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WEIGHTED EXPOSURE FORMULAS

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THE natural unit of counting the exposed to risk and the deaths in a mortality investigation is the number of lives involved. As it is desirable to give consideration to the financial effects of the experience, it is customary to use other units besides or instead of lives, such as amount of insurance, number of policies, number of selections, etc. In these cases the emerging rate is, in effect, a weighted average of the rates based on lives. It may be expected that the rates resulting from the use of different weights will be different from the rates based on number of lives alone. As a matter of fact, these differences sometimes provide important clues as to the experience in question.

Although investigations involving units other than lives may furnish valuable information about the financial impact of the rates involved, such analyses provide only qualitative information as to these effects, because other important factors, such as duration, are not considered. A select and ultimate mortality investigation based on amounts, and the select and ultimate mortality rates determined therefrom, will provide information about the effects of mortality rates on the cost of insurance. Many times, however, it is desirable to determine aggregate rates from the experience, which nevertheless should be representative of the actual costs involved. Then the unit of the investigation should be something other than the number of lives, amounts, etc.; it could be, for example, the amount at risk, or in pension fund investigations the earnings times service or other factors which in the judgment of the actuary will have a direct bearing on the costs. These and other similar factors have one thing in common, namely that the unit of counting varies during the analysis period. In order to distinguish between the different types of factors considered, "unit of counting" hereinafter refers to a factor which is constant and "weight" refers to a function which varies during the observation period.

It is possible by a year-by-year analysis to consider varying weights. However, such a procedure would be too cumbersome and all the advantages of the exposure formula method would be lost. A method will be developed here which, by a combination of exposed to risk formulas, provides a comparatively simple procedure to permit consideration of weights which vary during the observation period. After the development of the method, its application will be shown on two different weight functions.

NOTATION

The weight used for the investigation is variable and it is generally a function of attained age and other factors. Therefore, the weight function, $W(x)$, will be an expression which will contain:

Constants which are the same for all lives at all ages: $A, B, C, \ldots$

Constants which are the same at all ages but different for the different lives: $a, b, c, \ldots$

Functions of age only: $f(x), g(x), h(x), \ldots$

In the exposure formula and in its components, superscripts will indicate the units of counting, or the weights; thus $E_x^{W(x)}$ is the general expression for a weighted exposed to risk formula and E_x^S indicates exposure by amounts. The symbols $s_x^{W(x)}$, $n_x^{W(x)}$, $w^{W(x)}$, etc., are interpreted similarly. A symbol without any superscript indicates that the unit of counting is the number of lives. Otherwise, the notations of Mr. E. W. Marshall's paper, "Principles Underlying Exposed to Risk Formulae" (*TASA* XLVI, 10), are used.

GENERAL METHOD

The exposed to risk formula based on lives, underlying the weighted exposure formula, can be selected without any restriction; the choice of formula as always will depend upon the circumstances, as to the form in which the experience data are available. For illustrative purposes, a mortality investigation is assumed, using the policy year formula (formula (2) in Mr. Marshall's paper), but the method is applicable to any other investigation or formula. In order to avoid confusion in the use of the symbol a the summation is assumed to begin at age 0. The basic exposure formula:

$$E_{\underline{x}|} = \sum_0^x (s_y + n_y - w_{|y|} - e_y) - \sum_0^{x-1} \theta_{\underline{y}|}$$

or in recursive form the underlying basic formula:

$$E_{\underline{x}|} = E_{\underline{x-1}|} + s_x + n_x - w_{|x|} - e_x - \theta_{\underline{x-1}|} .$$

There are two basically different approaches in the development of the weighted exposure formulas; one may be called the "attained age method," the other the "recursive method."

1. Attained Age Method

The weight function generally is a linear function of constants and functions of the attained age. A simple weight function could be written in the following form for attained age x:

$$W(x) = A + a + B \cdot f(x) + b \cdot g(x) .$$

The weighted exposure formula for attained age x:

$$E_{\underline{x}|}^{W(x)} = E_{\underline{x-1}|}^{W(x)} + s_x^{W(x)} + n_x^{W(x)} - w_{|x|}^{W(x)} - e_x^{W(x)} - \theta_{\underline{x-1}|}^{W(x)}.$$

In this expression $W(x)$ means the weight at attained age x regardless of the age indicated in the subscript.

