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APPORTIONABLE BASIS FOR NET PREMIUMS AND RESERVES

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CECIL J. NESBITT:

The author states that a new function, the apportionable annuity due, will be defined and derived. This function is not really new. Spurgeon, in chapter ix, section 8, of his text entitled *Life Contingencies*, briefly refers to a "complete annuity-due" but says that there is no such thing. Quite a number of years ago, before Jordan's text appeared, we discussed the notion of an apportionable or refund annuity due in our classes at Michigan and suggested its use for calculation of net premiums and reserves on an apportionable basis. One definition that we used for the apportionable annuity due led to such formulas as the author's formulas (7) and (8). However, in discussing the paper "Complete Annuities," by E. A. Rasor and T. N. E. Greville (*TSA*, IV, 583), I was led to the defining relation

$$d^{(m)}\ddot{a}_x^{\{m\}} = \delta\bar{a}_x . \tag{1}$$

Here one thinks of both sides as representing the present value of interest payments on a principal of 1, interest to be paid throughout the exact whole of life of (x), on any basis that is equivalent to the force δ. In the case of the left member of this equation, in the $1/m$ interval of death, the payment $d^{(m)}/m$ at the beginning of the interval would overpay the interest $(1+i)^t - 1$ accrued to time of death, and a refund of $[d^{(m)}/m](1+i)^t - [(1+i)^t - 1] = 1 - v^{1/m-t}$ would be made. Thus, under this definition for $\ddot{a}_x^{\{m\}}$, the refund would be $(1 - v^{1/m-t})/d^{(m)}$ rather than $1/m - t$, as in the author's definition.

While this alternative definition has a more complex refund notion, it is strictly consistent with compound interest theory and the resulting premium and reserve relations are simple. Thus, for the ordinary life insurance case, we have

$$\begin{aligned} P^{\{m\}}(\bar{A}_x) &= \bar{A}_x/\ddot{a}_x^{\{m\}} \\ &= \bar{A}_x\Big/\left[\frac{\delta}{d^{(m)}}\,\bar{a}_x\right] \qquad (2) \\ &= \frac{d^{(m)}}{\delta}\bar{P}(\bar{A}_x). \end{aligned}$$

In other words, $P^{\{m\}}$ is here exactly equal to the author's discounted continuous premium. Moreover,

$$\begin{aligned} {}_tV^{\{m\}}(\bar{A}_x) &= \bar{A}_{x+t} - P^{\{m\}}(\bar{A}_x)\ddot{a}_{x+t}^{\{m\}} \\ &= \bar{A}_{x+t} - \frac{d^{(m)}}{\delta}\bar{P}(\bar{A}_x)\frac{\delta}{d^{(m)}}\bar{a}_{x+t} \qquad (3) \\ &= {}_t\bar{V}(\bar{A}_x). \end{aligned}$$

Thus, the apportionable basis reserve is here exactly equal to the continuous basis reserve.

If a definition of an apportionable annuity due is to be adopted, I would vote for the definition based on relation (1). Under this definition, net premiums on the apportionable basis are derived easily from continuous net premiums, and reserves on the apportionable basis exactly equal reserves on the continuous basis. It would then be best to tabulate continuous basis functions rather than apportionable basis functions as premiums, for the latter vary with m.

As a sidelight, it may be added that for apportionable premiums and reserves for m a fraction such as $\frac{1}{2}$ or $\frac{1}{5}$ would follow by the same formulas as before from relation (1). The author's formulas, however, might require re-examination.

I concur with the author that it is desirable to approach net premiums and reserves on an apportionable basis by means of a definition of an apportionable annuity due. I urge him to give consideration to the alternative definition of an apportionable annuity due that is presented here.

WILLIAM A. WHITE:

Mr. Lauer's paper on apportionable net premiums and reserves provides a different and interesting perspective on the subject of premium death benefits. Our traditional approach has been to treat premiums as though they were payable continuously during the lifetime of the insured. This approach is difficult to accept among people accustomed to thinking of premiums as payable annually in advance. While following through Lauer's derivation, I happened to journey along an interesting tangent in this connection.

A "New" Annual Apportionable Annuity Due

For simplicity I limited myself to the case where $m = 1$, although the formula can be generalized for more frequent payments. We start with a formula stating that the "new" apportionable annuity is our usual annuity due less a death benefit:

$$\ddot{a}^{\{1\}'}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \theta^{\{1\}'}_{x:\overline{n}|} \, . \tag{1}$$

The function $\theta^{\{1\}'}_{x:\overline{n}|}$ is identical to that defined by Lauer except that the present value of the premium refund benefit, B_t, is undefined.

$$\theta^{\{1\}'}_{x:\overline{n}|} = \sum_{r=0}^{n-1} v^r \, {}_r p_x \int_0^1 v^t \, {}_t p_{x+r} \mu_{x+r+t} B_t dt \, . \tag{2}$$

