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INTERPOLATION COMMUTATION COLUMNS

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The popularity of monthly payment insurance plans makes interpolated policy values more important than ever before. Also, electronic data processing machinery has made some unusual actuarial formulas useful. The interpolation commutation columns developed below can be used to calculate interpolated reserves and interpolated cash values by formulas which seem to fit into these recent developments. Interpolation commutation column formulas are analogous to regular commutation column formulas, and their use is therefore easy to explain and to "program."

Straight line interpolated values

The interpolated value at age $x+f$ of a single premium pure endowment, when f is a positive fraction, is

$$(1-f)\,A_{x:\overset{1}{\overline{n|}}}+fA_{x+1:\overset{1}{\overline{n-1|}}}=(1-f)\frac{D_{x+n}}{D_x}+f\cdot\frac{D_{x+n}}{D_{x+1}}.$$

If we derive a value of a special "D" such that this interpolated value for $A_{x+f:\overset{1}{\overline{n-f|}}}$ is equal to D_{x+n}/"D", then this value of "D" can be used with the usual commutation columns to produce pure endowments maturing at any integral age after x. The value of any life, endowment, term or annuity benefit *commencing at age $x+1$* is equivalent to a *corresponding* amount of pure endowment maturing at age $x+1$, and the *corresponding* amounts are in the proportions that D_{x+1}, M_{x+1}, $M_{x+1}-M_{x+n}+D_{x+n}$, $M_{x+1}-M_{x+n}$, and N_{x+n} bear to D_{x+n}. Hence the interpolated value at age $x+f$ of benefits *commencing at age $x+1$* can be expressed, in terms of this "D", as M_{x+1}/"D" (Life, commencing at $x+1$), $(M_{x+1}-M_{x+n})$/"D", $(M_{x+1}+D_{x+n}-M_{x+n})$/"D", etc.

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A special value of "C" can be added to M_{x+1} to produce a special value "M" which when combined with values such as M_{x+n} and D_{x+n} and divided by the special "D" will give the interpolated single premium insurance values at age $x + f$. These single premiums can then be used to obtain interpolated cash values and interpolated reserves on premium paying insurance as well as on paid-up policies.

Notation

Interpolated values obtained by straight line interpolation generally are accepted as being exact values of single premiums, but the equation is properly expressed in the form $A_{x+f:\overline{n-f}|}^{\,1} \doteqdot (1-f)\,A_{x:\overline{n}|}^{\,1} + fA_{x+1:\overline{n-1}|}^{\,1}$ to indicate that it is not an exact equality. Furthermore, the commutation columns which produce straight line interpolated values are not straight line interpolated commutation columns. Hence a special notation is needed. For this purpose we have borrowed the two dots from the $\doteqdot$ symbol to give the symbols $\dot{D}^{\cdot}_{x+f}$, $\dot{C}^{\cdot}_{x+f}$, $\dot{M}^{\cdot}_{x+f}$, $\dot{N}^{\cdot}_{x+f}$, $\dot{A}^{\cdot\,1}_{x+f:\overline{n-f}|}$, $\dot{A}^{\cdot}_{x+f:\overline{n-f}|}^{\;\;1}$, and $\dot{A}^{\cdot}_{x+f:\overline{n-f}|}$.

The special commutation columns are such that they can be used to reproduce the usual interpolated single premiums and values. Thus

$$\frac{\dot{C}^{\cdot}_{x+f}}{\dot{D}^{\cdot}_{x+f}} = (1-f)\,A_{x:\overline{1}|}^{\,1} = \dot{A}^{\cdot\,1}_{x+f:\overline{1-f}|}$$

$$\frac{\dot{M}^{\cdot}_{x+f}}{\dot{D}^{\cdot}_{x+f}} = (1-f)\,A_x + fA_{x+1} = \dot{A}^{\cdot}_{x+f}$$

$$\frac{\dot{N}^{\cdot}_{x+f}}{\dot{D}^{\cdot}_{x+f}} = (1-f)\,\ddot{a}_x + f\ddot{a}_{x+1}\,.$$

$(1-f)\ddot{a}_x + f\ddot{a}_{x+1} - (1-f)$, as used in mean reserve calculations, is $\frac{N_{x+1}}{\dot{D}^{\cdot}_{x+f}}$, where $f = \frac{1}{2}$.

