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# THE UNFUNDED PRESENT VALUE FAMILY OF PENSION FUNDING METHODS

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## INTRODUCTION

In an earlier paper[1] published in the 1952 *Transactions*, the author has described and classified various methods of pension funding. Other writers[2] have also considered pension funding methods and outlined the methods commonly employed today. It is obvious that these traditional methods do not exhaust the possibilities, and that the ingenious human mind can, without too much difficulty, devise other funding arrangements. Some of these might well have highly satisfactory characteristics.

The introduction of new funding methods just to add to an already long list would serve no useful purpose. However, it appears that a particular funding concept, to be described in this paper, has much to recommend it as to simplicity, generality and flexibility.

The concept under consideration is more than a single funding method. It is perhaps more accurately described as a *family* of funding methods, since the formula defining the concept includes a parameter $k$ which can be set at any point within a range. The choice of parameter determines just where the particular member of the family falls within the classification of funding methods, Classes I through V, set out in the 1952 paper.

Since in many ways this paper is an extension of the earlier one, it will prove convenient to use similar notation and the same illustrations. The reader may find it convenient to review the earlier paper before going further.

## THE CONCEPT

Let $C_t$ represent the $t$th annual *contribution* to the pension plan, payable annually in advance.

Let $F_t$ represent the *fund* built up after $t$ years (before contribution or benefits then due).

Let $B_t$ represent *benefits* for the $t$th year, assumed to be payable annually in advance.

[1] "Fundamentals of Pension Funding," Trowbridge, *TSA* IV, 17.

[2] *Fundamentals of Private Pensions*, McGill, chap. 4, 1955. *Pensions*, Hamilton and Bronson, chap. 11, 1958.

Let $V_t$ represent the present *value* of benefits, for both active and retired lives, at the beginning of the $t$th year, including the $B_t$ payments then due. Benefits for future entrants are not included within $V_t$.

Let $B_t/V_t$ and $F_{t-1}/V_t$, two ratios with the same denominator, be hereafter called the ***benefit ratio*** and the ***fund ratio***,[3] respectively.

Then the funding family proposed is completely defined by the formula

$$C_t = (k + d)(V_t - F_{t-1}) , \tag{1}$$

where $k$ is a positive number less than 1, the exact range of which will be developed later, and where $d$ is the rate of discount corresponding to $i$, the assumed rate of interest.

Stating (1) above in words, the contribution for the $t$th year is 100 $(k + d)\%$ of the amount by which the present value of benefits exceeds the funds previously accumulated. The particular percentage chosen is unique to a particular member of the family.

Hereafter we shall refer to $V_t - F_{t-1}$ as the *unfunded present value*, and the family of funding methods indicated by (1) above as the *unfunded present value family*.

### THE INITIALLY MATURE SITUATION

*Assumptions and Notation*

In order to develop the characteristics of the proposed family of funding methods, it is convenient to look first at the situation where the population is mature from the beginning of the plan. We reserve for a later section the more practical case where the population is initially immature. We therefore assume for the moment the rigorous conditions set forth in Section II of the earlier paper, and adopt its notation.

*Derivation of $C_\infty$ and $F_\infty$*

Since the population is already mature, $B_t$ and $V_t$ are from the beginning at their ultimate levels

$$B = \sum_r^\omega l_x$$

$$V = \sum_a^{r-1} l_x \cdot {}_{r-x|}\ddot{a}_x + \sum_r^\omega l_x \cdot \ddot{a}_x .$$

The initial contribution, since we begin with $F_0 = 0$, is

$$C_1 = (k + d)V$$

[3] Note the denominator $V_t$ in the fund ratio here defined. The fund ratio in this paper must be distinguished from an otherwise similar function, used by some actuaries, with the accrued or past service liability in the denominator.

and the first year fund

$$F_1 = (C_1 - B)(1 + i) = [(k + d)V - B](1 + i) .$$

If we now define a quantity $b$, such that $b = B/V - d$, the expression for $F_1$ takes the form

$$F_1 = V(1 + i)(k - b) .$$

Since the quantity $b$ will be found to be important in the mathematical development to follow, it might be well to note here that $b$ is closely related to the *benefit ratio* $B/V$. $b$ is actually the excess of the benefit ratio over $d$, the rate of discount. The arithmetical value of $b$ is likely to be in the neighborhood of 1%, as will be shown later. Demonstration I, to be found in the appendix to this paper, carries forward the development started above and shows that as $t$ increases, $F_t$ and $C_t$ approach the positive and finite limits

$$F_\infty = \frac{V(k-b)}{k} \tag{2}$$

$$C_\infty = \frac{Vb(k+d)}{k}, \tag{3}$$

provided that $b \leq k \leq 1 - d$.

The Equation of Maturity (see earlier paper) for the unfunded present value family is expressed by the identity $C_\infty + dF_\infty \equiv B_\infty$ or

$$\frac{Vb(k+d)}{k} + d \cdot \frac{V(k-b)}{k} \equiv V(b+d) \equiv B .$$

*The Limiting Situations*

It is apparent that when $k$ is at its minimum $k = b$, $F_\infty = 0$, $C_\infty = V(b + d) = B$, and we have exactly pay-as-you-go or Class I funding.

It is also clear that when $k$ is at its maximum $k = 1 - d$, the fundamental equation of this funding concept becomes $C_t = V - F_{t-1}$. Then the funds *after* payment of the contribution become $F_{t-1} + C_t = V$. From this we recognize the Class V or the initial funding method described in the earlier paper. $F_\infty$ becomes

$$\frac{V(1-d-b)}{1-d} \quad \text{and} \quad C_\infty = \frac{Vb}{1-d} .$$

*F as a Function of k*

Since $F_\infty = V(k - b)/k$ is a continuous function of $k$ throughout the range $b \leq k \leq 1 - d$, the ultimate $F$ for any of the traditional funding methods can be reproduced by the proper choice of $k$.

Although $k$ can be chosen to get to the same place ultimately as any of the established funding methods, in general this concept does not travel over the same route. The two extremes of the range of $k$ are exceptions, in the initially mature situation under consideration, since this method actually duplicates pay-as-you-go (Class I) and initial funding (Class V), not only in the ultimate situation but for all values of $C_t$ and $F_t$. Aggregate funding is also an exception and hence a member of the family, if the population is mature from the beginning. It is the special case where

$$k + d = \frac{\sum_a^{r-1} l_x}{\sum_a^{r-1} l_x \ddot{a}_{x:\overline{r-x|}}},$$

the reciprocal of the average temporary annuity.

*An Illustration*

The actual operation of the unfunded present value concept in the initially mature situation for various values of $k$ is illustrated in Table I. The illustration employed is exactly the same as that of Tables I and II of the previous paper, to facilitate comparisons between this family of funding methods and the traditional methods of the earlier paper. The interest assumption, $2\frac{1}{2}\%$, is not as appropriate today as it appeared in 1952, but is used in Table I nonetheless to preserve comparability. Table I shows up some of the characteristics of this family:

(1) The smoothly decreasing progression of the contributions $C_t$ over the years, for all values of $k$.

(2) The proportionality of the initial contribution to $k + d$, and the tendency of the contribution curves to cross some 15 to 30 years in the future, as the smaller unfunded present value, for the high value of $k$, tends to outweigh the larger $k + d$.

(3) The large effect in the low end of the range of $k$ (up to say 3%) of a small change in $k$ on the ultimate fund; the relatively small effect in the range of $k$ above 3% of a change in $k$ on the ultimate fund.

(4) The *slow* approach of $F_t$ to its ultimate value $F_\infty$, for small values of $k$ particularly. Note that $F_t$ is barely more than half-way there after 50 years for $k = 1.5\%$, but 93% of the way along for $k = 5\%$.

Note the exact reproduction of some of the funding methods of the earlier paper by a proper choice of $k$.

*a*) If $k = 1.20774\%$ we exactly reproduce "pay-as-you-go" or Class I funding.

## TABLE I

$V = 1,727,559 \quad d = .0243902 \quad b = .0120774$

| | $k=b$ (1.20774%) | $k=1.5\%$ | $k=1.70259\%$ | $k=2\%$ | $k=3\%$ | $k=4.00750\%$ | $k=5\%$ | $k=8.16017\%$ | $k=1-d$ (97.56098%) |
|---|---|---|---|---|---|---|---|---|---|
| | Class I | | Class II | | | Class III | | Class IV | Class V |
| | | | | | Contributions | | | | |
| Beg. of Year | | | | | | | | | |
| 1 | 63,000 | 68,049 | 71,549 | 76,687 | 93,962 | 111,367 | 128,513 | 183,109 | 1,727,559 |
| 2 | “ | 67,845 | 71,186 | 76,064 | 92,236 | 108,172 | 123,518 | 170,060 | 21,386 |
| 3 | “ | 67,644 | 70,829 | 75,454 | 90,563 | 105,107 | 118,779 | 158,103 | “ |
| 4 | “ | 67,447 | 70,479 | 74,856 | 88,942 | 102,168 | 114,282 | 147,145 | “ |
| 5 | “ | 67,252 | 70,135 | 74,271 | 87,370 | 99,350 | 110,016 | 137,104 | “ |
| 10 | “ | 66,323 | 68,501 | 71,520 | 80,207 | 86,904 | 91,750 | 98,178 | “ |
| 15 | “ | 65,463 | 67,006 | 69,040 | 74,080 | 76,812 | 77,708 | 73,026 | “ |
| 20 | “ | 64,668 | 65,636 | 66,804 | 68,838 | 68,630 | 66,915 | 56,775 | “ |
| 25 | “ | 63,931 | 64,382 | 64,787 | 64,355 | 61,995 | 58,617 | 46,274 | “ |
| 30 | “ | 63,250 | 63,234 | 62,969 | 60,519 | 56,616 | 52,239 | 39,489 | “ |
| 35 | “ | 62,619 | 62,182 | 61,330 | 57,239 | 52,255 | 47,337 | 35,105 | “ |
| 40 | “ | 62,036 | 61,219 | 59,852 | 54,432 | 48,718 | 43,568 | 32,273 | “ |
| 50 | “ | 60,996 | 59,530 | 57,319 | 49,978 | 43,526 | 38,444 | 29,260 | “ |
| Limit | 63,000 | 54,790 | 50,753 | 46,309 | 37,827 | 33,563 | 31,042 | 27,101 | 21,386 |
| | | | | | Funds | | | | |
| End of Year | | | | | | | | | |
| 1 | None | 5,175 | 8,763 | 14,029 | 31,736 | 49,576 | 67,151 | 123,112 | 1,706,173 |
| 2 | “ | 10,271 | 17,373 | 27,770 | 62,496 | 97,117 | 130,861 | 235,926 | “ |
| 3 | “ | 15,288 | 25,832 | 41,230 | 92,310 | 142,705 | 191,306 | 339,304 | “ |
| 4 | “ | 20,228 | 34,144 | 54,413 | 121,208 | 186,420 | 248,653 | 434,036 | “ |
| 5 | “ | 25,092 | 42,311 | 67,326 | 149,217 | 228,339 | 303,061 | 520,844 | “ |
| 10 | “ | 48,314 | 81,056 | 128,031 | 276,861 | 413,479 | 536,026 | 857,380 | “ |
| 15 | “ | 69,805 | 116,534 | 182,762 | 386,051 | 563,591 | 715,106 | 1,074,828 | “ |
| 20 | “ | 89,695 | 149,024 | 232,107 | 479,452 | 685,304 | 852,765 | 1,215,329 | “ |
| 25 | “ | 108,100 | 178,776 | 276,598 | 559,352 | 783,988 | 958,584 | 1,306,112 | “ |
| 30 | “ | 125,134 | 206,022 | 316,711 | 627,698 | 864,003 | 1,039,926 | 1,364,770 | “ |
| 35 | “ | 140,898 | 230,970 | 352,880 | 686,164 | 928,881 | 1,102,456 | 1,402,671 | “ |
| 40 | “ | 155,486 | 253,817 | 385,490 | 736,177 | 981,484 | 1,150,523 | 1,427,160 | “ |
| 50 | “ | 181,482 | 293,897 | 441,400 | 815,555 | 1,058,716 | 1,015,876 | 1,453,208 | “ |
| Limit | None | 336,598 | 502,104 | 684,338 | 1,032,078 | 1,206,924 | 1,310,271 | 1,471,873 | 1,706,173 |
| Fund Ratio $F_\infty/V_\infty$ | 0 | 19.48% | 29.06% | 39.61% | 59.74% | 69.86% | 75.85% | 85.20% | 98.76% |
| $F_{50}/V_{51}$ | 0 | 10.51 | 17.01 | 25.55 | 47.21 | 61.28 | 70.38 | 84.12 | 98.76 |

*b*) If $k = 8.16017\%$ we exactly reproduce "aggregate" funding, one of the forms of Class IV funding.