Using the assumed form of the weight function the weighted number of survivors can be expressed as:

$$s_x^{W(x)} = A \cdot s_x + s_x^a + B \cdot f(x) \cdot s_x + g(x) \cdot s_x^b .$$

Since $A, B, f(x)$ and $g(x)$ are the same for all lives at the same attained age the number of lives surviving may be multiplied by them. a and b, on the other hand, being different for the individual lives, have to be used as the unit of counting, and thus the expression for $s_x^{W(x)}$ becomes self-evident. Similarly

$$n_x^{W(x)} = A \cdot n_x + n_x^a + B \cdot f(x) \cdot n_x + g(x) \cdot n_x^b$$
$$w_{|x|}^{W(x)} = A \cdot w_{|x|} + w_{|x|}^a + B \cdot f(x) \cdot w_{|x|} + g(x) \cdot w_{|x|}^b$$
$$e_x^{W(x)} = A \cdot e_x + e_x^a + B \cdot f(x) \cdot e_x + g(x) \cdot e_x^b .$$

Although the expression of deaths appears in the formula at age $x - 1$, the weight function components for age x are to be used, as indicated before:

$$\theta_{\underline{x-1}|}^{W(x)} = A \cdot \theta_{\underline{x-1}|} + \theta_{\underline{x-1}|}^a + B \cdot f(x) \cdot \theta_{\underline{x-1}|} + g(x) \cdot \theta_{\underline{x-1}|}^b .$$

Collecting the terms with the same superscript, the weighted exposure formula can be written as:

$$\begin{aligned} E_{\underline{x}|}^{W(x)} = E_{\underline{x-1}|}^{W(x)} + [A + B \cdot f(x)] \cdot (s_x + n_x - w_{|x|} - e_x - \theta_{\underline{x-1}|}) \\ + (s_x^a + n_x^a - w_{|x|}^a - e_x^a - \theta_{\underline{x-1}|}^a) \\ + g(x) \cdot (s_x^b + n_x^b - w_{|x|}^b - e_x^b - \theta_{\underline{x-1}|}^b) . \end{aligned}$$

Remembering that $W(x)$ is the weight function for age x even if the subscript refers to $x - 1$ or any other age, similar expressions can be written for $E^{W(x)}_{\underline{x-1|}}$, $E^{W(x)}_{\underline{x-2|}}$, $E^{W(x)}_{\underline{x-3|}}$, etc., down to the first age in the analysis. Adding vertically both sides of these equations, the $E^{W(x)}_{\underline{x-1|}}$, $E^{W(x)}_{\underline{x-2|}}$, $E^{W(x)}_{\underline{x-3|}}$, etc., values will cancel on both sides and the following equation will result:

$$E^{W(x)}_{\underline{x|}} = [A + B \cdot f(x)] \cdot \left[\sum_{0}^{x} (s_y + n_y - w_{|y|} - e_y) - \sum_{0}^{x-1} \theta_{\underline{y|}}\right] + \left[\sum_{0}^{x} (s^a_y + n^a_y - w^a_{|y|} - e^a_y) - \sum_{0}^{x-1} \theta^a_{\underline{y|}}\right] + g(x) \cdot \left[\sum_{0}^{x} (s^b_y + n^b_y - w^b_{|y|} - e^b_y) - \sum_{0}^{x-1} \theta^b_{\underline{y|}}\right].$$

Recognizing that the expressions in the brackets are basic exposed to risk formulas, the first based on lives, the other two based on the constants a and b respectively, the final attained age expression for the weighted exposed to risk is:

$$E^{W(x)}_{\underline{x|}} = [A + B \cdot f(x)] E_{\underline{x|}} + E^a_{\underline{x|}} + g(x) \cdot E^b_{\underline{x|}}.$$

By this formula the exposed to risk using varying weights can be simply expressed as the algebraic sum of three exposure formulas of constant weights. Using punch card equipment, the a and b quantities are to be punched on the individual cards, then three separate columns of exposures are determined and finally using the proper coefficients, which vary by age, the three exposures are combined for each attained age to result in the desired weighted exposure. The weighted expression of deaths is similarly determined from the same worksheets.