Derivation

To derive the value of $\dot{D}^{\cdot}_{x+f}$ we start with the equations, $\dot{A}^{\cdot}_{x+f:\overline{1-f}|}^{\;\;1} = (1-f)\,A_{x:\overline{1}|}^{\;\;1} + f$ (by definition) and $\frac{D_{x+1}}{\dot{D}^{\cdot}_{x+f}} = A_{x+f:\overline{1-f}|}^{\;\;1}$ (by definition of $\dot{D}^{\cdot}_{x+f}$). Hence,

$$\dot{D}^{\cdot}_{x+f} = D_{x+1} \div [(1-f)\,A_{x:\overline{1}|}^{\;\;1} + f] = D_{x+1} \div [(1-f)\,vp_x + f]\,.$$

One method of calculating $\dot{D}^{\cdot}_{x+f}$ is to find its reciprocal by interpolating between consecutive reciprocals of D_x. However, by an operation comparable to multiplying the next year's reserve value (or dividing D_{x+1}, which appears in the denominator of that reserve value) by a discount factor in the process of producing a current value, D_{x+1} is divided by a discount factor for a fraction of the year to produce the current interpolated $^{\cdot}D^{\cdot}_{x+f}$ function. In essence, we simply incorporate the reciprocal of the discount factor in the denominator when we take $^{\cdot}D^{\cdot}_{x+f}$ as $D_{x+1} \div [(1-f)vp_x + f]$.

To derive the value of $^{\cdot}C^{\cdot}_{x+f}$ we have

$$\frac{^{\cdot}C^{\cdot}_{x+f}}{^{\cdot}D^{\cdot}_{x+f}} = \frac{(1-f)\,C_x}{D_x}, \qquad {}^{\cdot}C^{\cdot}_{x+f} = \frac{^{\cdot}D^{\cdot}_{x+f}}{D_x}(1-f)\,C_x\,.$$

To derive $^{\cdot}M^{\cdot}_{x+f}$ we either take $^{\cdot}M^{\cdot}_{x+f} = M_{x+1} + {}^{\cdot}C^{\cdot}_{x+f}$, or take

$$\frac{^{\cdot}M^{\cdot}_{x+f}}{^{\cdot}D^{\cdot}_{x+f}} = {}^{\cdot}A^{\cdot}_{x+f}$$

and solve for $^{\cdot}M^{\cdot}_{x+f}$ (the values are identical).

Calculation

Table 1 shows the calculations, on the 1941 CSO 3% basis, of $^{\cdot}D^{\cdot}_{35+f}$ and $^{\cdot}C^{\cdot}_{35+f}$, and of $^{\cdot}M^{\cdot}_{35+f}$ by both formulas.

For the annuity used in the annual premium assumption for obtaining interpolated Ordinary insurance reserve values, such as mean reserves, we have

$$(1-f)\,\ddot{a}_x + f\ddot{a}_{x+1} - (1-f) = \frac{N_{x+1}}{^{\cdot}D^{\cdot}_{x+f}}\,.$$

For the annuity used in values such as midterminal reserves in industrial insurance and interpolated cash values, we have

$$\frac{N_{x+1} + (1-f)\,{}^{\cdot}D^{\cdot}_{x+f}}{^{\cdot}D^{\cdot}_{x+f}} = \frac{^{\cdot}N^{\cdot}_{x+f}}{^{\cdot}D^{\cdot}_{x+f}}\,.$$

The difference between $^{\cdot}N^{\cdot}{}_{+f}/{}^{\cdot}D^{\cdot}_{x+f}$ and $N_{x+1}/{}^{\cdot}D^{\cdot}_{x+f}$ is simply $1-f$. An $^{\cdot}N^{\cdot}_{x+f}$ table for general purposes would therefore not be very useful, and might be confusing.

Illustrations

Table 2 illustrates the use of interpolation commutation columns in calculating interpolated reserves and interpolated minimum cash values. For these illustrations the 1941 CSO 3% values are calculated for a Term to Age 65 policy issued at age 30. The reserves are net level. The attained age is $35 + f$ so that the policy is in its sixth policy year.

