i] = E[Y^i], \; i = 1, 2, \cdots, n \), then \( X <_{sl(n)} Y \).

**Proof**

For convinence, we let \( n \) be an even number. When \( n \) is an odd number, we can
apply the same method to arrive at the result. Let

\[ h_i(t) = \Pi_Y^{(i)}(t) - \Pi_X^{(i)}(t), \quad i = 0, 1, \ldots, n, \]

then from Theorem 1, we have \( h'_i(t) = -ih_{i-1}(t) \). We only need to show that

\[ h_n(t) \geq 0, \quad \text{for all } t > 0. \quad (10) \]

First we know that

\[ (-1)^j[F_Y(t) - F_X(t)] \cdot 0, \quad j = 1, 2, \ldots, n, \]

and

\[ h'_1(t) \cdot 0, \quad h_1(t) \downarrow \text{ on } I_0, \]

\[ h'_1(t) \geq 0, \quad h_1(t) \uparrow \text{ on } I_1, \]

\[ \vdots \]

\[ h'_1(t) \cdot 0, \quad h_1(t) \downarrow \text{ on } I_n. \]

On the other hand, from \( h_n(0) = h_n(\infty) = 0 \), we know that there exists \( a_1 \in (0, \infty) \) such that \( h'_n(a_1) = 0 \). Using Rolle’s theorem and repeating this process, we know that there exist \( b_1 < b_2 < \cdots < b_{n-1} \) such that\n
\[ h_1(0) = h_1(b_1) = \cdots = h_1(b_{n-1}) = h_1(\infty) = 0. \]

Combin the discussions above, the following conclusion must be true: There exist \( c_1 \in I_1, \cdots, c_{n-1} \in I_{n-1} \) such that

\[ h_1(t) \cdot 0 \text{ on } [0, c_1) = I_0^{(1)}, \]
\[ h_1(t) \geq 0 \text{ on } [c_1, c_2) = I_1^{(1)}, \]

\[ \vdots \]

\[ h_1(t) \geq 0 \text{ on } [c_{n-1}, \infty) = I_{n-1}^{(1)}. \]

Repeat the same process, we finally have (10). \qed

We can see that the condition of Theorem 6 implies: \( F_X(t) = F_Y(t) \) at least at \( n \) different points in \((0, \infty)\).

Theorem 5 and 6 are two useful tools to help us find out or construct the risk pairs which have \( n \)th stop-loss orders.

Compound risk was discussed in Theorem 16 of Cheng and Pai (1999a) where the counting variables \( N_1 \) and \( N_2 \) have identical probability distributions. Now we give another result where \( N_1 <_{sl(1)} N_2 \) but \( X_i \) and \( Y_i \) are two sequences of independent and identically distributed risks.

**Theorem 7. (Compound Risks)** Let \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) be two sequences of independent and identically distributed risks, \( N_j (j = 1, 2) \) be counting variables independent of \( X_i \) and \( Y_i \). In the collective risk models, \( S_1 \) and \( S_2 \) are defined as

\[ S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i. \]

If \( X_i <_{sl(n)} Y_i \), \( N_1 <_{sl(1)} N_2 \), then we have \( S_1 <_{sl(n)} S_2 \).
Proof

According to Definition 2, we need to prove

$$E[S^i_1] \cdot E[S^i_2] \quad i = 1, 2, \cdots, n - 1,$$

and

$$\Pi^{(n)}_{S_1}(u) \cdot \Pi^{(n)}_{S_2}(u), \quad \text{for all } u \geq 0.$$

First we prove (12). From Theorem 4, we have for all $u \geq 0$,

$$\Pi^{(n)}_{S_1}(u) = E[(S_1 - u)_+]^n$$

$$= \sum_{k=0}^{\infty} E[(S_1 - u)_+^n | N_1 = k] \cdot \Pr(N_1 = k)$$

$$\cdot \sum_{k=0}^{\infty} E[(S_2 - u)_+^n | N_1 = k] \cdot \Pr(N_1 = k)$$

$$= \sum_{k=0}^{\infty} E[(\sum_{i=1}^{k} Y_i - u)_+]^n \cdot \Pr(N_1 = k).$$

(Define $E[(\sum_{i=1}^{k} Y_i - u)_+]^n = 0$ when $k = 0$.)

Let

$$w_1(k) = E[(\sum_{i=1}^{k} Y_i - u)_+]^n.$$

It is obvious that $w_1(k) \cdot w_1(k + 1), \quad k = 0, 1, \cdots$. If

$$2w_1(k + 1) \cdot w_1(k) + w_1(k + 2), \quad k = 0, 1, \cdots,$$
we can construct a convex function $w_2(t)$, such that $w_2(k) = w_1(k)$, and $w_2(t) \geq 0$, and non-decreasing in $t$. Then from Theorem 3, we have $E[w_2(N_1)] \cdot E[w_2(N_2)]$, and (13) becomes

$$
\Pi_{S_1}^{(n)}(u) \cdot \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_1 = k) \\
\cdot \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_2 = k) \\
= \Pi_{S_2}^{(n)}(u).
$$

Now we only need to show (14). Let $A_k = \sum_{i=1}^{k} Y_i$. (14) is equivalent to the following inequality

$$
E[\{(A_k + Y_{k+1} - u)_+\}^n] + E[\{(A_k + Y_{k+2} - u)_+\}^n] \\
\cdot E[\{(A_k - u)_+\}^n] + E[\{(A_k + Y_{k+1} + Y_{k+2} - u)_+\}^n],
$$

and this follows directly if we look at the conditional distribution with $A_k = a, Y_{k+1} = y, Y_{k+2} = z$, and use the following inequality

$$
(a + y - u)_+^n + (a + z - u)_+^n \cdot (a - u)_+^n + (a + y + z - u)_+^n.
$$

(15)

When $u \geq a$, (15) is obvious; when $u < a$, we can get (15) by using Binomial Theorem.