2. Recursive Method

It is possible that the weight function cannot be expressed conveniently as a function of the attained age but a more simple expression can be found in recursive form. In such cases the attained age method cannot be used, but it is possible to find an expression for the exposed to risk with varying weights.

A general recursive form of the weight function could be:

$$W(x) = h(x) \cdot W(x-1) + A + a + B \cdot f(x) + b \cdot g(x).$$

The weighted exposure formula for attained age x is again:

$$E^{W(x)}_{\underline{x|}} = E^{W(x)}_{\underline{x-1|}} + s^{W(x)}_x + n^{W(x)}_x - w^{W(x)}_{|x|} - e^{W(x)}_x - \theta^{W(x)}_{\underline{x-1|}}.$$

Substituting the recursive form of the weight function in $E^{W(x)}_{\underline{x-1|}}$,

$$E^{W(x)}_{\underline{x-1|}} = h(x) \cdot E^{W(x-1)}_{\underline{x-1|}} + [A + B \cdot f(x)] E_{\underline{x-1|}} + E^a_{\underline{x-1|}} + g(x) \cdot E^b_{\underline{x-1|}},$$

and the desired formula:

$$E_{\underline{x}|}^{W(x)}=h(x)\cdot E_{\underline{x-1}|}^{W(x-1)}+[A+B\cdot f(x)]E_{\underline{x-1}|}+E_{\underline{x-1}|}^{a}+g(x)\cdot E_{\underline{x-1}|}^{b}$$

$$+s_{x}^{W(x)}+n_{x}^{W(x)}-w_{|x|}^{W(x)}-e_{x}^{W(x)}-\theta_{\underline{x-1}|}^{W(x)}.$$

Thus the weighted exposure at attained age x may be determined by the recursive method from the weighted exposure at attained age $x-1$ by adjusting the weights for age x and then adjusting the exposure for age x. According to this formula all entries and exits are to be tabulated with their weights at the time of the event, except the deaths which are to be tabulated with their weights at the age following the age at death. In order to conveniently tabulate the data, it is desirable that all events should be tabulated with their weights at their time of occurrence; therefore the recursive form of weight function is applied to $\theta_{\underline{x-1}|}^{W(x)}$ also:

$$E_{\underline{x}|}^{W(x)}=h(x)\cdot E_{\underline{x-1}|}^{W(x-1)}+[A+B\cdot f(x)]E_{\underline{x-1}|}+E_{\underline{x-1}|}^{a}+g(x)\cdot E_{\underline{x-1}|}^{b}$$

$$+s_{x}^{W(x)}+n_{x}^{W(x)}-w_{|x|}^{W(x)}-e_{x}^{W(x)}-h(x)\cdot\theta_{\underline{x-1}|}^{W(x-1)}$$

$$-[A+B\cdot f(x)]\,\theta_{\underline{x-1}|}-\theta_{\underline{x-1}|}^{a}-g(x)\cdot\theta_{\underline{x-1}|}^{b}.$$

The exposed to risk formula with varying weights is expressed again as the combination of three exposure formulas—one of them based on lives, the two others based on the constants a and b respectively—and a tabulation of the entries and exits. The exposure formula based on lives can be selected in any form. The weighted exposure formulas can be developed for any form on the lines given here.

Generally, if the attained age method can be used, the resulting weighted exposure can be expressed in terms of constants, functions of attained age, and exposures based on some unit of counting, so the choice of the underlying formula will affect the exposed to risk formulas but not the manner in which they have to be combined.

If the recursive method has to be used, the resulting formulas may be somewhat more complicated; nevertheless, the work required is still less than that for a year-by-year analysis.