*c*) If $k = 97.56098\%$ we exactly reproduce "initial" or Class V funding (with the initial accrued liability immediately funded).

Note also the exact reproduction of the other funding classes, but in the ultimate situation only.

*d*) If $k = 1.70259\%$ we get the same $C_\infty$ and $F_\infty$ as "terminal" or Class II funding, but in the course of reaching this point the unfunded present value method does not fully fund for retired lives.

*e*) If $k = 4.00750\%$ we get the same $C_\infty$ and $F_\infty$ as "unit credit" or Class III funding, but with an initial contribution at about the level of normal cost, plus 20 year funding of the initial past service liability.

THE INITIALLY IMMATURE SITUATION

In the more realistic situation of an initially immature group (see section IV of earlier paper) where a gradual approach to the limiting mature group is assumed, we find $V_t$ is no longer constant, but increases quite smoothly from $V_1$ to $V_\infty$. No new conclusions need to be reached with respect to the ultimate situation.

$$F_\infty = \frac{V_\infty(k-b)}{k} \quad \text{and} \quad C_\infty = \frac{V_\infty b(k+d)}{k},$$

as before, where $b$ is defined in terms of the *ultimate* benefit ratio—*i.e.*, $b = B_\infty/V_\infty - d$. $C_1$ for the initially immature situation will bear a ratio of $V_1/V_\infty$ to the $C_1$ for the initially mature situation, and the entire graph of $C_t$ initially immature will be lower but asymptotic to the $C_t$ initially mature, for the same value of $k$.

$C_t$ still follows a smooth progression from year to year for a particular value of $k$, but in the initially immature situation the trend is *not* necessarily downward. For small values of $k$ the trend of $C_t$ is upward, for large values of $k$ downward.

If we specify that $C_1 = C_\infty$ and solve for $k$, we determine that $k = (V_\infty/V_1)\cdot b$. For this particular member of the family $C_t$ is nearly (but not exactly) level, and $F_\infty = V_\infty - V_1$. In general, if $k > (V_\infty/V_1)\cdot b$ the trend of $C_t$ is downward; if $k < (V_\infty/V_1)\cdot b$ the trend of $C_t$ is upward.

As in the initially mature situation, the ultimate $C_\infty$ and $F_\infty$ of any of the traditional funding methods can be reproduced by proper choice of $k$. Actually the correct choice of $k$ is the same as in the initially mature case. None of the traditional methods, however, can be exactly reproduced throughout the entire range of $t$.

Table II illustrates the unfunded present value family applied to an

## TABLE II

$V_1 = 922,974 \quad V_\infty = 1,727,559 \quad d = .0243902 \quad b = .0120774$

| | $k=b$ (1.20774%) | $k=1.70259\%$ | $k=2\%$ | $k=(V_\infty/V_1)\cdot b$ (2.26056%) | $k=3\%$ | $k=4.00750\%$ | $k=5\%$ | $k=8.16017\%$ | $k=1-d$ (97.56098%) |
|---|---|---|---|---|---|---|---|---|---|
| | Class I | Class II | | | | Class III | | Class IV | Class V |
| | **Contributions** | | | | | | | | |
| Beg. of Year | | | | | | | | | |
| 1 | 33,659 | 38,226 | 40,971 | 43,376 | 50,201 | 59,500 | 68,660 | 97,828 | 922,974 |
| 2 | 34,042 | 38,467 | 41,104 | 43,401 | 49,850 | 58,469 | 66,772 | 91,969 | 21,926 |
| 3 | 34,412 | 38,694 | 41,225 | 43,416 | 49,497 | 57,466 | 64,965 | 86,577 | 21,703 |
| 4 | 34,767 | 38,907 | 41,331 | 43,416 | 49,140 | 56,487 | 63,229 | 81,606 | 21,424 |
| 5 | 35,110 | 39,107 | 41,426 | 43,408 | 48,784 | 55,535 | 61,567 | 77,030 | 21,224 |
| 10 | 36,698 | 39,980 | 41,791 | 43,283 | 47,061 | 51,216 | 54,324 | 59,115 | 20,779 |
| 15 | 38,169 | 40,755 | 42,094 | 43,144 | 45,556 | 47,680 | 48,716 | 47,489 | 20,743 |
| 20 | 39,574 | 41,492 | 42,397 | 43,052 | 44,305 | 44,853 | 44,453 | 40,041 | 20,941 |
| 25 | 40,921 | 42,197 | 42,701 | 43,003 | 43,271 | 42,605 | 41,224 | 35,292 | 21,014 |
| 30 | 42,217 | 42,876 | 43,011 | 42:997 | 42,431 | 40,834 | 38,801 | 32,307 | 21,415 |
| 35 | 43,553 | 43,632 | 43,434 | 43,142 | 41,884 | 39,595 | 37,163 | 30,676 | 22,008 |
| 40 | 44,781 | 44,292 | 43,781 | 43,236 | 41,373 | 38,542 | 35,846 | 29,539 | 21,531 |
| 50 | 46,908 | 45,327 | 44,245 | 43,255 | 40,411 | 36,823 | 33,864 | 28,095 | 21,301 |
| Limit | 63,000 | 50,753 | 46,309 | 43,376 | 37,827 | 33,563 | 31,042 | 27,101 | 21,386 |
| | **Funds** | | | | | | | | |
| End of Year | | | | | | | | | |
| 1 | 34,500 | 39,182 | 41,995 | 44,460 | 51,456 | 60,988 | 70,377 | 100,274 | 946,048 |
| 2 | 69,395 | 78,729 | 84,315 | 89,197 | 102,978 | 121,582 | 139,717 | 196,188 | 991,312 |
| 3 | 104,250 | 118,206 | 126,526 | 133,776 | 154,134 | 181,372 | 207,647 | 287,682 | 1,036,188 |
| 4 | 138,861 | 157,409 | 168,422 | 177,990 | 204,724 | 240,174 | 274,016 | 374,889 | 1,080,421 |
| 5 | 172,861 | 195,970 | 209,635 | 221,474 | 254,387 | 297,643 | 338,513 | 457,758 | 1,123,727 |
| 10 | 324,725 | 370,122 | 396,450 | 418,943 | 479,960 | 556,750 | 625,767 | 807,510 | 1,318,593 |
| 15 | 435,643 | 502,465 | 540,494 | 572,556 | 657,506 | 760,096 | 848,019 | 1,058,759 | 1,469,332 |
| 20 | 506,659 | 594,059 | 642,903 | 683,559 | 788,897 | 911,224 | 1,011,481 | 1,232,238 | 1,578,418 |
| 25 | 546,536 | 653,702 | 712,552 | 760,942 | 883,671 | 1,021,028 | 1,129,020 | 1,349,649 | 1,654,772 |
| 30 | 564,885 | 691,027 | 759,139 | 814,493 | 952,075 | 1,100,813 | 1,213,357 | 1,428,902 | 1,708,214 |
| 35 | 549,704 | 694,114 | 770,831 | 832,488 | 982,844 | 1,140,237 | 1,255,262 | 1,464,010 | 1,729,222 |
| 40 | 509,954 | 671,956 | 756,686 | 824,061 | 985,445 | 1,149,419 | 1,265,571 | 1,467,349 | 1,722,733 |
| 50 | 432,063 | 627,109 | 726,149 | 803,354 | 982,304 | 1,154,700 | 1,270,541 | 1,459,541 | 1,702,437 |
| Limit | None | 502,104 | 684,338 | 804,583 | 1,032,078 | 1,206,924 | 1,310,271 | 1,471,873 | 1,706,173 |
| Fund Ratio $F_\infty/V_\infty$ | 0 | 29.06% | 39.61% | 46.57% | 59.74% | 69.86% | 75.85% | 85.20% | 98.76% |
| $F_{50}/V_{51}$ | 25.07% | 36.38 | 42.13 | 46.61 | 56.99 | 66.99 | 73.71 | 81.67 | 98.76 |

initially immature situation. Again the example is taken without change from the earlier paper (Tables III and IV thereof) to facilitate comparison.

Indications from this example which the author finds particularly interesting are these:

1. For $k = b = 1.20774\%$, the contributions stay ahead of pay-as-you-go funding in the early years, fall behind after 20 years, and eventually level out at the pay-as-you-go rate. $F_t$ builds up for 30 years, then very slowly falls to zero.
2. For $k = 1.70259\%$ the proposed method stays ahead of terminal funding. Recall that it fell behind in the initially mature situation.
3. If $k = (V_\infty/V_1)\cdot b = 2.26056\%$, the proposed method almost (but not quite) duplicates entry age normal, with interest only paid toward the accrued liability.
4. If $k = 4.00750\%$ the proposed method duplicates Class III funding in the ultimate situation, and starts out at about the level of normal cost, and 15+ year funding of the past service liability.
5. If $k = 8.16017\%$ the proposed method duplicates all Class IV methods in the ultimate situation, and is strongly similar to the aggregate method for the entire range of $t$. The departure from an exact duplicate of the aggregate method is due to the variation, in the aggregate method, of the average temporary annuity as the population matures.

Compared to the initially mature situation, this initially immature example shows a more rapid approach to the ultimate $F_\infty$. For small values of $k$ the approach is down from a higher value of $F_t$ instead of up from below.

## ADJUSTMENT FOR ACTUARIAL GAINS AND LOSSES

The mathematical development to this point has assumed that the actuarial assumptions are exactly realized. In practice this is never the case, and some method of adjusting for actuarial gains or losses is needed.

The proposed family of funding methods automatically adjusts for actuarial gains and losses. It uses the "spread" technique described in the earlier paper, which is a characteristic of aggregate funding, attained age normal, and the frozen initial liability form of entry age normal.

An actuarial gain $\Delta_t$ during the $t$th year will affect the unfunded present value $V_{t+1} - F_t$ at the end of the year. The unfunded portion of $V_{t+1}$ will be smaller by $\Delta_t$ than expected in accordance with the actuarial assumptions. The contribution $C_{t+1}$ for the following year will therefore be decreased by $(k + d)\Delta_t$.

Each $C_{t+u}$ thereafter will include $-(k + d)S^{u-1}\Delta_t$ [where $S = 1 -$

$(1 + i)k]$ as a component to adjust for $\Delta_t$. Each $V_{t+u+1} - F_{t+u}$ will be smaller, by $S^u\Delta_t$, than it would have been if actuarial assumptions had exactly worked out during the $t$th year. The analysis of actuarial losses is identical to the above, but with a negative value for $\Delta_t$.

Gains or losses are thus spread in a decreasing asymptotic fashion. The adjustment for the gain or loss of any particular period is never completed, but approaches zero as that period falls farther and farther into the past. In the meantime additional gains or losses have been experienced and spread in the same fashion. Gains thus serve both to reduce contributions (to extent that gain is recognized) and to increase the fund ratio (to the extent that gain is spread into the future). Losses increase contributions and decrease the fund ratio.