Lauer defined B_t as $(1 - t)$, on the assumption that equity will be done if a proportionate part of the annual premium (or, in this case, the unit annual payment) is returned at death. My departure was to assume that greater equity might be done if we were to allow interest from the date of receipt to the date of return on that portion of the annual premium which was not needed. Following the line of reasoning established by Boermeester and cited in Lauer's paper, we find that, for death at time t,

$$B_t = \frac{\bar{a}_{\overline{1-t}|}}{\bar{a}_{\overline{1}|}} \, . \tag{3}$$

By means of algebraic manipulation consistent with that performed by Lauer, the "new" annual apportionable annuity due can be simplified to

$$\ddot{a}^{\{1\}'}_{x:\overline{n}|} = \frac{\delta}{d} \, \bar{a}_{x:\overline{n}|} \, . \tag{4}$$

I then went ahead and calculated a "new" apportionable net annual premium payable for n years, using the notation of Lauer's paper:

$$\begin{aligned} {}_nP^{\{1\}'} &= \frac{PVFB_x}{\ddot{a}^{\{1\}'}_{x:\overline{n}|}} \\ &= \frac{d}{\delta} \cdot \frac{PVFB_x}{\bar{a}_{x:\overline{n}|}} \\ &= {}_n\bar{P}^d \, . \end{aligned} \tag{5}$$

This result is not really too startling, but it is interesting. It implies that our traditional *continuous net yearly* premiums and reserves are actually just *apportionable net annual* premiums and reserves, with an extra little (and not illogical) ingredient of interest in the definition of what is to be apportioned. Since the contractual benefit will normally be based on the gross premium—as often as not that portion applicable to the period *after the end* of the policy month of death—it may not be practical to suggest changing reserves simply because our company practices do not provide for interest in the premium refund. In fact, maybe our practices might be changed.

In any event, Lauer's paper gives us a valuable insight into an area too often neglected in our literature. I am sure that many of us would welcome additional such excellent studies of the theoretical and, especially, the practical problems posed by apportionable and continuous actuarial functions.

LAVERNE W. CAIN:

I was particularly interested in the relationship between continuous premiums and apportionable premiums as defined in Mr. Lauer's paper. Several times in the past few years, since our adoption of the continuous functions basis for reserve determination, I have been asked to explain how continuous functions provide for a premium refund at death. Also I have noted, as Lauer has, some confusion because of the existence of two net premiums associated with the use of continuous functions. This confusion has existed even among actuarial students and Fellows.

I was also interested in the resulting premium formula developed by Lauer compared to that developed by general reasoning in Jordan. The following development is very similar to that used in Lauer's paper. Using an ordinary life policy with immediate payment of claims and an annual premium, the following general equation holds where $f(t)$ is a function defining the refund of premium at death:

$$P\cdot \ddot{a}_x = \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_s p_x \int_0^1 v^t\, {}_t p_{x+s}\mu_{x+s+t}\cdot f(t)\,dt\,.$$

This refund is expressed as a portion of P, the annual premium. If $f(t)$ is set equal to $1-t$, I obtain the result that Lauer obtained:

$$P \doteq \frac{\bar{A}_x}{\ddot{a}_x - (1/2+\delta/12)\,\bar{A}_x}\,.$$

If $f(t) = \bar{a}_{\overline{1-t}|}/\bar{a}_{\overline{1}|}$, then the premium P is exactly equal to ${}_n\bar{P}^d$, the discounted continuous yearly premium. This short demonstration has proved very helpful to me in explaining the nature of continuous functions.

$$P\cdot \ddot{a}_x = \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_s p_x \int_0^1 v^t\, {}_t p_{x+s}\mu_{x+s+t}\cdot \frac{\bar{a}_{\overline{1-t}|}}{\bar{a}_{\overline{1}|}}\,dt\,;$$

$$P\cdot \ddot{a}_x = \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_s p_x\cdot \left(\frac{\bar{a}_{\overline{1}|} - \bar{a}_{x+s:\overline{1}|}}{\bar{a}_{\overline{1}|}}\right);$$

$$P\cdot \ddot{a}_x = \bar{A}_x + P\ddot{a}_x - P\,\frac{\bar{a}_x}{\bar{a}_{\overline{1}|}}\,;$$

$$\therefore \qquad P = \frac{\bar{A}_x}{\bar{a}_x}\cdot \bar{a}_{\overline{1}|} = {}_n\bar{P}^d\,.$$

If $f(t) = (1+i)^t(1-t)$, an interesting result is obtained. This refund could be described as pro rata plus interest from the beginning of the year. This is certainly a reasonable refund, although I know of no company using this practice. Using this, we have

$$P\cdot \ddot{a}_x = \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_sp_x \int_0^1 v^t\, {}_tp_{x+s}\mu_{x+s+t}\cdot (1+i)^t(1-t)\,dt;$$

$$P\cdot \ddot{a}_x = \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_sp_x \int_0^1 {}_tp_{x+s}\mu_{x+s+t}(1-t)\,dt.$$

Using uniform distribution of deaths, we have

$$P\cdot \ddot{a}_x \doteq \bar{A}_x + P\cdot \sum_{s=0}^{\infty} v^s\, {}_sp_x q_{x+s} \int_0^1 (1-t)\,dt;$$

$$P\cdot \ddot{a}_x \doteq \bar{A}_x + \tfrac{1}{2}(1+i)P\cdot A_x;$$

$$\therefore \qquad P \doteq \frac{\bar{A}_x}{\ddot{a}_x - \frac{1}{2}(1+i)A_x}.$$

This formula is similar to the formula used in Jordan's textbook. Jordan's formula could be derived similarly to the above if the refund were