The most complicated formula will result if the general exposed to risk formula is used:

$$E_{\underline{x}|}=E_{\underline{x-1}|}+(1-{}_{s}f)\cdot s_{\underline{x-1}|}+(1-{}_{n}f)\cdot n_{\underline{x-1}|}-(1-{}_{w}f)\cdot w_{x-1|}$$

$$-(1-{}_{e}f)\cdot e_{x-1|}+{}_{s}f\cdot s_{\underline{x}|}+{}_{n}f\cdot n_{x|}-{}_{w}f\cdot w_{x|}-{}_{e}f\cdot e_{\underline{x}|}-\theta_{x-1|}.$$

Assuming the same form of the recursive weight function as above, the functions for age $x - 1$ are to be changed to incorporate the weights for that age:

$$\begin{aligned}
E_{\underline{x}|}^{W(x)} = {} & h(x) \cdot E_{\underline{x-1}|}^{W(x-1)} + [A + B \cdot f(x)] E_{\underline{x-1}|} + E_{\underline{x-1}|}^{a} + g(x) \cdot E_{\underline{x-1}|}^{b} \\
& + (1 - {}_{s}f) \{ h(x) \cdot s_{\underline{x-1}|}^{W(x-1)} + [A+B \cdot f(x)]\, s_{\underline{x-1}|} + s_{\underline{x-1}|}^{a} + g(x) \cdot s_{\underline{x-1}|}^{b} \} \\
& + (1 - {}_{n}f) \{ h(x) \cdot n_{\underline{x-1}|}^{W(x-1)} + [A+B \cdot f(x)]\, n_{\underline{x-1}|} + n_{\underline{x-1}|}^{a} + g(x) \cdot n_{\underline{x-1}|}^{b} \} \\
& - (1 - {}_{w}f) \{ h(x) \cdot w_{\underline{x-1}|}^{W(x-1)} + [A+B \cdot f(x)]\, w_{\underline{x-1}|} + w_{\underline{x-1}|}^{a} + g(x) \cdot w_{\underline{x-1}|}^{b} \} \\
& - (1 - {}_{e}f) \{ h(x) \cdot e_{\underline{x-1}|}^{W(x-1)} + [A+B \cdot f(x)]\, e_{\underline{x-1}|} + e_{\underline{x-1}|}^{a} + g(x) \cdot e_{\underline{x-1}|}^{b} \} \\
& - \{ h(x) \cdot \theta_{\underline{x-1}|}^{W(x-1)} + [A+B \cdot f(x)]\, \theta_{\underline{x-1}|} + \theta_{\underline{x-1}|}^{a} + g(x) \cdot \theta_{\underline{x-1}|}^{b} \} \\
& \qquad + {}_{s}f \cdot s_{\underline{x}|}^{W(x)} + {}_{n}f \cdot n_{\underline{x}|}^{W(x)} - {}_{w}f \cdot w_{\underline{x}|}^{W(x)} - {}_{e}f \cdot e_{\underline{x}|}^{W(x)} .
\end{aligned}$$

APPLICATION OF THE METHOD

The practical application of the methods of developing weighted exposure formulas will be illustrated on two examples. Although in both cases the attained age method is applicable, for illustrative purposes the solution by the recursive method will be indicated also.

1. Mortality Rates Weighted by Duration for Pension Fund Valuation Purposes

In many pensions plans the benefit accrued can be expressed as the product of accrued service and annual earnings. If the benefit is based on final earnings, then this product is to be multiplied by an attained age factor which forecasts the final earnings from the current earnings. As, however, the deaths in the numerator of the mortality rate are to be multiplied by the same factor, the multiplication need not be carried through at all. In pension plans where the benefit is based on the total earnings, this product represents the accrued benefits only if it can be assumed that earnings were constant in the past. This is generally not true, and especially not true now after an inflationary period. However, for the purposes of deriving mortality or withdrawal rates for pension fund valuation purposes, it may be assumed that the present level of earnings will prevail in the future. The present level of individual earnings is known for all persons existing at the end of the observation period, but for persons terminated for any reason during the analysis period this information is not available. In order that the salary information should refer to the same level whether an individual is existing or terminated, the salaries of persons terminated are to be adjusted. This adjustment may be made by the use of a single adjustment factor which is a function of the calendar year of termination and which can be derived from general statistical data or from the experience of the fund itself. Actually, by using the accrued benefits as a weight, the cost of such benefit which will be released is considered, because it is the product of the accrued benefits and the deferred annuity factor; the latter, however, need not be considered since it would appear both in the numerator and in the denominator of the resulting rate.