*Example:* If gain arises each year because of interest earnings 1% higher than assumed, results of Table II for $k = 3\%$ are shown below, together with results of actuarial losses of 1% of the fund each year.

| | Actuarial Gain 1% of Fund Each Year | Actuarial Loss 1% of Fund Each Year |
|---|---|---|
| $C_1$ | $ 50,201 | $ 50,201 |
| $C_{10}$ | 45,931 | 48,135 |
| $C_{20}$ | 40,306 | 47,882 |
| $C_{35}$ | 32,739 | 49,360 |
| $C_{50}$ | 27,024 | 50,419 |
| $F_1$ | 51,958 | 50,954 |
| $F_{10}$ | 504,963 | 456,318 |
| $F_{20}$ | 868,536 | 718,071 |
| $F_{35}$ | 1,156,981 | 841,317 |
| $F_{50}$ | 1,232,763 | 796,107 |
| $F_{50}/V_{51}$ | 71.46% | 46.15% |

The relative speed in recognizing a gain or loss is a function of $k$. Both gains and losses are recognized more rapidly if $k$ is large than if it is small. This gives rise to an interesting and important interplay between the value of $k$ selected and the choice of actuarial assumptions.

In the absence of actuarial gains or losses the ultimate fund ratio $F_\infty/V_\infty = (k - b)/k$ is increased by an increase in $k$. If actuarial assumptions are conservative, so that net gains develop, the effect of the increase in $k$ on the fund ratio will have a negative component due to the more rapid recognition of gains. Hence gains tend to dampen the effect of a change in $k$. Conversely losses tend to accentuate such an effect.

### THE DETERMINATION OF $k$

It should be now apparent that the important key to the funding concept proposed lies in the choice of the parameter $k$. With a set of rigorous actuarial assumptions and considerable detailed calculation one can (as has been done for Tables I and II) lay out both $C_t$ and $F_t$ over a long span of time for various values of $k$. If the actuary has faith in the underlying assumptions, an intelligent choice of $k$ then offers no particular difficulty. For day-to-day usage we cannot expect such detail and some guides are needed, lest the choice of $k$ be distinctly unscientific.

Considerable experience with this funding concept may be necessary before the actuary feels entirely comfortable about his techniques for determining $k$. The author does not claim to have all the answers in this regard. He does offer the following as reasonable approaches to the determination of the parameter.

*From the Long Range Objective*

Given a long range objective and the value of $b$, $k$ can be determined. Let us postpone discussion of the determination of $b$ and for the moment treat $b$ as fixed. $d$ is of course also known. Then the determination of $k$ can be illustrated by any of the following:

1. Perhaps the long range objective is expressed in terms of the fund ratio $F_\infty/V_\infty$. If we decide this ratio should be $p$, then

$$\frac{F_\infty}{V_\infty}=\frac{k-b}{k}=p$$

$$k=\frac{b}{1-p}\,.$$

For example, if $b = 1\%$ and we desire an ultimate fund ratio of 2/3, $k = 3\%$.

2. The long range objective might be expressed in terms of the relationship of the ultimate contribution $C_\infty$ to ultimate benefit payments $B_\infty$. This particular relationship is suggested by the Equation of Maturity. If $C_\infty$ is to be $fB_\infty$, hence $(1-f)B_\infty$ is to come from interest on $F_\infty$, then

$$C_\infty = fB_\infty = f(b+d)V_\infty$$

$$\frac{b(k+d)}{k}=f(b+d)$$

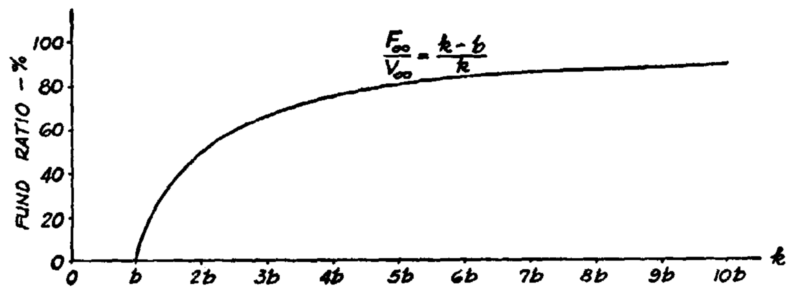
$$k=\frac{bd}{f(b+d)-b}\,.$$

For example, if $f$ were set at $\frac{1}{2}$, $k = 2bd/(d-b)$.

3. The long range objective might be expressed in terms of one or another of the traditional funding methods. This approach does not lend itself to rigorous analysis. However, if the illustration is reasonably typical, we might expect funding ratios and corresponding $k$'s about as follows:

| Class | Fund Ratio | $k$ |
|---|---|---|
| I—Pay-As-You-Go | None | $b$ |
| E.A.N. with int. only | 0%–50% | $b$ to $2b$ |
| II—Terminal | 30%–35% | $1.5b$ |
| III—Unit Credit | 70%–75% | $3b$ to $4.5b$ |
| IV—Aggregate | 80%–90% | $5b$ to $10b$ |
| E.A.N. with A.L. funded | | |
| Attained Age Normal | | |
| Individual Level Premium | | |

A graph of the fund ratio against $k$, by means of the relationship $F_\infty/V_\infty = (k - b)/k$, is useful in getting a feel for the long range determinations suggested above.




We see that the fund ratio, as a function of $k$, is a sharp breaking hyperbola. The quadratic nature of this curve is responsible for the phenomenon noted earlier. At the low end of the range of $k$ a small change has a big effect on the fund ratio, but above $k = 3b$ or so the additional effect is relatively small. The author views $k = 3b$ as a sort of boundary between the thin and the firm ice.

The preceding development rather clearly establishes the importance of $b$ in determining an appropriate value of $k$. It should be remembered that $b$ is, by definition, the excess of the ultimate benefit ratio $B_\infty/V_\infty$ over the rate of discount.

For a feeling of confidence in setting $k$, and for a rigorous mathematical demonstration, it would be highly satisfactory if $b$ were entirely independent of both the rate of interest assumed and the characteristics of the underlying service table. Such is not entirely the case, though $b$ is independent of the service table to a surprising degree.

For the service table illustrating the earlier paper (see Table I thereof), the values of $b$ under three different interest assumptions are as follows:

| Interest Rate | $b$ |
|---|---|
| $2\frac{1}{2}\%$.......... | 1.20774% |
| $3\frac{1}{2}\%$.......... | 0.93241% |
| $4\frac{1}{2}\%$.......... | 0.71375% |

$b$ is thus in the general area of 1% at about $3\frac{1}{2}\%$ interest, but varies inversely with $i$. The approximate formula $b = (.075 - i)/4$ fits reasonably well.

Surprisingly enough, under the assumptions outlined in II of the earlier paper, $b$ can be shown (see Demonstration II in the Appendix) to be absolutely independent of the preretirement $l_x$'s and hence of preretirement death, withdrawal, and salary increase assumptions. On the other

TABLE III

| Int. Rate | Ret. Age | Entry Age | Postretirement Mortality | $b$ |
|---|---|---|---|---|
| $3\frac{1}{2}\%$...... | 65 | 30 | As in illustration | 0.93241% |
| $2\frac{1}{2}\%$...... | " | " | " " " | 1.20774 |
| $4\frac{1}{2}\%$...... | " | " | " " " | 0.71375 |
| $3\frac{1}{2}\%$...... | 70 | " | " " " | 0.86655 |
| "........ | 60 | " | " " " | 1.09061 |
| "........ | 65 | 25 | " " " | 0.75228 |
| "........ | " | 35 | " " " | 1.16781 |
| "........ | " | 30 | G 51 | 0.94589 |
| "........ | " | " | G 51 (C)-generation age 65 in 1975 | 0.91359 |

hand, $b$ is a function of the interest rate, retirement age $r$, the hiring age $a$, and the mortality assumption after retirement.

The variation of $b$ as a function of $i$ has been previously indicated.

The variation of $b$ as a function of retirement age is in the direction that the higher the retirement age the lower the $b$.

The variation of $b$ as a function of $a$ is in the direction that the higher the hiring age the larger the $b$.

The variation of $b$ as a function of postretirement mortality is the smaller the $q$'s the smaller the $b$.

To determine an appropriate $b$ for any combination of these variable factors, the formula for $b$ developed in Demonstration II of the Appendix, or Table III which illustrates the action of this formula, will be useful.

*From the Initial Data*

It may be difficult or even impossible to get agreement on a long range objective; or it may not appeal to the actuary to emphasize the long range future. In such circumstances it may be possible to do a reasonable job of setting the parameter $k$ from consideration of the original situation only. An approach to the setting of $k$ from calculations on the initial data only is outlined below:

1. A pretty good idea of the $k$ required to give eventually Class IV results can be obtained by calculating the weighted average temporary annuity on the original data. The result might be reduced slightly to allow for future maturing of the active life group. Then the reciprocal of this annuity, less $d$, should pin down a sort of "practical maximum" value of $k$ ($k$ max.). [The correct value for $k$ to reproduce Class IV results does vary with the $l_x$'s below retirement age. This is not true of Class II or Class III funding.]
2. An entry age normal calculation could be made, based on original employee data. A $k$ could then be determined to reproduce, for the first year, normal cost plus interest on the initial accrued liability. This value of $k$ ($k$ min.) might be viewed as a "practical minimum."
3. Within the range established by $k$ max. and $k$ min. the employer and actuary together might settle on a value of $k$, with the following ideas in mind.
   *a*) The security of employee's pension expectations and the employer's long range competitive position will both be enhanced by keeping $k$ up.
   *b*) Conservative actuarial assumptions make a high $k$ *less* necessary; too liberal assumptions make a high $k$ *more* necessary.
   *c*) The $k$ min. above is pay-as-you-go only, in the initially mature situation, and is far from adequate funding in the initially immature situation if the security of employee's pension expectations is important.

The author's own somewhat limited experience with the unfunded present value concept leads him to the conclusion that a $k$ of 5% to 8% is pretty solid, of 3% to 5% is marginal, and anything under 3% should be considered acceptable only as a temporary expedient, and only if substantial actuarial gains are likely.

## ADVANTAGES

Whether the unfunded present value family will ever be widely accepted and widely used must in the long run be determined by its inherent advantages and disadvantages, as compared with other funding methods, *all* important factors being taken into account. The author may be naturally prejudiced in its favor. In the introduction to this paper a claim was made *for* the method on grounds of simplicity, generality, and flexibility. Each of these will now be discussed in turn.

*Simplicity*

1. Once the $k$ has been established the actuarial valuation for any year requires only (i) the calculation of the present value of future benefits for present and former employees, and (ii) a valuation of the assets. The contribution for the next year becomes a simple percentage of the unfunded present value (i) − (ii).
2. A change in the benefit formula, an extension of the plan to additional groups of employees, or a change in actuarial assumptions, requires no modification of technique. The new benefits or new assumptions are used in the present value calculation, with no adjustment needed for the fact that other benefits or other assumptions may have been employed last year. The funding ratio will generally be changed, but will thereafter resume its progression toward its ultimate goal.
3. Even a change in funding method, from another method formerly used to the unfunded present value concept, causes no complications. One simply starts with whatever unfunded present value the old method has to date produced.
4. The explanation to employers becomes extremely simple. Confusing concepts such as normal cost, accrued liability, actuarial gains or losses, etc., can be largely ignored. The emphasis is on the fund ratio, and its progress from year to year.
5. If former employees (pensioners and vested withdrawals) are fully funded by ear-marked assets, omitting the benefits for such persons from (i), and omitting the corresponding assets from (ii), will not generally distort (i) − (ii). This may be a further simplification in certain circumstances, particularly under deposit administration or split-funded plans with fully purchased retired life benefits.