NET LEVEL PREMIUMS PER $1,000 INSURANCE BASED ON 1958 CSO MORTALITY TABLE AND 3 PER CENT INTEREST

Age at Issue	Plan	Discounted Continuous Yearly Premium	Pro Rata Annual Premium	Pro Rata plus Interest Annual Premium
5.....	Whole life	$ 6.138	$ 6.138	$ 6.138
	20-pay life	11.498	11.498	11.498
	20-year term	1.443	1.443	1.443
	20-year endowment	36.934	36.934	36.934
35.....	Whole life	16.671	16.670	16.671
	20-pay life	24.652	24.652	24.653
	20-year term	5.271	5.271	5.271
	20-year endowment	38.597	38.597	38.598
65.....	Whole life	67.968	67.956	67.979
	20-pay life	70.890	70.880	70.901
	20-year term	61.331	61.321	61.340
	20-year endowment	72.143	72.132	72.154

equal to $(1-t)\cdot v^{1-t}$, which is not a reasonable refund of premium. Obviously, $(1+i)^t(1-t)$ is greater than $(1-t)$. It is fairly easy to show that

$$(1+i)^t(1-t) \geq \frac{\bar{a}_{\overline{1-t}|}}{\bar{a}_{\overline{1}|}} \geq (1-t) \quad \text{for} \quad 0 \leq t \leq 1\,,$$

but the effect on premium calculation is quite small. The tabulation on page 141 reproduces Table 1 of Lauer's paper with an additional column showing premiums with a pro rata refund plus interest.

The confusion caused by the existence of two net premiums under continuous functions can probably be reduced by a clear demonstration of the nature of the refund feature assumed. If not, then some consideration to adopting either Lauer's formulas or those derived above should be given.

JOHN A. MEREU:

Mr. Lauer's paper presents an interesting study of premiums on the apportionable basis and shows how closely these compare with premiums on the discounted continuous basis.

Under both bases a part of the last premium paid prior to the date of death is refundable but, as Lauer demonstrates, the portion refundable under the discounted continuous basis is slightly greater than the portion refundable under the apportionable basis. It follows, therefore, that discounted continuous premiums should be slightly larger than the corresponding apportionable premiums; this is borne out by Lauer's calculations.

To facilitate the comparison of apportionable premiums and discounted continuous premiums, Lauer has assumed a uniform distribution of deaths between integral ages. He has also ingeniously defined an apportionable annuity due which can be used to find the present value of apportionable premiums to be paid with due recognition given to the refund of the pro rata part of the last premium paid.

The author shows that the apportionable annuity-due is equal to the standard annuity due less a death benefit

$$\theta^{\{m\}}_{x:\overline{n}|} = \left[\frac{1}{d^{(m)}} - \frac{1}{\delta}\right] \bar{A}^{\,1}_{x:\overline{n}|}\,.$$

He then goes on to express the apportionable annuity due in terms of the continuous annuity in order to make a comparison between the apportionable premium and the discounted continuous premium.

Another way to facilitate the comparison of premiums on the two bases is to define a "discounted continuous" annuity due $\ddot{a}^{(\overline{m})}_{x:\overline{n}|}$, which would play the same role with respect to discounted continuous premiums that the apportionable annuity due plays with respect to apportionable premiums. It is shown below that the discounted continuous annuity due may be represented analogously to Lauer's equation (1) by

$$\ddot{a}^{(\overline{m})}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \theta^{(\overline{m})}_{x:\overline{n}|} ,$$

where $\theta^{(\overline{m})}_{x:\overline{n}|}$ is the present value of the refund due at the moment of death.

Corresponding to Lauer's equation (2) we have

$$\theta^{(\overline{m})}_{x:\overline{n}|} = \sum_{r=0}^{n-1} v^r \cdot {}_rp_x \sum_{s=0}^{m-1} v^{s/m} \cdot {}_{s/m}p_{x+r} \int_0^{1/m} \frac{1 - v^{1/m-t}}{d^{(m)}} \cdot v^t \cdot {}_tp_{x+r+s/m+t} \cdot \mu_{x+r+s/m+t} dt .$$

Under the assumption of a uniform distribution death, this simplifies to

$$\theta^{(\overline{m})}_{x:\overline{n}|} = \sum_{r=0}^{n-1} v^r \, {}_rp_x \sum_{s=0}^{m-1} v^{s/m} \cdot q_{x+r} \cdot \frac{1}{m d^{(m)}} \left[\frac{d^{(m)}}{\delta} - v^{1/m} \right]$$

$$= \sum_{r=0}^{n-1} v^r \cdot {}_rp_x \cdot q_{x+r} \cdot \frac{d}{d^{(m)}} \left[\frac{1}{\delta} - \frac{1}{i^{(m)}} \right]$$

$$= \sum_{r=0}^{n-1} v^{r+1} \, {}_{r|}q_x \cdot \frac{i}{d^{(m)}} \left[\frac{1}{\delta} - \frac{1}{i^{(m)}} \right]$$

$$= \frac{\delta}{d^{(m)}} \left[\frac{1}{\delta} - \frac{1}{i^{(m)}} \right] \bar{A}^{1}_{x:\overline{n}|}$$

$$= \left[\frac{1}{d^{(m)}} - \frac{\delta}{i^{(m)} \cdot d^{(m)}} \right] \bar{A}^{1}_{x:\overline{n}|} .$$