In the following development, S represents the salary of an individual, which has been adjusted to the level of earnings at the end of the analysis period if the individual terminated during the period; n is the average duration of employment during the year of exposure; and x_e is the age at entrance into the fund.

The weight function:

$$W(x) = n \cdot S = (x + \tfrac{1}{2} - x_e)\, S = (x + \tfrac{1}{2})\, S - x_e \cdot S$$

The attained age exposure formula is:

$$E^{W(x)}_{x]} = (x + \tfrac{1}{2}) \cdot E^{S}_{x]} - E^{(x_e \cdot S)}_{x]} .$$

The practical procedure is simple. Only the earnings and the product of earnings (adjusted for the current level) and entry age are to be tabulated, then the two exposures are determined and combined according to the formula.

The weight function may be defined alternatively by a recursive relationship:

$$W(x) = W(x-1) + S$$

and then the employment anniversary exposure formula for mortality rates is:

$$E^{W(x)}_{\underline{x}]} = E^{W(x-1)}_{\underline{x-1}]} + E^{S}_{\underline{x-1}]} + s^{W(x)}_{x} + n^{W(x)}_{x} - w^{W(x)}_{|x|} - e^{W(x)}_{x} - \theta^{W(x)}_{\underline{x-1}|} .$$

If the weights are used as of the middle of the age, then the product of the duration and earnings of survivors, new entrants, existing and withdrawals is to be entered for half a year higher duration than at the last anniversary and for deaths for one and one-half years higher than at the last anniversary, and $n^{W(x)}_{x} = \frac{1}{2} n^{S}_{x}$. The formula can be further simplified if $\theta^{W(x)}_{\underline{x-1}|}$ is expressed by the recursive form of the weight function:

$$E^{W(x)}_{x]} = E^{W(x-1)}_{\underline{x-1}]} + E^{S}_{\underline{x-1}]} - \theta^{S}_{\underline{x-1}]} + s^{W(x)}_{x} + \tfrac{1}{2} n^{S}_{x} - w^{W(x)}_{|x|} - e^{W(x)}_{x} - \theta^{W(x-1)}_{\underline{x-1}]} .$$

In this expression, the weights are to be used for the age when the event occurred; *i.e.*, the product of the duration and earnings are taken for half a year higher than at the last anniversary.

The actual calculations may be made on one worksheet, containing the following columns:

(1) x

(2) s_x^S

(3) n_x^S

(4) $w_{|x|}^S$

(5) e_x^S

(6) $\theta_{\underline{x-1}}^S$

(7) $E_x^S = (7)_{x-1} + (2)_x + (3)_x - (4)_x - (5)_x - (6)_x$

(8) $s_x^{W(x)}$

(9) $w_{|x|}^{W(x)}$

(10) $e_x^{W(x)}$

(11) $\theta_{\underline{x-1}|}^{W(x-1)}$

(12) $E_{\underline{x}|}^{W(x)} = (12)_{x-1} + \frac{1}{2}(3)_x - (6)_x + (7)_{x-1} + (8)_x - (9)_x - (10)_x - (11)_x$

Using the product of service and earnings as the weight, as in almost all cases, the attained age method results in a more simple procedure than the recursive method.

2. Mortality Rates Weighted by the Amount at Risk

The financial effects of mortality can be expressed by the cost of insurance for the amount at risk. It appears to be reasonable that this amount should be used as the weight of the mortality rate. As the assumed mortality rate in the expression of the cost of insurance is to be the same in both the numerator and the denominator of the mortality rate to be derived, the weight to be considered is the amount at risk.

If the Karup attained age valuation method is used, the valuation constants are available on the individual valuation cards and the weighted exposed to risk may be conveniently calculated by the attained age method. The amount at risk:

$$W(x) = (1 - A_{x+1})\, S + \ddot{a}_{x+1} \cdot P^S - \frac{1}{D_{x+1}} \cdot T ,$$

where T is the valuation constant. The exposed to risk formula:

$$E_{\underline{x}|}^{W(x)} = (1 - A_{x+1})\, E_{\underline{x}|}^S + \ddot{a}_{x+1} \cdot E_{\underline{x}|}^{P^S} - \frac{1}{D_{x+1}} \cdot E_{\underline{x}|}^T .$$

The objection may be raised that the reserve already includes a mortality assumption and actually the mortality rate to be derived should be used in the weight function. Remembering, however, that the amount at risk is being used only as a weight, the reserve calculated by some mortality assumptions will be acceptable to indicate the relative magnitude of the individual weights.