*Generality*

1. Since in the ultimate situation the traditional methods are special cases of the unfunded present value concept, this concept is extremely general. It permits of all gradations between the traditional methods. It appeals to the mathematically trained mind because of its generality.
2. The method is likewise general in another sense. So far as the author can see today this concept has no particular limitations. It seems to be

as convenient for one benefit formula as another, for contributory plans as for noncontributory, for salaried groups as for hourly. It appears to have good characteristics under a wide range of circumstances.

*Flexibility*

Up to this point we have treated $k$ as a variable in some respects, as a constant in others. We have recognized that $k$ can be established at any point within a fairly wide range when the funding arrangements are being worked out at the establishment of the plan; but once $k$ is set we have thought of it as a fixed constant thereafter.

There is of course a substantial degree of flexibility inherent in the right to set the $k$ initially. This allows the method to fit many different financial situations. The "pay-more-now-to-pay-less-later" philosophy will find itself comfortable with one member of the family, the "pay-as-little-now-as-possible-and-let-the-future-take-care-of-itself" school with another.

The astute reader has by now noted another sort of flexibility to which this concept gives rise. Although we have treated $k$ as a constant once the funding has begun, ***there is no necessity that this be so.*** Mathematically $k$ can vary from year to year without any particular inconvenience.

Reasons why $k$ might not remain constant throughout the life of the plan are not at all difficult to visualize.

*a*) The long range funding objective might change. Employer A, who originally established $k = 3b$ with the idea of building up an ultimate fund of $\frac{2}{3}$ of the value of benefits, finds himself more concerned with security of his employees than before and his financial position stronger. He sets a new objective of $\frac{3}{4}$ rather than $\frac{2}{3}$, and raises his $k$ to $4b$.

*b*) Any error in the original $k$ can be corrected. Actuary B recommended a $k$ of 5% based on an 80% funding objective and an estimate that $b = 1\%$. A study ten years after the plan has started indicates that $b$ is more likely 0.9%. He recommends that $k$ be lowered to 4.5%.

*c*) Emergence of capital gains or losses may indicate a change in $k$. Employer C uses no turnover discount in his hourly plan, under which pensions are computed at $2.50 per month for each year of service. His objective is to build up funds sufficient to provide full accrued benefits in event of plan termination, which indicates no withdrawal assumption, a fund ratio of 75%, and a $k$ of $4b$. His plan is only partially vested, however, and heavy nonvested terminations in early years produce substantial actuarial gains. He prefers to retain the no-turnover assumption, despite its lack of realism, in order that the resulting fund ratio will measure his progress toward his specific objective. After a few years it becomes plain that actuarial gains will soon cause the fund ratio to exceed 75%. He cuts the $k$ to $3.5b$.

*d*) Temporary changes in the financial picture may be recognized. Employer D is perfectly satisfied with his established $k = 6\%$ from a long range point of view. He has occasional poor profit years, however, in which it is difficult to find the cash for his pension contribution. In good years he has extra funds to make up back shortages or get a little ahead toward his funding objective. He varies his $k$ somewhat from year to year, but in such a fashion that his average $k$ to date is always close to 6%.

*e*) A high fund ratio objective needn't necessarily require extremely high initial contributions. Employer E wants to arrive at Class IV funding eventually. The actuary estimates the eventual average temporary annuity at 10, thereby setting the ultimate $k + d$ at 10%. The employee group is reasonablymature and an extremely high first year contribution results if $k + d = 10\%$ is used initially. The actuary recommends an initial $k + d$ of $7\frac{1}{2}\%$,building up at 0.1% per year to 10% after 25 years.

The foregoing examples illustrate the high degree of flexibility possible under the unfunded present value method.

The $k$ initially chosen is like a direction arrow pointing out the direct route to the chosen objective. If later on the objective changes, the pointer moves to indicate the new objective. But the traveler need not always take the direct route. If he chooses to wander a little along the way, the pointer goes with him, constantly indicating the path to follow when the reason for deviation no longer exists. In addition to the direction arrow $k$ the traveler also has available the distance measure $F_{t-1}/V_t$ indicating how far he has been and how far he has yet to go. With these two tools he can travel as he chooses, yet he should never get lost.

### DISADVANTAGES

In an objective presentation of any new concept there is an obligation on the part of its proponent to point out the weak points as well as the strong. The most important weak points that the author sees in the unfunded present value family come under the two headings of (1) lack of seasoning, and (2) danger of excessive flexibility. There may of course be other weaknesses of which the author is as yet unaware.

#### *Lack of Seasoning*

The unfunded present value concept proposed breaks somewhat with tradition. It looks more to the future and less to the past than some of the earlier approaches. In doing so it ignores the concept of past service or accrued liability, a concept which has been around for many years and which is a part of the training and experience of many actuaries. It is more of an over-all or group approach than many, and its results are not particularly relatable to a single individual. It uses a parameter which to some will appear arbitrary, or at best empirical. The best methods of setting $k$ have probably not yet been devised.

Like any new method that departs from tradition, only time will tell as to its acceptance by actuaries, employers, unions, Treasury officials, accountants, lawyers and others interested in pension funding. Such acceptance will presumably come in time if the concept is meritorious, will never come if it isn't.

Until such time as this family may be approved by the Treasury as an acceptable and recognized method (subject perhaps to limitations on the value of $k$) it would be the bold actuary who would use it in actual actuarial valuations without an additional calculation by a more traditional method for purposes of justifying the contribution in the tax return. The author sees no reason why the Treasury, after due consideration, might not approve the method, but the fact remains that it has not yet been proposed to Treasury officials, so their attitude for the present remains unknown.

*Danger of Excessive Flexibility*

At the risk of appearing inconsistent, after the earlier claim that the flexibility in this concept is an advantage, the author feels he must point out the dangers that may be lying within this flexibility. There are really two aspects that may concern the actuary.

First, given such a wide choice, the employer may succumb to the siren's song of low initial outlay and set his funding objective and hence his $k$ too low. Such a course is particularly hazardous if, as is likely, the pressure is on at the same time for liberal actuarial assumptions. Like high blood pressure and overweight, the combination of liberal assumptions and a low $k$ is worse than the sum of their individual effects. The result will surely be an underfunded plan, with all its attendant evils. Another look at the graph on page 161, particularly its steepness up to $k = 3b$ or so, will help the actuary in keeping plans from falling over the precipice.

Second, with so much room to wander, even the employer with a sound funding objective may find that, though never lost, he never arrives. The right to wander somewhat when the occasion really demands should not deter the plan from the most direct route in a majority of situations. Even the direct route is long, particularly if $k$ is small.

## CONCLUSION

This paper has presented a concept of pension funding somewhat different from the traditional approaches.

The method proposed is appealing in its simplicity. It is extremely flexible, which is an advantage; but if this flexibility is abused it may prove to be a serious disadvantage.

This paper is presented in the hope that the method will prove useful to pension actuaries, if not immediately, then later on as, and if, it earns acceptance.

## APPENDIX

*Demonstration I*

$$C_1 = (k + d)V$$

$$F_1 = (C_1 - B)(1 + i) = [(k + d)V - B](1 + i) = V(1 + i)(k - b).$$

Similarly

$$C_2 = (k + d)(V - F_1)$$
$$= (k + d)V[1 - (k - b)(1 + i)]$$

and

$$F_2 = (F_1 + C_2 - B)(1 + i)$$
$$= V(1 + i)(k - b)[1 + (1 + i)(1 - k - d)].$$

Continuing the above process

$$F_t = V(1 + i)(k - b)[1 + s + s^2 + \ldots + s^{t-1}]$$

where

$$s = (1 + i)(1 - k - d) = 1 - (1 + i)k$$

$$F_t = V(1 + i)(k - b)\frac{1 - s^t}{1 - s}.$$

Now as long as

$$0 \leq s < 1 , \qquad \underset{t\to\infty}{\mathrm{L}} F_t = \frac{V(1 + i)(k - b)}{1 - s} = \frac{V(k - b)}{k}.$$

$s$ will be within this range if $0 < k \leq 1 - d$.

$$F_\infty = \frac{V(k - b)}{k}$$

will be $\geq 0$ if $k \geq b$. Therefore the conditions under which the limit of $F_t$, as $t$ increases, will be finite and positive are $b \leq k \leq 1 - d$.

Under these circumstances

$$F_\infty = \frac{V(k-b)}{k}$$

and

$$C_\infty = \frac{Vb(k+d)}{k}.$$

*Demonstration II*

In the ultimate state

$$B = \sum_r^\omega l_x = l_r \sum_r^\omega \frac{l_x}{l_r} = l_r \sum_0^\omega {}_tp_r = l_r(e_r+1)$$

$$V = \sum_a^{r-1} l_x \cdot {}_{r-x|}\ddot{a}_x + \sum_r^\omega l_x \cdot \ddot{a}_x$$

$$= \sum_a^{r-1} l_x \cdot v^{r-x} \frac{l_r}{l_x} \ddot{a}_r + l_r \sum_r^\omega \frac{l_x \ddot{a}_x}{l_r}$$

$$= l_r \ddot{a}_r \sum_a^{r-1} v^{r-x} + l_r \sum_0^\omega {}_tp_r \ddot{a}_{r+t}$$

$$\therefore \frac{B}{V} = b + d = \frac{e_r+1}{\ddot{a}_r \sum_a^{r-1} v^{r-x} + \sum_0^\omega {}_tp_r \ddot{a}_{r+t}}$$

$$= \frac{e_r+1}{\ddot{a}_r a_{\overline{r-a|}} + \sum_0^\omega {}_tp_r \ddot{a}_{r+t}}$$

$$b = \frac{e_r+1}{\ddot{a}_r a_{\overline{r-a|}} + \sum_0^\omega {}_tp_r \ddot{a}_{r+t}} - d.$$

$b$ is independent of the $l_x$'s below age $r$, since these cancelled out in the third line above.

$b$ is a function of $r$, the mortality table after age $r$, and the interest rate $i$, and the hiring age $a$.

## DISCUSSION OF PRECEDING PAPER

CECIL J. NESBITT:

The basic concept of this paper has been in the air for some while. I first heard of it as the notion of "perpetual amortization" proposed by an actuary with long experience with public employee retirement funds. His idea was to determine on each valuation date the annual amount required to amortize over the $n$ years following the valuation date the difference between the present value of benefits and the sum of the present value of future contributions during the remaining service of participants and the fund on hand. Since the $n$ years is translated forward at each valuation date, the method involves "perpetual amortization." By varying $n$ and the level of contributions during service, one would, in fact, have a two-parameter family of funding methods. Also, ten years ago I proposed 75 per cent of the full annual cost determined by the aggregate funding method as an intermediate funding procedure for a small and struggling public employee fund. I was going to add that I never heard from the fund again, but, as a matter of fact, I did five years ago and repeated the notion. If the cycle continues, I may have occasion to review the matter soon. In addition, in our pension mathematics course at Michigan, which starts off with a discussion of Mr. Trowbridge's paper on fundamentals of pension funding (*TSA* IV, 17), we explore a *modified aggregate* method defined by the continuous analogue of Mr. Trowbridge's equation (1) in his present paper. But, as before, it has remained for Mr. Trowbridge to bring the concept into full focus and to explore it thoroughly.

In this discussion, I will follow up the amortization concept previously mentioned. We are used to studying funding methods on a continuous basis (which has some advantage and some disadvantage over the discrete basis) but for consistence with the paper will follow a discrete model. For simplicity, we shall discuss the mature situation, but I do not believe much change would have to be made to adapt to the immature situation. In all cases, the formulas are for the simple illustrative plan discussed in Mr. Trowbridge's paper. Also, out of habit, I shall refer to "modified aggregate method" rather than "unfunded present value method" which would agree with the author's terminology. Of course there is really a family of methods.