Now it can be shown that

$$\left[\frac{1}{d^{(m)}} - \frac{\delta}{i^{(m)} \cdot d^{(m)}} \right] = \frac{1}{2m} + \frac{\delta}{6m^2} - \frac{\delta^3}{180m^4} + \cdots .$$

Therefore it follows, making reference to Lauer's equation (8), that

$$\theta^{(\overline{m})}_{x:\overline{n}|} - \theta^{\{m\}}_{x:\overline{n}|} \doteq \frac{\delta}{12m^2} \cdot \bar{A}^{1}_{x:\overline{n}|} .$$

Hence $\theta^{(\overline{m})}_{x:\overline{n}|}$ is slightly greater than $\theta^{\{m\}}_{x:\overline{n}|}$. Therefore $\ddot{a}^{(\overline{m})}_{x:\overline{n}|}$ is slightly smaller than $\ddot{a}^{\{m\}}_{x:\overline{n}|}$. Therefore ${}_n\bar{P}^{d}$ is slightly larger than ${}_n\bar{P}^{\{m\}}$, which agrees with the conclusions reached by Lauer.

JOHN H. COOK AND ALLAN R. JOHNSON:

Mr. Lauer is to be commended for his perseverance in search of theoretical accuracy. However, we must take issue with his recommendation that the apportionable basis for net premiums and terminal reserves be adopted the next time a new mortality standard is published.

Let us consider the consequences of what Lauer proposes. Tables 1 and 2 in his paper quote numerical data which illustrate the small changes in net premiums and reserves that result from use of the apportionable basis. If these apportionable values represent absolute accuracy, then the continuous values represent a reasonable approximation to accuracy.

If one looked at Lauer's Table 2 and made an "armchair" estimate of the difference in the average reserve factor per $1,000 of insurance for a typical model-office distribution, it would not be unreasonable to assume that such difference would be less than 1 cent per thousand dollars of insurance. To change from the continuous basis with its inherent advantages to a less convenient basis producing essentially the same reserves is a questionable move if the only reason for the change is because of theoretical differences. Numerical refinements as a result of the apportionable basis are far less significant than the differences that could result from minor variations in tabular mortality rates themselves. Most recent mortality tables which serve as the basis for reserves and nonforfeiture benefits are graduated experience rates which contain mortality margins. The mortality margins alone can affect average reserve factors by far more than 1 cent per $1,000 of insurance. Such a statement does not argue against theoretical accuracy. It does emphasize, however, that theoretical accuracy for its own purpose is a will-o'-the-wisp. If we achieve such accuracy at an expense that is unreasonable, the choice seems to be obvious.

Looking directly at the weakness of the continuous basis, Lauer categorizes it as (generally) overstating both net premiums and reserves. This conclusion is based on an assumption that the premium refund should include a pro rata share of the net premium. Conversely, the premiums and reserves can be accepted as accurate if we shift the burden of approximation to the calculation of the refund at death (see Lauer's formula [26]). All premium refund provisions which we have reviewed provide for a pro rata portion of the gross premium. The reserve calculation takes into account the disposition of only the unearned net premium. The disposition of the loading on the unearned net premium would not affect the reserve. In theory, the premium refund can be analyzed as providing the appropriate share of the net premium in accordance with formula (26). The balance of the refund simply represents somewhat less than a pro rata share of the loading.

Lauer indicates in his introduction that the derivations in his paper are based on the assumption of a uniform distribution of deaths within each year of age. This particular assumption is commonly used in the continuous basis. Such an assumption, however, is obviously unrealistic and is inconsistent with the concept of a continuous progression in the force of mortality. Nowhere in the paper has Lauer given any indication of the magnitude of the distortion in premiums and policy values which results from this convenient but false assumption.

As a mathematical excursion, we have investigated in the Appendix to this discussion the consequences of an alternate assumption involving the distribution of deaths. The magnitude of the deviations as a result of that assumption is illustrated in the Appendix and can be seen to exceed many times the deviations resulting from the apportionable basis.

If we accept the principle that an apportionable premium represents greater accuracy, then consider the need and the consequences of achieving this greater accuracy. Lauer states that the apportionable basis in-

TABLE 1

q_x	r_x	Coefficient of D_x	Coefficient of D_{x+1}
0.00010	1.014926	0.495113	0.504960
0.00100	1.014929	.495015	0.505061
0.01000	1.014951	.494292	0.505805
0.10000	1.015189	.486475	0.513857
1.00000	1.03	0.0	1.014926

TABLE 2

NET LEVEL ANNUAL PREMIUMS PAYABLE AT THE BEGINNING OF THE YEAR BASIS: $1,000 OF INSURANCE; 1958 CSO MORTALITY; 3 PER CENT INTEREST

Age at Issue	Plan	Constant Force of Mortality	Continuous Basis	Apportionable Annual Basis
5.....	Whole life	$ 6.139	$ 6.138	$ 6.138
	20-pay life	11.500	11.498	11.498
	20-year term	1.443	1.443	1.443
	20-year endowment	36.935	36.934	36.934
35.....	Whole life	16.675	16.671	16.670
	20-pay life	24.657	24.652	24.652
	20-year term	5.271	5.271	5.271
	20-year endowment	38.598	38.597	38.597
65.....	Whole life	68.028	67.968	67.956
	20-pay life	70.938	70.890	70.880
	20-year term	61.368	61.331	61.321
	20-year endowment	72.184	72.143	72.132