Applying the same method, we can prove (11).
In the following Corollary, we generalized the result of Theorem 3.2.5 in Goovaerts et al. (1990) from stop-loss orders to \( n \)-th stop-loss orders.

**Corollary 8. (Conditional Compound Poisson Distribution)** Let \( \Lambda_j \) be a non-negative structure variable, and \( N_j \) be an integer valued non-negative random variable. Their conditional distribution given \( \Lambda_j = \lambda \) of \( N_j \) is Poisson(\( \lambda \)) distributed, \( j = 1, 2 \). Let \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) be two sequences of independent and identically distributed risks, \( N_j(j = 1, 2) \) be counting variables independent of \( X_i \) and \( Y_i \). In the collective risk models, \( S_1 \) and \( S_2 \) are defined as

\[
S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.
\]

If \( X_i \prec_{sl(n)} Y_i, \quad i = 1, 2, \cdots \), and \( \Lambda_1 \prec_{sl(1)} \Lambda_2 \), then \( S_1 \prec_{sl(n)} S_2 \).

**Proof**

In view of Theorem 7, we only need to know \( N_1 \prec_{sl(1)} N_2 \). From the proof of Theorem 3.2.5 of Goovaerts et al. (1990), \( \Lambda_1 \prec_{sl(1)} \Lambda_2 \) implies \( N_1 \prec_{sl(1)} N_2 \).
4 The Application in Risk Premium Principles

Now we cite some concepts of risk premium principles in Goovaerts et al. (1990).

We make three assumptions.
1. If \( X \preceq_{s(0)} Y \), then \( \pi[X] \cdot \pi[Y] \), with equality only if \( F_X = F_Y \).
2. If \( P[X = c] = 1, 0 \cdot c \), then \( \pi[X] = c \).
3. Let \( X, X' \) be risks such that \( \pi[X] = \pi[X'], p \in [0, 1] \), then
   \[
   \pi[pF_X + (1 - p)F_Y] = \pi[pF_{X'} + (1 - p)F_Y].
   \]

These assumptions lead to the Mean Value Principle. The premium is calculated from the formula
\[
\pi[X] = f^{-1}(E[f(X)]),
\]
for some suitable increasing continuous valuation function \( f \). For example, \( f(x) = -u(w - x) \) where \( u(x) \) is a utility function and \( w \) is the wealth of the decision maker. We can narrow the class of premium principles even further by adding the fourth requirement of additivity.

4. A premium principle \( \pi \) is called additive if for independent risk \( X \) and \( Y \),
   \[
   \pi(X + Y) = \pi(X) + \pi(Y).
   \]

From Theorem 6.2.2 in Goovaerts (1990), we can see that by the four requirments mentioned above the set of feasible premium principles is reduced to the net premium principles \( f(x) = x \) and the exponential principles \( f(x) = e^{\alpha x} \).
For net premium principle, we can not distinguish the risk $X$ and $Y$ if $E[X] = E[Y]$ but $X <_{sl(1)} Y$ (that is $Var(X) < Var(Y)$ by Theorem 5), the situation is different if we use exponential principle, from the following theorem we can see that the exponential premium principle can differentiate between losses more finely than the net premium principle under some conditions.

**Theorem 9.** Let $X$ and $Y$ be two risks. If $E[X^k] = E[Y^k], k = 1, 2, \cdots, n - 1,$ and $X <_{sl(n)} Y,$ then $\pi(X) < \pi(Y),$ under the exponential premium principle for the same $\alpha.$

**Proof**

From Theorem 5, we know that $E[X^j] < E[Y^j], j = n, n + 1, \cdots.$ Consequently,

$$\pi(X) = \frac{1}{\alpha} \ln[E[e^{\alpha X}]]$$

$$= \frac{1}{\alpha} \ln(1 + \alpha E[X] + \frac{\alpha^2}{2!}E[X^2] + \cdots + \frac{\alpha^n}{n!}E[X^n] + \cdots)$$

$$< \frac{1}{\alpha} \ln(1 + \alpha E[Y] + \frac{\alpha^2}{2!}E[Y^2] + \cdots + \frac{\alpha^n}{n!}E[Y^n] + \cdots)$$

$$= \pi(Y). \blacksquare$$
5 Concluding Remarks

The theory of partial orders of risks is interesting and useful in many fields. This paper discussed the properties of $n$th stop-loss orders. The necessary condition and the sufficient condition for the $n$th stop-loss order are convenient tools to construct risk pairs that can have $n$th stop-loss orders. The applications of these partial orders in evaluating existing risk premium principles and setting up new risk premium principles are worth further study.
References