First, we obtain an expression for $c = k + d$ if modified aggregate funding, according to the author's equation (1), is to produce ultimately the same fund as a standard funding method with normal cost $^*N$ and ultimate fund $^*F_\infty = (B - {}^*N)/d$. From equations (2) and (3) of Demonstration I of the 1952 paper, one has

$$V = \sum_a^{r-1} l_x \cdot {}_{r-x|}\ddot{a}_x + \sum_r^{\omega} l_x \ddot{a}_x = \Big(\sum_r^{\omega} l_x - v l_a \cdot {}_{r-a|}\ddot{a}_a\Big)/d \qquad (a)$$

$$= (B - v\,{}^1N)/d\,,$$

where $^1N$ is the normal cost for the initial funding method. Equating the author's expression (2) for $F_\infty$ to $^*F_\infty$, we find $(Vc - B)/(c - d) = (B - {}^*N)/d$ or

$$c[Vd - B + {}^*N] = {}^*Nd\,.$$

Finally, substitution from ($a$) for $V$ yields $c[{}^*N - v\,{}^1N] = {}^*Nd$ or

$$c = \frac{d}{1 - v({}^1N/{}^*N)}\,. \qquad (b)$$

As a check, if * denotes initial funding, then $c = 1$, which agrees with the author's result that $k = 1 - d$ in this case. Further, we note that if $({}^1N/{}^*N) = v^z$ then

$$c = \frac{1}{\ddot{a}_{\overline{z+1|}}}\,. \qquad (c)$$

That is, the modified aggregate funding method may be interpreted as "perpetual amortization" with $n = z + 1$ and no contributions other than for amortization.

It may be of interest to tabulate for the illustrative plan values of $z$ if the modified aggregate funding method is to produce the same ultimate fund as develops under a standard method. For this purpose, it is useful to tabulate the normal costs as in Table A.

By substituting these expressions for the normal costs in equation ($b$), we obtain Table B.

From Table B, it is observed that the amortization period $n = z + 1$ extends from 1 to more than the active service period of $r - a$ years according to choice of the funding method to which the modified aggregate method is to be equivalent ultimately.

It may be noted that another family of modified aggregate funding methods may be obtained by splitting off an amount $L_0$ of initial accrued liability on which only interest will be paid, and with contribution (for the illustrative plan) determined by

$$C_t = (V_t - F_{t-1} - L_0)/y + dL_0\,,$$

where

$$y = \sum_a^{r-1} l_x \ddot{a}_{x:\overline{r-x}|} \Big/ \sum_a^{r-1} l_x$$

as in the aggregate method. By various choices of $L_0$, one obtains funding equivalent ultimately to that under given standard methods.

It is not entirely surprising that for the illustrative plan the benefit

TABLE A

NORMAL COSTS FOR THE ILLUSTRATIVE PLAN UNDER VARIOUS FUNDING METHODS

| Funding Method | Normal Cost |
|---|---|
| Initial. . . . . . . . . . . . . . | $l_a \cdot {}_{r-a\|}\ddot{a}_a = (1+i)^a N_r$ |
| Entry age normal. . . . | $\frac{N_r}{N_a - N_r}\sum_a^{r-1} l_x = \frac{\sum_a^{r-1}(1+i)^x D_x}{\sum_a^{r-1} D_x} N_r = (1+i)^\xi N_r, \quad a < \xi < r$ |
| Unit credit. . . . . . . . . | $\frac{1}{r-a}\sum_a^{r-1} l_x \cdot {}_{r-x\|}\ddot{a}_x = \frac{\sum_a^{r-1}(1+i)^x}{r-a} N_r = (1+i)^\eta N_r, \quad \xi < \eta < r$ |
| Terminal. . . . . . . . . . . | $l_r \ddot{a}_r = (1+i)^r N_r$ |
| Pay-as-you-go. . . . . . | $\sum_r^\omega l_x = \frac{\sum_r^\omega l_x}{\sum_r^\omega v^x l_x} N_r = \frac{N_r}{v^\tau}, \quad \tau > r$ |

TABLE B

VALUES OF $z$ IF THE MODIFIED AGGREGATE FUNDING METHOD WITH $c = k + d = 1/\ddot{a}_{\overline{z+1}|}$ IS TO PRODUCE SAME ULTIMATE FUND FOR THE ILLUSTRATIVE PLAN AS SPECIFIED FUNDING METHOD

| Funding Method | $z$* |
|---|---|
| Initial. . . . . . . . . . . . . . . . . . . . | 0 |
| Entry age normal. . . . . . . . . . | $\xi - a$ |
| Unit credit. . . . . . . . . . . . . . . . | $\eta - a$ |
| Terminal. . . . . . . . . . . . . . . . . . | $r - a$ |
| Pay-as-you-go. . . . . . . . . . . . . | $\tau - a$ |

* Here $a < \xi < \eta < r < \tau$ (See Table A).

ratio $b$ is independent of the preretirement experience in regard to death, withdrawal, and salary increase, since the retirement benefit is simply an annuity of 1 per annum in case of survival to age $r$. In a practical plan, the retirement benefits would normally be a function of the preretirement experience, and in such a case $b$ would likely depend on such experience.

I wish to acknowledge the assistance of Mr. Robert Bucknell in the preparation of this discussion. He presented Mr. Trowbridge's paper in our actuarial seminar, and this discussion incorporates some of that presentation. Also, I wish to congratulate the author on his interesting and valuable contribution to the basic knowledge of pension funding.

MALCOLM D. MACKINNON:

A characteristic of funding methods in which the initial accrued liability is amortized through a series of level instalments is the sharp discontinuity in the progression of contributions which occurs at the end of the amortization period. An employer might prefer the smoothly decreasing contributions generated by the aggregate method (as illustrated in Mr. Trowbridge's earlier paper) but would probably select one of the other methods because of the lower initial contributions they permit.

Mr. Trowbridge's current paper offers an excellent alternative to such an employer. By generalizing from the aggregate method to his unfunded present value family, Mr. Trowbridge has achieved a smoothly decreasing progression of contributions for any reasonable initial contribution. In so doing, however, he has swept away the concepts of normal cost and accrued liability.

While often confusing, these concepts have proved helpful in the past, and much worthwhile educational work (of which Mr. Trowbridge's earlier paper is an important part) has been done to make them intelligible to actuaries, other pension technicians, employers, and union officials. They are embedded in the Internal Revenue Service publications. Since, as Mr. Trowbridge points out, it will probably be necessary for many years to carry out calculations by traditional methods to substantiate tax deductions, it seems desirable to see if contribution progressions similar to those produced by the unfunded present value family can be derived in terms of the sum of a normal cost payment and an instalment toward amortization of an unfunded accrued liability.

The purpose of this discussion is to offer the results generated by a slight variation on traditional funding methods for comparison with the results of the unfunded present value family. The variation will be referred to by the term "moving amortization period."

The initial contribution under this variation is determined in the traditional manner as the sum of the normal cost and the first of $n$ level annual instalments in the amount required to amortize the initial accrued liability over a period of $n$ years. By a suitable choice of $n$ between 1 and infinity, any reasonable initial contribution can be obtained.

At the beginning of the second year, the traditional contribution is the sum of the normal cost and the second of the $n$ level instalments determined at the inception of the plan. Under the variation, however, the contribution is the sum of the normal cost and the first of a new series of $n$ level annual instalments in the amount required to amortize the unfunded accrued liability at the beginning of the second year over the next $n$ years.

Similarly, the contribution at the beginning of any subsequent year is the sum of the normal cost and the first of a new series of $n$ level instalments in the amount required to amortize the current unfunded accrued liability over the next $n$ years. Although by staying on the method an employer never completely funds his accrued liability, he knows that, by leaving the method at any time and continuing to pay the most recently computed level instalment for $n - 1$ more years, he can reduce his unfunded accrued liability to zero.

For the mature population described by Mr. Trowbridge, the initial contribution under the moving amortization period variation (identified by the superscript $M$) is

$$^{M}C_1 = N + \frac{IAL}{\ddot{a}_{\overline{n}|}},$$

where $N$ and $IAL$ are the normal cost and initial accrued liability and $n$ is the amortization period. The fund at the end of the first year is

$$^{M}F_1 = (^{M}C_1 - B)(1 + i),$$

and subsequent contributions are given by

$$^{M}C_t = {^{M}C_1} - \left(\frac{1}{\ddot{a}_{\overline{n}|}}\right)^{M} F_{t-1}.$$

Using methods similar to those of Demonstration I in the Appendix to the paper we obtain

$$^{M}F_t = {^{M}F_1}\left(\frac{1 - R^t}{1 - R}\right)$$

where

$$R = (1 + i)\left(1 - \frac{1}{\ddot{a}_{\overline{n}|}}\right)$$

and

$$ {}^{M}C_t = {}^{M}C_1 - \frac{{}^{M}F_1}{\ddot{a}_{\overline{n}|}} \left( \frac{1-R^{t-1}}{1-R} \right). $$

In the limiting case

$$ {}^{M}F_\infty = \frac{{}^{M}F_1}{1-R} = IAL. $$

By the equation of maturity, ${}^{M}C_\infty$ is of course equal to $N$.

Table 1 shows the contributions generated by applying the moving amortization period variation to the facts of Mr. Trowbridge's example for the initially mature population. The normal cost and accrued liability were determined by the entry age normal method. The amortization periods selected are those which duplicate the initial contributions shown in Table I of Trowbridge's paper.

The following observations are suggested by a comparison of our Table 1 with Table I of the paper:

1. In two cases, the progression of contributions under the moving amortization period variation is identical to that under the unfunded present value method. These are $n = \infty$ for which both methods degenerate to "pay as you go," and $n = 10.594$ years (the term of the average temporary annuity), for which both methods are equivalent to aggregate funding.

2. For all intervening values of $n$ (this includes the whole range of practical choices) the early contributions under the moving amortization period variation are slightly in excess of those under the unfunded present value method. For the durations and amortization periods tabulated, the amount of this excess is always less than 4 per cent. At some duration, ranging from 40 to 67 for the amortization periods shown, the contributions under the two methods meet. Thereafter, the contributions under the moving amortization period variation are lower than those under the unfunded present value method, tending slowly to their ultimate level—the normal cost.

3. The fund at each duration equals or exceeds that under the unfunded present value method. Thus, if the contributions under the unfunded present value method are measured by the entry age normal concepts of normal cost and unfunded accrued liability, the implied amortization period (measured from the current date) increases steadily from year to year for all choices of $k$ between 1.20774 and 8.16017 per cent.