TABLE 3

NET LEVEL TERMINAL RESERVE BASIS: $1,000 OF INSURANCE; 1958 CSO MORTALITY; 3 PER CENT INTEREST

Age at Issue	Duration	Constant Force of Mortality	Continuous Basis	Apportionable Annual Basis	Constant Force of Mortality	Continuous Basis	Apportionable Annual Basis
		Whole Life			20-Pay Life		
5	1	$ 4.96	$ 4.95	$ 4.95	$ 10.48	$ 10.48	$ 10.48
	5	26.80	26.79	26.79	56.20	56.20	56.19
	10	58.02	58.01	58.01	121.75	121.73	121.73
	15	92.83	92.81	92.81	196.74	196.71	196.71
	20	132.30	132.27	132.27	283.35	283.30	283.30
	40	353.24	353.16	353.16	....	....	....
	60	637.02	636.81	636.80	....	....	....
	80	860.52	859.76	859.75	....	....	....
	90	940.85	938.01	937.99	....	....	....
35	1	14.64	14.64	14.64	22.87	22.87	22.87
	5	76.50	76.48	76.47	120.48	120.46	120.46
	10	160.01	159.96	159.95	256.23	256.18	256.18
	15	249.28	249.20	249.19	408.90	408.82	408.82
	20	342.20	342.07	342.06	581.69	581.57	581.57
	30	528.57	528.32	528.30	....	....	....
	50	818.84	817.87	817.84	....	....	....
	60	923.18	919.49	919.45	....	....	....
65	1	37.93	37.88	37.87	40.98	40.94	40.93
	5	182.65	182.37	182.34	200.40	200.19	200.17
	10	345.97	345.33	345.27	392.67	392.23	392.19
	15	493.68	492.54	492.45	599.36	598.69	598.64
	20	615.73	613.87	613.76	884.80	884.17	884.17
	30	837.05	829.30	829.16	....	....	....
		20-Year Term			20-Year Endowment		
5	1	$ 0.12	$ 0.12	$ 0.12	$ 36.70	$ 36.70	$ 36.70
	5	1.04	1.04	1.04	195.74	195.74	195.74
	10	2.21	2.21	2.21	424.11	424.11	424.11
	15	1.81	1.81	1.81	689.77	689.77	689.77
35	1	2.88	2.88	2.88	37.25	37.25	37.25
	5	13.66	13.66	13.66	197.30	197.30	197.30
	10	22.52	22.52	22.52	424.30	424.29	424.29
	15	21.21	21.21	21.21	687.71	687.70	687.70
65	1	30.96	30.93	30.93	42.28	42.25	42.25
	5	142.04	141.89	141.87	208.00	207.83	207.81
	10	239.11	238.81	238.77	412.67	412.33	412.29
	15	251.85	251.46	251.43	644.60	644.17	644.13

volves only one net premium rather than the two that are involved in the continuous basis. We consider that the additional net premium is a small concession to avoid the extra labor of the calculations required by Lauer's formula (8).

Lauer states in his opening remarks that many companies provide in their policies that, when death occurs, a pro rata premium refund will be made. We would like to point out that six of the ten largest companies do not have such a provision in their current policy forms. Furthermore, a large percentage of the current issue is being placed on a monthly mode of payment basis. Of the four companies out of the ten largest which do provide a pro rata premium refund on death, three of them return only that portion of the gross premium applicable to the period beyond the month of death.

In the case of whole life and coterminous endowment policies, terminal reserves calculated on the continuous basis can be produced from net single premiums by use of the following convenient formula:

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = (\bar{A}_{x+t:\overline{n-t}|} - \bar{A}_{x:\overline{n}|}) \div (1 - \bar{A}_{x:\overline{n}|}) . $$

In accordance with Lauer's assumption, net single premiums have not been changed. Since he has changed terminal reserves, he has destroyed the convenience of this formula for the calculation of reserves. Of course, the formula quoted above is also lost to a company which assumes annual premiums without refund and immediate payment of claims. In this case, however, there is a contract provision which is more radically at variance with the full continuous basis.

In addition to the administrative inconvenience, the apportionable basis would require the explanation of another reserve calculation method to the federal income tax auditors. It might also raise the question of mandatory use for minimum reserves and nonforfeiture benefits in the event that the insurance departments should accept the new procedures.

In summary, we say that increased accuracy is fine if the resultant condition represents an improvement. It has no other justification. Theoretical accuracy for the sheer benefit of theory is of little value. In fact, if it is achieved at a net cost, it is less than worthless.

We enjoyed reading Lauer's paper and in following the logic which he developed. We appreciate the fact that he published his paper even though we are opposed to adopting the practice that he proposes. We believe that investigations of the type conducted by Lauer are essential to the effective continuation of the actuarial profession. If we did not agree with this, we would be at a loss to defend the mathematical excursion which we append to this discussion.