If the valuation constants are not available, the recursive method can be used. The calculation is simplified if first an exposed to risk formula based on the terminal reserve is derived:

$$W(x) = V_{x+1}^{S} = (V_x^S + P^S)\, u_x - S \cdot k_x ,$$

where V_{x+1}^{S} indicates the individual reserve at age $x+1$; u_x and k_x are the Fackler valuation functions. The basic formulas are as follows:

The recursive weight function:

$$W(x) = u_x\,[W(x-1) + P^S] - k_x \cdot S .$$

The exposed to risk formula based on the terminal reserve:

$$E_{\underline{x}}^{W(x)} = u_x\,[E_{\underline{x-1}|}^{W(x-1)} + E_{\underline{x-1}|}^{P^S}] - k_x \cdot E_{\underline{x-1}|}^{S} + s_x^{W(x)} + n_x^{W(x)} - w_{|x|}^{W(x)} - e_x^{W(x)} - \theta_{\underline{x-1}}^{W(x)} .$$

As it is desirable that the weights for age $x-1$ should be used for the deaths, $\theta_{\underline{x-1}|}^{W(x)}$ is expressed in recursive form also:

$$\theta_{\underline{x-1}|}^{W(x)} = u_x\,[\theta_{\underline{x-1}|}^{W(x-1)} + \theta_{\underline{x-1}}^{P^S}] - k_x \cdot \theta_{\underline{x-1}}^{S} .$$

Substituting this in the exposure based on the terminal reserve:

$$E_{\underline{x}}^{W(x)} = u_x\,[E_{\underline{x-1}}^{W(x-1)} - \theta_{\underline{x-1}}^{W(x-1)} + E_{\underline{x-1}|}^{P^S} - \theta_{\underline{x-1}|}^{P^S}] - k_x\,[E_{\underline{x-1}|}^{S} - \theta_{\underline{x-1}}^{S}] + s_x^{W(x)} + n_x^{W(x)} - w_{|x|}^{W(x)} - e_x^{W(x)} ,$$

and the formula for the exposed to risk based on the amount at risk:

$$E_{\underline{x}|}^{A.R.} = E_{\underline{x}|}^{S} - E_{\underline{x}|}^{W(x)} .$$

In the expression of $E_{\underline{x}|}^{W(x)}$ the weight for all entrants and exits is the terminal reserve at the end of the policy year in which the event occurred. The actual computations can be made on one worksheet along the same lines as indicated in the previous example.

CONCLUSION

An attempt has been made here to develop methods by which exposure formulas with varying weights may be determined. The methods indicated here do not cover all possibilities as to the forms of weight functions and exposure formulas. These methods, however, suggest means of approach for determining weighted exposure formulas. It is not intended that the weights used in the examples be considered as being definitely desirable to use for actual analyses. They should be considered only as illustrations of the application of the methods.

In general, the procedure for determining weighted experience rates consists of the following steps:

1. Select a basic exposure formula using lives as units of counting E_x.

2. After a decision is made as to the weights to be used, determine the weight function, $W(x)$, as a function of age if possible.

3. If the weight function can be expressed as a function of age, the weighted exposure formula, $E_x^{W(x)}$, may be obtained by the attained age method as a composite of exposure formulas, using (*a*) the constants which are different for different lives as units of counting to determine exposures based thereon, and (*b*) the constants which are the same for all lives, together with the functions of age, as factors to multiply the exposures thus derived.

4. If the weight function cannot be expressed as a function of age, but a recursive relationship between the different weights can be established, $E_x^{W(x)}$ may be obtained by substituting the recursive expression of the weight function in $E_{x-1}^{W(x)}$, and if practicable, by substituting such expression in any other function where the subscript and superscript of the symbol refer to different ages.

5. After the weighted exposure is determined, if the attained age method is used the numerators of the rates to be derived may be obtained by the same relationship as used for the weighted exposure, or, alternatively, if the recursive method is used the numerators will be a by-product of the derivation of the exposures.