Similar results are obtained for the initially mature population under the moving amortization period variation if the normal cost and accrued

TABLE 1

| Contribution at Beginning of Year | $k=b=1.20774\%$ ($n=\infty$) | $k=1.5\%$ ($n=84.735$ Yrs.) | $k=1.70259\%$ ($n=66.766$ Yrs.) | $k=2\%$ ($n=52.134$ Yrs.) | $k=3\%$ ($n=31.180$ Yrs.) | $k=4.00750\%$ ($n=22.486$ Yrs.) | $k=5\%$ ($n=17.698$ Yrs.) | $k=8.16017\%$ ($n=10.594$ Yrs.) |
|---|---|---|---|---|---|---|---|---|
| 1 | 63,000 | 68,049 | 71,549 | 76,687 | 93,962 | 111,367 | 128,513 | 183,109 |
| 2 | 63,000 | 67,905 | 71,284 | 76,214 | 92,520 | 108,529 | 123,886 | 170,060 |
| 3 | 63,000 | 67,762 | 71,021 | 75,746 | 91,110 | 105,786 | 119,471 | 158,103 |
| 4 | 63,000 | 67,619 | 70,760 | 75,283 | 89,730 | 103,136 | 115,257 | 147,145 |
| 5 | 63,000 | 67,476 | 70,500 | 74,823 | 88,379 | 100,575 | 111,235 | 137,104 |
| 10 | 63,000 | 66,772 | 69,224 | 72,592 | 82,052 | 89,007 | 93,716 | 98,178 |
| 15 | 63,000 | 66,079 | 67,985 | 70,465 | 76,377 | 79,260 | 79,845 | 73,026 |
| 20 | 63,000 | 65,399 | 66,782 | 68,438 | 71,289 | 71,048 | 68,862 | 56,775 |
| 25 | 63,000 | 64,731 | 65,615 | 66,505 | 66,726 | 64,129 | 60,166 | 46,274 |
| 30 | 63,000 | 64,074 | 64,482 | 64,663 | 62,635 | 58,299 | 53,281 | 39,489 |
| 35 | 63,000 | 63,429 | 63,383 | 62,907 | 58,965 | 53,388 | 47,829 | 35,105 |
| 40 | 63,000 | 62,795 | 62,315 | 61,232 | 55,675 | 49,249 | 43,513 | 32,273 |
| 50 | 63,000 | 61,560 | 60,274 | 58,115 | 50,079 | 42,824 | 37,389 | 29,260 |
| Limit | 63,000 | 27,101 | 27,101 | 27,101 | 27,101 | 27,101 | 27,101 | 27,101 |
| Year in which progressions cross | ............ | 67 | 65 | 61 | 51 | 45 | 40 | ............ |
| Fund at end of 50th year | 0 | 237,612 | 379,893 | 560,037 | 976,951 | 1,206,491 | 1,329,347 | 1,453,208 |
| $F_{50}/V_{51}$ | 0 | 13.75% | 21.99% | 32.42% | 56.55% | 69.84% | 76.95% | 84.12% |
| $F_\infty/V_\infty$ | 0 | 85.20% | 85.20% | 85.20% | 85.20% | 85.20% | 85.20% | 85.20% |

liability are determined by the unit credit method. In this situation the ultimate fund ratio takes the Class III value (69.86 per cent) for all amortization periods. For an amortization period of 19.254 years, the values shown for Class III in Table I of the paper are exactly duplicated.

A feature of the unfunded present value family is its ability to move toward any chosen fund ratio. Similar flexibility may be obtained through a moving amortization period by funding only a portion of the initial accrued liability determined according to the entry age normal method. Suppose, for example, a funding ratio of 75.85 per cent is a long-range objective in Mr. Trowbridge's initially mature population. (Under the unfunded present value method this would imply $k = 5$ per cent.) The portion of the initial accrued liability to be left unfunded under the moving amortization period variation is

$$(0.8520 - 0.7585)\ V \doteq 161{,}600$$

and each contribution is increased by

$$d(161{,}600) = 3{,}941\,.$$

This amount, when added to the entry age normal cost of 27,101, produces an ultimate contribution of 31,042 (which equals the value of $C_\infty$ for $k = 5$ per cent). If an amortization period of 16.091 years is chosen, the contributions at all durations are those shown in Table I for $k = 5$ per cent.

In the initially immature population, quite similar relationships may be observed between the contributions generated by the two methods, although it no longer appears possible to duplicate the unfunded present value family contributions exactly at all durations.

In summary, the moving amortization period variation, when applied to the traditional methods, produces contributions which are reasonably close to those of the unfunded present value family. Thus an employer who wishes to do so may achieve the smooth progression of contributions implied by the unfunded present value family without abandoning the traditional benchmarks of normal cost and accrued liability.

DONALD R. SONDERGELD:

I would like to compliment the author on a most interesting and well-written paper. When I was studying the various funding methods for actuarial exam use, I—and I am sure many others—carefully read Mr. Trowbridge's 1952 paper on "Fundamentals of Pension Funding" in spite of the fact that it was not required reading. I believe this paper will also be well received.

There is one rather minor point that I shall discuss. This is regarding the definitions of $k$ and $b$, which are of course arbitrary, but why was the choice not $k'$ and $b'$, where $k' = k + d$ and $b' = b + d$?

Table 1 compares some equations using the two definitions.

It seems that even more simplicity is suggested in the unfunded present value method if we have only a benefit ratio, and not $b$ in addition to the benefit ratio. Since the benefit ratio and $b$ are both partially a function

TABLE 1

| | Trowbridge | Sondergeld |
|---|---|---|
| (1) | Benefit Ratio $= \frac{B}{V} = b + d$ | Benefit Ratio $= \frac{B}{V} = b'$ |
| (2) | Fund Ratio $= \frac{F}{V} = \frac{k-b}{k} = p$ | Fund Ratio $= \frac{F}{V} = \frac{k'-b'}{k'-d} = f'$ |
| (3) | $k = \frac{b}{1-p}$ | Contribution Ratio $= \frac{b'-f'd}{1-f'} = k'$ |
| (4) | $C_t = (k+d)(V_t - F_{t-1})$ | $C_t = (k')(V_t - F_{t-1})$ |
| (5) | $F_\infty = \frac{V(k-b)}{k}$ | $F_\infty = \frac{V(k'-b')}{k'-d}$ |
| (6) | $C_\infty = \frac{Vb(k+d)}{k}$ (if $b \le k \le 1-d$) | $C_\infty = \frac{V(b'-d)(k')}{k'-d}$ (if $b' \le k' \le 1$) |

of the interest rate, I question the need for the use of $b$ in addition to the "benefit ratio."

If $k'$ is the "contribution ratio," i.e., ratio of contribution to "unfunded present value," and since $k'$ as well as $k$ is partially a function of the interest rate, a similar question is raised.

Irrespective of the $k$ and $b$ versus $k'$ and $b'$ definitions, which is merely an actuarial detail, the actual presentation of the unfunded present value funding method could be partially reported to Mr. Contributor as follows: "Your benefit ratio will eventually be 0.05. You indicated your fund ratio objective to be 0.80. This indicates a contribution ratio of 0.1524 for the year beginning 11/1/63 and a contribution of $160,300."

CURRENT COST INFORMATION (11/1/63)

| | |
|---|---|
| 1. Total liability | $1,236,000 |
| 2. Funds | $ 184,327 |
| 3. Unfunded liability (1–2) | $1,051,673 |
| 4. Contribution ratio | 0.1524 |
| 5. Recommended contribution (3×4) | $ 160,300 |

ADDITIONAL COST INFORMATION (LONG RANGE VIEW)

| | | | |
|---|---|---|---|
| 1. Fund ratio | 0.80 | 0.90 | 0.50 |
| 2. Benefit ratio | 0.05 | 0.05 | 0.05 |
| 3. Contribution ratio | 0.1524 | 0.2804 | 0.0756 |
| 4. Total liability | $2,000,000 | $2,000,000 | $2,000,000 |
| 5. Funds | 1,600,000 | 1,800,000 | 1,000,000 |
| 6. Unfunded liability (4–5) | 400,000 | 200,000 | 1,000,000 |
| 7. Benefits (2×4) | 100,000 | 100,000 | 100,000 |
| 8. Contributions (3×6) | 60,960 | 56,080 | 75,600 |
| 9. Interest earnings (7–8) | 39,040 | 43,920 | 24,400 |

I agree with the author that the main disadvantages of the proposed method are its excessive flexibility, its newness, and its not having Treasury approval, which might indicate an extra calculation for tax purposes. These disadvantages would of course be eliminated if this method were officially blessed by the Treasury, with a low $k'$ defined to prevent possible disqualification, and a high $k'$ defined for determination of the maximum deduction. An additional condition would be necessary so that a contribution of zero, in certain instances, would not disqualify the plan. Perhaps the Treasury could use this method as a standard, irrespective of the funding method actually employed. Since the total single premium for ultimate benefits is generally determined regardless of the funding method, the major exception being unit credit, it would be no extra work to determine the "unfunded present value" and applying $k'$ min and $k'$ max to determine the limitation on contributions for tax purposes.

ROBERT C. TOOKEY:

If the gap that separates the pension thinking of actuaries and accountants is ever bridged, much of the credit should go to what can be termed "the Trowbridge bridge." Our professional brethren in the accounting field who write the dictionary of the "language of business" should be very grateful for this generalized concept of pension funding. In addition to the advantages of generality, simplicity, and flexibility pointed out by the author, it embodies more of generally accepted accounting principles than some other funding methods.

No one is sure, and sometimes I think accountants least of all, just what generally accepted accounting principles are. The textbooks list the following principles of fundamental importance:

1. The accounts and statements should give expression, as far as possible, to facts evidenced by completed transactions and supportable by objective data.

2. Cost is the proper basis for the accounting for assets and asset expirations, subject to an occasional modification in those instances where there is convincing evidence that cost cannot be recovered, either through use or sale, whichever is normal for the asset.

3. Conservatism, while generally desirable, is not a justification for the understatement of the owners' equity or the misstatement of periodic net income.

4. Consistency should be maintained between the statements prepared at the end of one period and between the statements of successive periods. However, a proper regard for consistency need not preclude a desirable change in procedure. If a change of material consequence is made, the fact should be mentioned and the effect thereof on the statements should be indicated, if determinable.

5. The determination of net income requires a proper matching of revenues and expenses. (*a*) Revenues should not be regarded as earned until an increment has been realized or until its realization is reasonably assured. (*b*) Expenses are expired costs.

6. Statements should not be misleading and should make full disclosure of significant information.

It has been the hope of the accounting profession that some day a uniform method or standard of accruing pension costs could be established independent of the funding method actually used. Whether or not this is possible remains to be seen, but the $k$ method of funding proposed by the author possesses certain features that are highly desirable. The failure to make an annual charge to operations substantially equivalent to the amount of significant accruing pension costs appears to many accountants to be a violation of one of the most basic concepts of accrual accounting. They have been disturbed by the fact that some companies have been able drastically to reduce or eliminate contributions as a result of appreciation of pension fund assets. The $k$ method spreads gains and losses over the future years and thereby bears a close similarity to the "future adjustment" method used in depletion accounting which estimates ultimate recoverable oil or mineral reserves. These are subject to revision each year, and the effect of the new recoverable reserve estimates is spread over the remaining production life. Consistency in the *k* method can be maintained by requiring that, once arrived at, *k* could not be changed by more than, say, 1 per cent in any given year. By basing the annual cost on the difference between all benefits and the value of the fund, and not earmarking anything for past service liabilities, this method avoids the controversy and misunderstanding surrounding past service costs. The method is so simple that the problem of matching revenues and expenses is effectively extinguished. The actuary could merely establish a minimum *k* that could be permitted upon the installation of a plan.

Both the layman and the nonpension actuary have been confused by the prolifera of pension funding methods and perplexing pension terminology. To the extent that the author has defined all methods in terms of a single parameter *k*, much clarification on the subject appears likely. Our firm will probably experiment with the *k* method in pension valuation work to determine how it works out in practice. We assume others will be doing likewise and hope that any necessary debugging will be easily accomplished.

DORRANCE C. BRONSON:

The author gives us another excellent pension paper. Elegant development of how to fund—not from the gamut of alpha to omega but, say, from gamma to rho.

My remarks are directed to some practical aspects not mentioned by previous speakers. Certain nonactuarial influences are at work which could narrow, or even pinpoint, the choice within the laissez faire of the Trowbridgean scale.

The author mentioned one of these limiting possibilities, and I would like to make notations on his margin of a few more to bear in mind before putting this valuable paper on our handy reference shelf.

1. *IRS Old.* The author warned of the potential IRS limiting influence in establishing maximum funding for tax-deductible employer contributions—as things stood when he wrote the paper.
2. *IRS New.* Since then, Revenue Ruling 63-11 has appeared and seems possibly to invade not only the funding structure area (of Mr. Trowbridge) but, indeed, the area of basic actuarial assumptions (of actuarial judgment and free choice).
3. *The accounting profession.* In a zeal for better financial comparisons between companies, accountants are proposing something called "current costs" to charge to operations each year, paid or not. While not *seeming* to control the actuaries as to determination of actual contributions recommended, such a move might turn into just that, and, indeed, to become a standard or maximum for IRS and others to grab hold of, or a *minimum* when that is the desirable ax to grind.