APPENDIX

ANALYSIS OF NONUNIFORM DISTRIBUTION OF DEATHS

A uniform distribution of deaths within each year of age requires that l_{x+t} be linear between x and $x + 1$. Since $l_{x+t}\mu_{x+t}dt = -dl_{x+t}$, a constant value of dl_{x+t} means that $l_{x+t}\mu_{x+t}dt$ is constant, and μ_{x+t} must be increasing. Furthermore, $\mu_{x+\epsilon} \doteq d_x \div l_x$, whereas $\mu_{x-\epsilon} \doteq d_{x-1} \div l_x$. For that range of ages where d_x is an increasing function, we have a sudden increase in the force of mortality at the integral age.

Whatever curve represents the force of mortality, the integral under the curve of $l_{x+t}\mu_{x+t}$ is fixed by the value of d_x. A discontinuity with an increase at the integral age represents a smaller slope within each year of age than would occur in a continuous curve. Conversely, a discontinuous drop in the value of μ_x, in the age range where d_x is a decreasing function, results in a steeper slope than for a continuous function.

A decrease in the slope of the force of mortality within a year of age at a point where the level of mortality is low and the change in the slope is even smaller makes an insignificant difference in insurance costs. At the higher ages an increase in the slope of the force of mortality is far more significant because it anticipates more premium income from the relatively large number who die within the year and it anticipates more interest income before making payment of the claim.

Compared with a continuously increasing curve for the force of mortality, the assumption of a constant force of mortality throughout a year of age overstates insurance costs at all ages. The amount of the overstatement at the younger and middle ages is insignificant, as it is with uniform distribution of deaths. At the older ages, however, it tends to overstate costs much the same as uniform distribution of deaths understates costs. The following analysis is developed for the purpose of giving some measure of the financial effect of these features.

Assume a net single premium at age x for one year of coverage providing a death benefit of unity during the year with a pure endowment payable at the end of the year equal to the reserve at that time. Further, assume the value of the net single premium, symbolized as A_0' to be equal to $\mu_x \div (\delta + \mu_x)$. Also, assume that the force of mortality remains constant between ages x and $x + 1$. Under these conditions, the reserve remains constant throughout the year, since the interest increment is exactly equal to and offset by the mortality decrement. Accordingly A_0' is equal to A_1'.

Consider a new symbol $\bar{C}_x'$, such that $\bar{C}_x' \div D_{x+1}$ is the accumulated value, at age $x+1$, of a unit death benefit payable at death between ages x and $x+1$, reflecting a constant force of mortality during that year. Also consider r_x such that $\bar{C}'_x = r_x C_x$.

It is now true that the terminal reserve can be calculated from the initial reserve by the following formula:

$$\frac{A'_0 D_x - \bar{C}'_x}{D_{x+1}} = A'_1 .$$

It has already been demonstrated that, if A'_0 is equal to $\mu_x \div (\delta + \mu_x)$, A'_1 is equal to A'_0. Making the appropriate substitution and solving for the value of r_x, we find that

$$r_x = \frac{\mu_x}{q_x}\,\frac{i+q_x}{\delta+\mu_x}.$$

The value of r_x varies as the value of q_x varies. Reflecting the assumption that μ_x remains constant throughout a year of life, μ_x must be equal to the negative of $\log_e (1 - q_x)$.

By referring to the identity $\bar{A}_x = 1 - \delta\bar{a}_x$ and defining new symbols $\bar{M}'_x$, $\bar{D}'_x$, and $\bar{N}'_x$, it can be demonstrated that

$$\bar{D}'_x = \frac{1 - v r_x}{\delta} D_x + \frac{r_x - 1}{\delta} D_{x+1} .$$

Illustrating the effect of the mortality level and 3 per cent interest on the value of r_x and the coefficients of D_x in calculating $\bar{D}'_x$, the values are shown in Table 1.

Based on these formulas for $\bar{C}'_x$ and $\bar{D}'_x$, values have been calculated for premiums and reserves corresponding to those in Lauer's Tables 1 and 2. The results of these calculations are shown in Tables 2 and 3, along with the continuous and the apportionable results. The magnitude of the differences speaks for itself.

HARWOOD ROSSER:

This discussion of Mr. Lauer's excellent paper will deal mainly with the relationship between his apportionable annuity due and a complete annuity. A little thought shows that the following relationships hold:

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} - \mathring{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}(1 - \bar{A}_{x:\overline{n}|}) . \qquad (1)$$

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} = (1+i)^{1/m}\mathring{a}^{(m)}_{x:\overline{n}|} . \qquad (2)$$

The first of these recognizes that, except at the ends, the two series of payments are identical. The single-premium endowment is called for, since, in subtracting the complete annuity, we must account for the final full payment at the end of n years if the recipient is then alive.

In formula (2), we note that the total of the annuity amounts actually retained, including the fractional portion for the period when death occurs, is the same for both. The difference is that Lauer's annuity makes each payment $1/m$ sooner (except possibly the final one).

Using these, I made an unsuccessful attempt to confirm his formula (8) by a shorter route, since there are several published formulas for complete annuities. The important terms were confirmed, but full substantiation was lacking.

Numerical verification along these lines was also considered. However, Lauer gives no figures directly for his annuity, and the figures that he shows for the related premiums are only to three decimals. Thus, a reproduction of these would be inconclusive.

The bristling algebra tells us why this very logical approach has not been widely used before this. We must applaud Lauer for undertaking so formidable a task.