TABLE 1

	$f=\frac{1}{4}$	$f=\frac{1}{2}$
(1) D_{36}	311,354.85	311,354.85
(2) vP_{35}	.96641756	.96641756
(3) $(1-f)vP_{35}+f$	.97481317	.98320878
(4) $(1)\div(3)={}^{\cdot}D^{\cdot}_{35+f}$	319,399.51	316,672.16
(5) $C_{35}\div D_{35}$	.00445623	.00445623
(6) ${}^{\cdot}D^{\cdot}_{35+f}$	319,399.51	316,672.16
(7) $(1-f)\cdot(5)\cdot(6)={}^{\cdot}C^{\cdot}_{35+f}$	1,067.49	705.58
(8) M_{36}	126,301.77	126,301.77
(9) ${}^{\cdot}C^{\cdot}_{35+f}$	1,067.49	705.58
(10) $(8)+(9)={}^{\cdot}M^{\cdot}_{35+f}$	127,369.26	127,007.35
(11) A_{35}	.39648559	.39648559
(12) A_{36}	.40565217	.40565217
(13) $(1-f)\cdot(11)+f\cdot(12)={}^{\cdot}A^{\cdot}_{35+f}$	.39877724	.40106888
(14) ${}^{\cdot}D^{\cdot}_{35+f}$	319,399.51	316,672.16
(15) $(13)\cdot(14)={}^{\cdot}M^{\cdot}_{35+f}$	127,369.26	127,007.35

TABLE 2

	$f=\frac{1}{4}$	$f=\frac{1}{2}$
Single Premium Term Insurance		
(16) M_{65}	60,522.47	60,522.47
(17) $({}^{\cdot}M^{\cdot}_{35+f}-M_{65})\div{}^{\cdot}D^{\cdot}_{35+f}={}^{\cdot}A^{\cdot\,1}_{35\ f:\overline{30-f}\rceil}$	.2092889	.2099486
(18) $A^{1}_{35:\overline{30}\rceil}$	.2086293	.2086293
(19) $A^{1}_{36:\overline{29}\rceil}$	.2112679	.2112679
(20) $(1-f)\cdot(18)+f\cdot(19)={}^{\cdot}A^{\cdot\,1}_{35+f:\overline{30-f}\rceil}$	.2092889	.2099486
Single Premium Annuity		
(21) N_{36}	6,353,489.0	6,353,489.0
(22) N_{65}	826,990.9	826,990.9
(23) $(N_{36}-N_{65})\div{}^{\cdot}D^{\cdot}_{35+f}$	17.302 776	17.451 796
(24) $a_{35:\overline{29}\rceil}=\ddot{a}_{35:\overline{30}\rceil}-1.00$	17.153 755	17.153 755
(25) $\ddot{a}_{36:\overline{29}\rceil}$	17.749 838	17.749 838
(26) $(1-f)a_{35:\overline{29}\rceil}+f\ddot{a}_{36:\overline{29}\rceil}$	17.302 776	17.451 796
Interpolated Reserve		
(27) $1{,}000P^{1}_{30:\overline{35}\rceil}$	9.69656	9.69656
(28) $1{,}000\cdot(17)-(23)\cdot(27)$	41.51	40.73
(29) $1{,}000\,{}_5V^{1}_{30:\overline{35}\rceil}$	32.60	32.60
(30) $1{,}000\,{}_6V^{1}_{30:\overline{35}\rceil}$	39.16	39.16
(31) $(1-f)[(27)+(29)]+f\cdot(30)=1{,}000\,{}_{5+f}{}^{\cdot}V^{\cdot\,1}_{30:\overline{35}\rceil}$	41.51	40.72

In calculating an interpolated Minimum Cash Value from interpolation commutation columns it is possible to calculate an ${}^{\cdot}N^{\cdot}_{x+f}$ and hence an "$\ddot{a}^{\cdot}_{x+f}$." However, there are various kinds of interpolated annuities and, as previously mentioned, it seems inadvisable in general to calculate an ${}^{\cdot}N^{\cdot}_{x+f}$ (which could be misused) except for special purposes. Table 3 shows the necessary calculations, but with the ${}^{\cdot}N^{\cdot}_{x+f}$ not actually shown in the illustration.

Interpolation commutation columns can be used to produce interpolated values of accumulated costs and forborne annuities for calculation of reserves. This is illustrated in Table 4.