4. Disclosure acts are *continuing to implement Parkinson's law and* threaten to go into (Wisconsin already) complex measures of *regulation*, an inverted pyramid standing on the apex, or point, of the original objective of *disclosure* and *publicity* for employees and beneficiaries. This advance into *regulation* could well cut more limits into Mr. Trowbridge's scale.
5. *Canada*. The new Ontario law will force a "do-it-yourself" social security on employers of fifteen or more employees. The funding for this will be regulated (at least by minimum), and "standards of valuation" will be set, presumably at some point within the Trowbridgean scale (but using, I assume, directed assumptions).

Thus some of these practical or political parameters may well interplay with both themselves and, of course, with the author's parameters and those of us all. Indeed, I am concerned with the way the pension actuaries (wherever employed) are threatened in their freedom. I hope I do not see, down the road, the absolute fixation of all factors (including benefits) which seems to have become the Brave New Pension World of certain European and South American countries.

WILLIAM A. DREHER:

Once again Mr. Trowbridge deserves our thanks for a valuable addition to actuarial literature. The cluster of ideas underlying his unfunded present value family of pension funding methods will refresh the memories and stimulate the imagination of those of us whose professional practice focuses on these problems.

Mr. Trowbridge's new funding method has, in my opinion, two substantial advantages:

1. The concept that each year's contribution is a percentage of the quantity, $V_t - F_t - 1$, emphasizes that all funding methods are merely means of budgeting the employer's periodic payments to finance the benefits of his pension plan. Consulting and group actuaries have been using this concept in discussions with clients at the time a pension plan is being adopted, but Mr. Trowbridge's presentation of the new funding method will help us improve our explanations and should remind us of the need to continue to apply this test in future plan years.

2. It illustrates very clearly the ultimate implications of all the common funding methods, in terms of the funding ratio, $F_\infty/V_\infty$, and the level of annual contributions.

I believe there are a few practical disadvantages to Mr. Trowbridge's funding method.

1. Throughout the theoretical development of the funding method, $k$

is treated as a constant. However, a constant $k$ large enough to give a high ultimate funding ratio produces a pattern of annual contribution that declines rapidly.

Of course, it is not difficult to select a series of $k$ values that increases with duration and ultimately produces a high funding ratio through a pattern of annual contributions that is substantially level in dollars or as a per cent of payroll of employees participating in the pension plan. Mr. Trowbridge anticipates this possibility in Item (e) under his discussion of "Flexibility." One other possibility would be to select $k_t$ by solving the formula

$$(k_t + d)[V_t - F_{t-1}] = \frac{\frac{F_\infty}{V_\infty} V_t - F_{t-1}}{\ddot{a}_{\overline{n-t}|}}.$$

This would require selection of the desired ultimate funding ratio and the period of years, $n$, over which this goal would be reached.

Unfortunately, to define $k$ by devices such as the one suggested above is rather contorted and does violence to the principle of simplicity that is inherent in a general theory.

2. Too much stress is placed on the ultimate condition of the pension fund and not enough is given to important immediate questions, such as, "What is the actuarial value of accrued benefits? and what is the actuarial value of benefits attributable to the current year of plan participation?" These answers are and will continue to be important because:

*a*) Private pension plans will not all have perpetual existence. Companies fail, are sold, or merged; corporate policy in respect of employee security plans may change. Any of these influences may lead to the curtailment or discontinuance of the pension plan. Most employers want and should be given information about the current status of liabilities under the plan before selecting the current contribution.

*b*) There is increasing pressure to grant employees vested rights under pension plans and guarantee that those rights are secure.

*c*) The Internal Revenue Service, in testing the reasonableness of employers' claims for tax deductions, has shown a tendency to base its opinion, at least in part, on a comparison between the assets of the pension fund and the actuarial value of accrued benefits under the plan.

*d*) Investors and accountants evaluate a company's current operating results by comparing them with past performance. This requires a matching of income and expense and a consistent method of handling comparable items.

3. A wide range from which the value of $k$ may be selected could lead to confusion. Many employers—and some of us actuaries—need a peg on

which to hang a decision about the current contribution into a pension plan. The traditional funding methods offer a series of such benchmarks.

Also, all funding methods that include the concept of past-service liability give the employer substantial flexibility in the budgeting of his contributions. Perhaps the best way to use Mr. Trowbridge's new funding method would be to use his technique to analyze the condition of a pension fund and the desired level of current contributions but to present the results of this analysis to the client or policyholder in terms of a traditional funding method.

HARWOOD ROSSER:

Mr. Trowbridge is to be earnestly complimented on another excellent contribution in this field. His generalized approach, which intersects existing practice at several points, would appear to have a good chance of fairly widespread adoption by the pension fraternity, and perhaps even by the Treasury Department. Thus, the range of deductible contributions in a year might be defined by two values of $k$, possibly set in advance. In this case, there would be few mourners for such phrases as "unfunded past service liability."

As the author implies, the fixing of $k$ alone does not determine the degree of actuarial soundness. This will also depend on the actuarial assumptions chosen. Within any given set of such assumptions, Mr. Trowbridge's $k$ is a sort of common denominator for making much needed quantitative comparisons among various funding methods. It is to be hoped that his forging of this new tool may help to divert the Treasury from a tendency to concentrate on a single assumption at a time and toward a broader viewpoint, with more attention to the effect of the funding method.

This reviewer shares the author's qualms about too much flexibility.[1] Part of a professional man's duty—whether he be doctor, lawyer, or actuary—is to steel his client's soul against the temptations of delay and of the easy path. A most useful aid in such an effort is a set of projections, such as the author's first two tables. Here, again, being able to equate a particular level of funding, and hence a funding method, to a specific value of $k$ (at least in the ultimate situation) greatly shortens the numerical work, especially if an electronic computer is used.

One of the things that simplifies the analysis is Mr. Trowbridge's ingenious terminology, just as the adoption of a symbol for zero, in the Arabic numeral system, revolutionized multiplication and division. (It

[1] Cf. *TSA* X, 9.

staggers the imagination to consider the problem of programming on an electronic computer, using Roman numerals.) While he is primarily concerned with $k$, his defining of an auxiliary function $b$ enables him to observe a number of the latter's characteristics directly, such as its independence of withdrawal rates or salary scales. From personal experience, I have learned that such auxiliary functions can be extremely helpful.[2] Of course, $b$ has considerable meaning in its own right.

I will make one comment on external aspects. Mr. Trowbridge is continuing to uphold the tradition that, at least in terms of papers presented, insurance company actuaries are as much interested in pensions as are their consulting brethren. This is borne out by a brief tally of the indexes of the first thirteen volumes of the *Transactions*. Under the heading, "Retirement Plans," twenty-four formal papers are listed. Twelve of these were by actuaries then in consulting work, eleven by company men, and one by an insurance department official. Parenthetically, there has been some interchange between categories by the authors since these were written. As to any possible connection between authorship and such fence-hopping, I have no comment.

Returning to Mr. Trowbridge's paper, I would advise him, if he has not already decided to do so, to order a substantial number of additional reprints. I predict a considerable demand for them.

JAMES C. HICKMAN:

I will not attempt to hide my enthusiasm for this paper. I agree completely with Mr. Trowbridge's list of advantages for this family of funding methods. It seems to me that the disadvantages he lists are simply a restatement of the obvious admonition that a new and extremely flexible funding method may conceivably be misused.

It is clear that currently we have no specific rule for fixing the upper limit on annual federal income tax deductions allowable for contributions made to plan being funded by a member of this family. However, it does not seem to me that the establishment of an appropriate limit will be an extremely difficult problem. For many years we have faced a somewhat similar problem in the use of the aggregate funding method. Under the aggregate method the annual contribution is $(V_t - F_{t-1})/(\text{weighted average temporary annuity})$. As Mr. Trowbridge pointed out in his 1952 paper, the limit on annual deductions under aggregate funding (a method which is a member of the new family) has been stated in terms of a lower limit on the weighted temporary annuity. Since the contribution under a member of the new family is $[V_t - F_{t-1}]/[1/(k + d)]$, it appears as if a

[2] Cf. *TSA* XIV, 211 ff., esp. p. 217.

lower limit on $1/(k+d)$ might be a workable method for solving the problem.

The development of the mathematical properties of the members of this family of funding methods under the assumption that we have an initially stationary population may be very succinctly carried out using a continuous model. In this model we will use the following definitions.

$\bar{C}_t$ = annual contribution rate at time $t$.

$\bar{F}_t$ = fund at time $t$.

$\bar{B}_t$ = annual benefit payment rate at time $t$. Under our stationary population assumption $\bar{B}_t = T_r$ a constant which will be denoted by $\bar{B}$.

$\bar{V}_t$ = present value of future benefits. Under our assumption

$$\bar{V}_t = \int_a^r l_x \cdot {}_{r-x}|\bar{a}_x dx + \int_r^\infty l_x \bar{a}_x dx = \frac{1}{\delta}[T_r - l_a \cdot {}_{r-a}|\bar{a}_a],$$

a constant which will be denoted by $\bar{V}$.

$\bar{b} = (\bar{B}/\bar{V})$.

Then for members of the new family of funding methods we have

$$\bar{C}_t = (k+\delta)(\bar{V} - \bar{F}_t), \qquad k > 0.$$

Using the differential equation analogue of equation (1) in Mr. Trowbridge's 1952 paper, we have

$$\frac{d\bar{F}_t}{dt} = \bar{C}_t + \delta\bar{F}_t - \bar{B}$$

$$= (k+\delta)\bar{V} - k\bar{F}_t - \bar{B}.$$

Solving this linear differential equation and imposing the condition that $\bar{F}_0 = 0$, we have

$$\bar{F}_t = [(k+\delta)\bar{V} - \bar{B}][(1 - e^{-kt})/k]$$

$$= \bar{V}(\delta + k - \bar{b})(1 - e^{-kt})/k.$$

Note that for $k > \bar{b} - \delta > 0$ we will have a positive fund such that $\bar{F}_\infty = \bar{V}(\delta + k - \bar{b})/k$ and $\bar{C}_\infty = \bar{V}(\bar{b} - \delta)(k+\delta)/k$. These results help to supply the mathematical reasons for characteristics (1) and (4) of this family as Mr. Trowbridge lists them in his observations on Table I.

If a gain or loss of amount $\Delta_{t_0}$ is recognized at time $T_0$, for example a capital gain, the differential equation defining the subsequent progress of the fund (denoted now by $\bar{F}_t^*$ with corresponding contributions of $\bar{C}_t^*$) remains as before but now we have the condition that $\bar{F}_{t_0}^* = \bar{F}_{t_0} + \Delta_{t_0}$. Solving the differential equation and imposing the new condition, we have

$$\bar{F}_t^* = (\bar{F}_{t_0} + \Delta_{t_0})e^{-(t-t_0)k} + [(k+\delta)\bar{V} - \bar{B}][1 - e^{-(t-t_0)k}]/k, \qquad t > t_0.$$

From this equation for the subsequent fund, it is clear that the impact of $\Delta_{t_0}$ on the fund and also on the subsequent contributions, $\bar{C}_t^* = (k + \delta)(\bar{V} - \bar{F}_t^*)$, $t > t_0$, will diminish as $t$ increases and, as Mr. Trowbridge observes, the relative speed of completing the adjustment will depend on $k$.

EDWARD H. WELLS:

When Trowbridge writes a paper on pension funding, I read it before any other papers in the same galley. This dates back to his ingenious paper in *TSA* IV, 17, which is still sufficiently vivid in my recollection to make me marvel that it was published eleven years ago.