B. GEORGE ISEN:

This paper by Mr. Lauer has filled in a small gap in life contingency theory not usually covered in the training of our younger actuaries for our examinations. My discussion relates to a theoretical idea stimulated by Lauer's analysis.

On page 149 of *Life Contingencies*, by C. W. Jordan, Jr., there is a discussion of the complete life annuity, defined by the symbol $\mathring{a}_x^{(m)}$, representing the present value of $1/m$ at the end of each $(1/m)$th of a year that a life, now aged x, is alive plus a payment at the moment of death equal to the proportional part of that period lived prior to death. Jordan also refers to this annuity as an apportionable annuity. Lauer has defined the present value of an apportionable life annuity due as $\ddot{a}_x^{\{m\}}$.

These may be represented as follows:

$$\ddot{a}_x^{\{m\}} = \ddot{a}_x^{(m)} - \sum_{t=0}^{\infty} v^{t/m}\ {}_{t/m}p_x \int_0^{1/m} \left(\frac{1}{m} - s\right) v^s\ {}_s p_{x+t/m}\,\mu_{x+t/m+s}\,ds\ . \quad (1)$$

$$\mathring{a}_x^{(m)} = a^{(m)} + \sum_{t=0}^{\infty} v^{t/m}\ {}_{t/m}p_x \int_0^{1/m} s\, v^s\ {}_s p_{x+t/m}\,\mu_{x+t/m+s}\,ds\ . \quad (2)$$

By subtracting equation (2) from equation (1), we have

$$\ddot{a}_x^{\{m\}} - \mathring{a}_x^{(m)} = \ddot{a}_x^{(m)} - a_x^{(m)} - \frac{1}{m}\sum_{t=0}^{\infty} v^{t/m}\ {}_{t/m}p_x \int_0^{1/m} v^s\ {}_s p_{x+t/m}\mu_{x+t/m+s}\,ds$$

$$= \frac{1}{m}(1 - \bar{A}_x)$$

$$= \frac{\delta}{m}\,\bar{a}_x .$$

Then this relationship between the apportionable annuity (complete annuity) and the apportionable annuity due, as defined by Lauer, is as follows:

$$\ddot{a}_x^{\{m\}} = \mathring{a}_x^{(m)} + \frac{\delta}{m}\,\bar{a}_x . \tag{3}$$

Equation (3) may be considered analogous to

$$\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m} . \tag{4}$$

It is also interesting to note that $\delta \cdot \bar{a}_x$ may be reinterpreted as a ratio of annuities, $\bar{a}_x/\bar{a}_{\overline{\infty}|}$ and equation (3) may be further expressed as

$$\ddot{a}_x^{\{m\}} = \mathring{a}_x^{(m)} + \frac{1}{m}\left(\frac{\bar{a}_x}{\bar{a}_{\overline{\infty}|}}\right). \tag{5}$$

The form of equation (5) shows more clearly the analogy with equation (4), equation (5) reducing to equation (4) when the death benefit element is eliminated, and $\ddot{a}_x^{\{m\}} \rightarrow \mathring{a}_x^{(m)}$ as $m \rightarrow \infty$. It then follows that

$$P^{\{m\}}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x^{\{m\}}} = \bar{P}(\bar{A}_x) \cdot \frac{\bar{a}_x}{\mathring{a}_x^{(m)} + (\delta/m)\,\bar{a}_x} .$$

By using the approximation, $\mathring{a}_x^{(m)} \doteq (1 - \delta/2m)\bar{a}_x$,[1] it follows that

$$\ddot{a}_x^{\{m\}} \doteq (1 + \delta/2m)\bar{a}_x .$$

This results in the following approximation:

$$P^{\{m\}}(\bar{A}_x) \doteq \frac{1}{1 + (\delta/2m)}\,\bar{P}(\bar{A}_x) .$$

When m is large,

$$P^{\{m\}}(\bar{A}_x) \rightarrow \bar{P}(\bar{A}_x) .$$

When $m = 1$,

$$P^{\{1\}}(\bar{A}_x) \doteq \frac{1}{1 + \frac{1}{2}\delta}\,\bar{P}(\bar{A}_x) .$$

[1] Jordan, *Life Contingencies*, p. 150.

If the discounted continuous premium is considered as

$$\frac{d}{\delta}\bar{P}(\bar{A}_x) = {}^{d}\bar{P}(\bar{A}_x),$$

then

$$P^{\{1\}}(\bar{A}_x) \doteq \frac{1}{1+\frac{1}{2}\delta}\cdot\frac{\delta}{d}\cdot {}^{d}\bar{P}(\bar{A}_x).$$

Using 3 per cent interest and calculating to 5 decimal places, utilizing this approximation, it is found that

$$P^{\{1\}}(\bar{A}_x) \doteq 1.00007\,{}^{d}\bar{P}(\bar{A}_x).^{2}$$

Lauer, however, has established rigorously in his paper that the apportionable premium is slightly less than the discounted continuous premium. It is, therefore, seen that, to five or more decimal places, the approximation utilized in Jordan for the complete annuity yields a result inconsistent with Lauer's demonstration. However, in practice, this difference is certainly minimal.