The forborne annuity derived in Table 4 is the type used in calculating

TABLE 3

	$f=\frac{1}{4}$	$f=\frac{1}{2}$
Annuity		
(32) $N_{36}+(1-f)\cdot{}^{\cdot}D^{\cdot}_{35+f}-N_{65}$	5,776,047.7	5,684,834.2
(33) Annuity (32) $\div{}^{\cdot}D^{\cdot}_{35+f}$	18.052 776	17.951 796
Minimum Value		
(34) $(23)+(1-f)=(1-f)\ddot{a}_{35:\overline{30}\rceil}+f\ddot{a}_{36:\overline{29}\rceil}$	18.052 776	17.951 796
(35) $1{,}000\,{}^{A}P^{1}_{30:\overline{35}\rceil}$	11.05329	11.05329
(36) $1{,}000\cdot(17)-(33)\cdot(35)={}^{\text{min}}_{5+f}{}^{\cdot}V^{\cdot 1}_{30:\overline{35}\rceil}$	9.75	11.52
(37) ${}^{\text{min}}_{5}V^{1}_{30:\overline{35}\rceil}$	7.97	7.97
(38) ${}^{\text{min}}_{6}V^{1}_{30:\overline{35}\rceil}$	15.07	15.07
(39) $(1-f)\cdot(37)+f\cdot(38)$	9.74	11.52

TABLE 4

	$f=\frac{1}{4}$	$f=\frac{1}{2}$
Accumulated Costs		
(40) M_{30}	134,532.03	134,532.03
(41) $1{,}000(M_{30}-{}^{\cdot}M^{\cdot}_{35+f})\div{}^{\cdot}D^{\cdot}_{35+f}$	22.42574	23.76173
(42) $1{,}000\ {}_{5}k_{30}$	21.08975	21.08975
(43) $1{,}000\ {}_{6}k_{30}$	26.43367	26.43367
(44) $(1-f)\cdot(42)+f\cdot(43)=1{,}000\ {}_{5+f}k^{\cdot}_{30}$	22.42573	23.76171
Forborne Annuity		
(45) N_{30}	8,459,549.3	8,459,549.3
(46) $(N_{30}-N_{36})\div{}^{\cdot}D^{\cdot}_{35+f}$	6.59381	6.65060
(47) ${}_{5}u_{30}+1.00$	6.53702	6.53702
(48) ${}_{6}u_{30}$	6.76418	6.76418
(49) $(1-f)\cdot(47)+f\cdot(48)={}_{5+f}u^{\cdot}_{30}$	6.59381	6.65060
(50) $(27)\cdot(46)-(44)=1{,}000\ P^{1}_{31:\overline{35}\rceil}\,{}_{5+f}u^{\cdot}_{30}-1{,}000\,{}_{5+f}k^{\cdot}_{30}$	41.51	40.73

mean reserves, but not that used in calculating midterminal reserves or interpolated cash values. The mean reserves are the same reserves as were calculated above, in (31).

Special Cases

Interpolation commutation columns can be used to determine the term of extended insurance by solving for $t - f$ in the single premium term insurance formula,

$$\cdot V^{\cdot}_{x+f} = \frac{1,000\ (\cdot M^{\cdot}_{x+f} - M_{x+t})}{\cdot D^{\cdot}_{x+f}},$$

or to calculate the amount of pure endowment following extended insurance by solving for k in the equation

$$\cdot V^{\cdot}_{x+f} = \frac{1,000\ (\cdot M^{\cdot}_{x+f} - M_{x+n}) + kD_{x+n}}{\cdot D^{\cdot}_{x+f}}.$$

The attained age valuation formula for mean reserves, which uses $\frac{1}{2}(1/D_x + 1/D_{x+1})$ is a special case of the use of the $\cdot D^{\cdot}_{x+f}$ function.

Conclusion

Calculations by commutation columns are a common actuarial procedure. Interpolation commutation columns can be used with regular commutation columns in a "program" for clerical employees or for machines which (as compared to interpolation between single premiums or between reserve values) reduces reference to tables or to "memory." The "program," or the "instructions," are analogous to instructions using regular commutation columns. The machine operations are practically the same as for ordinary commutation column calculations. The relative availability of underlying factors is an important consideration in comparing the two methods. Because of variations in the meaning of interpolated annuity values, the calculation of a special value of $\cdot N^{\cdot}_{+f}$ for general use appears to be inadvisable.