There is an interesting relationship between Trowbridge's parameter $k$, or rather $k + d$, in the present paper, and the "funding factors" developed by the late Clifton L. Hickok, A.S.A., for our company's use in certain areas of pension fund calculation. Clif was "Director of Group Pensions" when he passed away last April, leaving his wife and three delightful young daughters. The responsibilities of his position made him more of a salesman than an actuary, although I had always kept close to him because he had formerly been one of my associates. He had designed more unusual methods of pension funding than perhaps even Trowbridge knows of. I became horrified more than once about the administrative complexities and valuation problems of some of Clif's tailor-made plans, many involving group life insurance benefits as well as group annuities, but Clif invariably assured me that it was really all very simple if I only bore in mind his funding factor approach.

I wish he were here to explain his funding factors, and conduct this discussion, but I am afraid I shall have to do it. Despite repeated urging, Clif never agreed to write it up in the form of a paper. Neither do I intend to explain the whole idea, but rather to restrict this discussion mainly to how the funding factors relate to Trowbridge's parameters.

In Clif's parlance a funding factor $f_x$ was an attained age function, representing that fraction of the unfunded pension benefit (or you can also use that fraction of the unfunded reserve), beginning at retirement age $r$, to be purchased at age $x$. Clif was not particularly concerned with deriving formula expressions for $f_x$ corresponding to various funding bases, since it was easier to work out a table suitable for the precalculated net premiums starting at a low entry age by progressing backward from the year when the last premium was due for completely funding the benefit. This is, in the usual case, age $r - 1$. Obviously $f_{r-1} = 1$, because 100 per cent of the unfunded benefit is to be purchased in that year. This corresponds to Trowbridge's initial funding method if all purchases are deferred to that age. At the next earlier age the net premium, if level, buys a larger amount of deferred annuity, say, $K_{r-2}$, as contrasted with $K_{r-1}$. Then

$$f_{r-2} = \frac{K_{r-2}}{K_{r-2} + K_{r-1}}, \text{ and so forth,}$$

to

$$f_x = \frac{K_x}{\sum_{t=x}^{r-1} K_t}.$$

Since $K_t$, in the level premium, pure deferred annuity case, is proportional to $D_t$

$$f_x = \frac{1}{\ddot{a}_{x:\overline{r-x}|}}.$$

Trowbridge's formula for $k + d$, as shown in his paper for the similar aggregate funding case, is the harmonic mean of these funding factors for all the lives involved. Hickok once proposed such a mean, as I recollect it, in a deposit administration case but ran into difficulty because of a peculiar vesting provision desired by the employer.

Hickok's funding factors have surprisingly broad applicability to most pension arrangements. He was dealing usually with a single table of attained age functions. Two more simple illustrations are:

1. Unit credit funding (for future service benefits):

$$f_x = \frac{1}{r - x}.$$

2. Endowment at age $r$:

$$f_x = 1 - \frac{\ddot{a}_{x+1:\overline{r-x-1}|}}{\ddot{a}_{x:\overline{r-x}|}}.$$

In the latter case, a little study will convince you that, even when the insurance purchase is on a different table from the reserve accumulation, it is possible to construct a single linear table that does the trick for every entry age. It also suffices when the benefit is increased from time to time by superimposing additional level premiums.

Even if the original or superimposed premiums are imperfectly calculated, through rounding discrepancies, or, say, the use of interpolation methods, you still wind up with the right answer, going through the table of funding factors, simply because the last factor is unity. So all the pension gets purchased after all—but with a slightly unlevel premium. Moreover, if the rate basis for future purchases is changed, it may prove unnecessary, in practice, to change the funding factors because of negligible differences.

From biology we learn that a linear string of genes along a chromosome in a cell completely describes the organism of which the cell is a part. In the same way, it may be said that Hickok's funding factors are the genes, and Trowbridge's parameters the chromosomes, that completely describe the pension plan organism.

Need I add that there are, of course, situations demanding a square table of funding factors, as for the retirement endowment, an invention of the devil, although unfortunately a popular pension funding vehicle.

(AUTHOR'S REVIEW OF DISCUSSION)

CHARLES L. TROWBRIDGE:

First let me express my sincere appreciation for the excellent discussion from so many different sources. It is particularly gratifying to find, among those who have shown enough interest in the paper to write about it, consulting actuaries, insurance company actuaries, and actuaries from the academic world. Mr. Rosser would seem to be right in his suggestion that several segments of the actuarial profession are interested in pension funding.

It is evident from the discussions of Dr. Nesbitt and Mr. MacKinnon that "the basic concept has been in the air for some time," as Dr. Nesbitt puts it. Frankly, the author did not realize the extent to which others were thinking along similar lines. Dr. Nesbitt's concept of "perpetual amortization" and Mr. MacKinnon's "moving amortization period" seem to be essentially alike, and both have much in common with the unfunded present value family.

In fact, Dr. Nesbitt's modified aggregate method and the unfunded present value family appear to be identical twins in slightly different dress. His Tables A and B are particularly enlightening. Who would have guessed, for example, that the $k = 1.70259$ per cent representing Class II or terminal funding could have been obtained by $k + d = 1/\ddot{a}_{\overline{36}|}$? It helps me in getting a feel for Dr. Nesbitt's $\xi$, $\eta$, and $\tau$ to compute each, and hence arrive at the amortization periods implied by each of the funding methods. For the example illustrated in the paper:

| Funding Method | Amortization Period in Years |
|---|---|
| Initial | $0+1=1$ |
| Entry age normal | $\xi-a+1=10+$ |
| Unit credit | $\eta-a+1=19+$ |
| Terminal | $r-a+1=36$ |
| Pay-as-you-go | $\tau-a+1=44+$ |

It would be worth somebody's time to investigate to what extent these amortization periods are functions of the retirement age $r$, the hire age $a$, the interest rate, and the underlying service table. It is clear, for example, that the period for terminal funding depends only on $r - a$ and is independent of everything else.

Dr. Nesbitt's second family of modified aggregate funding methods and Mr. MacKinnon's next but last paragraph both include the ingenious idea of leaving a part $L_0$ of the accrued liability unfunded. Dr. Nesbitt would let $L_0$ be set aside and pay interest only on it, while all the rest of $V$ is funded by the aggregate method. Mr. MacKinnon would vary not only $L_0$ but the "moving amortization period" $n$, thereby getting an extra dimension of flexibility.

Mr. Sondergeld feels that the algebraic presentation might have been improved if it had been in terms of $k' = k + d$ and $b' = b + d$ instead of in terms of $k$ and $b$. Dr. Nesbitt makes in effect the same translation, for his $c$ is also my $k + d$. In some ways I, too, prefer Mr. Sondergeld's forms to mine. His $k'$ and $b'$ are more straightforward and simpler in concept than $k$ and $b$. I did have a reason (maybe not a good reason) for my definitions, and Mr. Sondergeld rightly surmises that they involve an attempt (not entirely successful) to get the important formulas independent of the interest rate. The main virtues of $k$ and $b$ are that $d$ drops out of the important equation (2), and hence out of the fund ratio

$$\frac{F_\infty}{V_\infty} = \frac{k - b}{k}.$$

Mr. Sondergeld sees that I really have not gotten rid of the interest rate, however, since $b$ itself depends upon the interest rate.

The table below looks into the behavior of $k$, $b$, $k'$, and $b'$ as the interest rate changes, based on the illustration in the paper.

| Fund Ratio Objective | Interest Rate | $b$ | $k$ | $b'$ | $k'$ |
|---|---|---|---|---|---|
| 80%......... | 2½% | 1.20774% | 6.03870% | 3.64676% | 8.47772% |
| | 3½ | 0.93241 | 4.66205 | 4.31405 | 8.04369 |
| | 4½ | 0.71375 | 3.56875 | 5.01997 | 7.87497 |
| 50%......... | 2½% | | 2.41548% | | 4.85450% |
| | 3½ | | 1.86482 | | 5.24646 |
| | 4½ | | 1.42750 | | 5.73372 |

From the above I reach the conclusion that the $k'$ and $b'$ are probably as stable as $k$ and $b$ as the interest rate changes and hence my reason for using $k$ and $b$ not entirely sound. Both $b$ and $k$ diminish with increasing interest, and $b'$ goes up. The $k'$ has a little more complicated behavior, because $d$ as well as $b$ enters the formula. The direction of its change is downward with increasing interest for high fund ratio objectives but upward for low.

Mr. Tookey refers to the search by the accounting profession for a means of determining for a pension plan a "standard charge to operations." Mr. Bronson's third "limiting possibility" refers to the same accounting development, as does 2(*d*) of Mr. Dreher's remarks. The members of the Society's Committee To Study Pension Accounting know that the concept underlying the paper arose in the author's mind from an idea Frank Griffin presented to his committee in this same connection. If the unfunded present value family ever proves to be useful in solving the accountant's problem (as Mr. Tookey perhaps suggests), more than mere coincidence will have been involved.

Several of the discussors have alluded to the problem of limits on the choice of $k$ (or $k' = c = k + d$) in order to make the unfunded present value acceptable to the Internal Revenue Service. Mr. Rosser and Mr. MacKinnon see possibilities in a preset upper and lower limit. Dr. Hickman thinks one might get at the upper limit through analogies with the aggregate method. Dr. Nesbitt's "modified aggregate" terminology obviously leads in the same direction.

On the other hand, Mr. Dreher's last paragraph seems to indicate a feeling that a past-service liability concept may be necessary to obtain flexibility in contributions. I assume he has the I.R.S. in mind, because I am sure he would not argue that there is insufficient flexibility within the unfunded present value family, provided its inherent flexibility can actually be employed.

The author hopes to find time to look more closely into the possibility of establishing some criteria which would be acceptable to the I.R.S.

The author appreciates Dr. Hickman's expression of enthusiasm for the paper and also the development he has furnished, using a continuous model. Dr. Nesbitt is also more used to the continuous basis. The author finds himself a little too far away from his college days to feel as comfortable as he should in the continuous medium.

After reading Mr. Wells's discussion, one finds himself wishing that Mr. Hickok had published his concept of "funding factors." Mr. Wells has given us a glimpse of an idea that might otherwise be lost completely and might be persuaded to tell us more of what he knows about Mr. Hickok's methods.

I do not feel particularly qualified to comment on Mr. Bronson's remarks. "IRS Old," which we have more or less learned to live with, is no longer the only outside force putting some bounds on a pension actuary's freedom of action. Mr. Bronson would no doubt agree that those of us who work for life insurance companies have all the forces he mentions to deal with, and state insurance departments in addition.

Mr. Dreher mildly criticizes the unfunded present value method because of the stress it puts on the ultimate condition of the pension fund at the expense of more nearby objectives. His point is well taken. The method is not, however, limited to the extremely long view. As a suggestion for further investigation one might well propose questions of this nature: What is an appropriate value of $k$ such that in $n$ years $F$ shall be equal (at least approximately) to (1) the value ($V_1$) of accrued benefits or (2) the value ($V_2$) of vested benefits? The solution to this problem would seem to depend on better knowledge than we now have as to the relationships between $V_1$ and $V_2$ and the over-all value of benefits $V$.

There may be some point in evaluating the discussion from the point of view of the acceptability, among pension actuaries as a whole, of a method which ignores completely the concept of an accrued or past service liability. I get the feeling that Mr. Rosser, and perhaps Drs. Nesbitt and Hickman, could get along very nicely without an accrued liability. Mr. MacKinnon and Mr. Dreher quite properly suggest that tradition is strong, so that as a practical matter many actuaries and many employers will want to see results in terms of their previous orientation. The author certainly has no quarrel with the accrued liability idea. It has served a useful purpose in the past and may continue to do so in the future. I do feel that it is a source of many misconceptions and much confusion and that the present value family may lead to a desirable simplification in pension thinking.