As a final comment, Lauer seems to indicate that it is not necessary to calculate a discounted apportionable premium due, whereas this is necessary for the use of the continuous premium. On further consideration, he would probably agree that $P^{\{m\}}(\bar{A}_x)$ is the nominal annual premium for the apportionable benefit, $1/mP^{\{m\}}(\bar{A}_x)$ being payable at the beginning of each period of duration $1/m$. Therefore, in order to obtain the equivalent effective annual premium, a discounting factor will be needed in this case as well. This could very well be $\ddot{a}_{\overline{1}|}^{(m)} = d/d^{(m)}$, analogous to d/δ used for the continuous premium. For a premium payable annually, $d/d^{(m)}$, of course, reduces to unity. However, for every other mode of premium payment a different factor is needed. Even if one approximate factor is used for all modes of payment other than annually, the policy file will have to be sorted into two files for this purpose. It would almost seem simpler to apply the factor d/δ to all premiums calculated on the continuous basis.

(AUTHOR'S REVIEW OF DISCUSSION)

J. ALAN LAUER:

Nesbitt points out that the idea of an apportionable annuity due is not as new as I had thought. The paper by Rasor and Greville (*TSA*, IV,

[2] For $i = 0.03$,
$\delta = 0.02955880224$,
$d = 0.02912621359$,

$$\frac{\delta}{d(1+\delta/2)} = \frac{\delta}{0.02955668158} = 1.00007175.$$

574) to which he refers and Nesbitt's discussion of that paper are of interest because of the close relationship of the apportionable annuity due to the complete annuity.

There is general agreement that the apportionable annuity due is an annuity due with a partial return of the final payment at the death of the annuitant. It is clear from the various discussions that there are several possible definitions of the amount of the partial return, each of which has a corresponding definition of the amount of partial payment at death under a complete annuity.

Borrowing from White's notation, let $\ddot{B}^k_t$ be the amount of partial return at death for the apportionable annuity due and $\mathring{B}^k_t$ be the amount of partial payment at death for the complete annuity, both for definition k. In the table on page 154, the three definitions described by Cain are designated as Definitions I, II, and III. Table 1 compares formulas for the apportionable annuity due and the complete annuity under the three definitions.

Definition I is suggested by Mereu, White, and Nesbitt, as well as by Cain. Nesbitt arrives at Definition I by considering the defining relation

$$d^{(m)}\ddot{a}_x^{\{m\}} = \delta\bar{a}_x = i\mathring{a}_x^{(m)} , \tag{1}$$

where each term represents the present value of interest payments on a principal of 1 through the exact whole life of (x). Similarly, $\ddot{a}_x^{\{m\}}$ represents the present value of interest payments on a principal of $1/d^{(m)}$, with interest payable in advance and an adjustment to be made at death. When death occurs at time $t(0 < t < 1/m)$ after the final payment of $1/m$ has been made, the amount to be refunded is

$$\frac{1}{m}(1+i)^t - \frac{1}{d^{(m)}}[(1+i)^t - 1] = \frac{1 - v^{1/m-t}}{d^{(m)}} = \frac{1}{m}\,\frac{\bar{a}_{\overline{1/m-t}|}}{\bar{a}_{\overline{1/m}|}} . \tag{2}$$

The left side of this equation can be thought of as the return of the final payment of $1/m$ with compound interest minus compound interest on the principal of $1/d^{(m)}$.

If we work in terms of effective rate j per $(1/m)$th year, that is, if

$$j = (1+i)^{1/m} - 1 , \tag{3}$$

then the principal of $1/d^{(m)}$ can be expressed as $(1+j)/mj$ and equation (2) above can be restated as

$$\frac{1}{m}(1+j)^{mt} - \frac{1+j}{mj}[(1+j)^{mt} - 1] = \frac{1+i}{mj}(1 - v^{1-mt}) = \frac{1}{m}\,\frac{\bar{a}_{\overline{1-mt}|}}{\bar{a}_{\overline{1}|}} . \tag{4}$$

TABLE 1

k	$\ddot{B}^k_t$	Formula for Apportionable Annuity Due	$\mathring{B}^k_t$	Formula for Complete Annuity
I. . .	$\frac{1-v^{1/m-t}}{d^{(m)}}$	$\ddot{a}^{(m)}_{x:\overline{n}\rceil}-\left[\frac{1}{\delta}-\frac{1}{i^{(m)}}\right]\frac{\delta}{d^{(m)}}\bar{A}^{\,1}_{x:\overline{n}\rceil}$	$\frac{(1+i)^t-1}{i^{(m)}}$	$a^{(m)}_{x:\overline{n}\rceil}+\left[\frac{1}{d^{(m)}}-\frac{1}{\delta}\right]\frac{\delta}{i^{(m)}}\bar{A}^{\,1}_{x:\overline{n}\rceil}$
II. . .	$\frac{1}{m}-t$	$\ddot{a}^{(m)}_{x:\overline{n}\rceil}-\left[\frac{1}{d^{(m)}}-\frac{1}{\delta}\right]\bar{A}^{\,1}_{x:\overline{n}\rceil}$		$a^{(m)}_{x:\overline{n}\rceil}+\left[\frac{1}{\delta}-\frac{1}{i^{(m)}}\right]\bar{A}^{\,1}_{x:\overline{n}\rceil}$
III. . .	$\left(\frac{1}{m}-t\right)(1+i)^t$	$\ddot{a}^{(m)}_{x:\overline{n}\rceil}-\frac{1}{2m}\frac{\delta}{d^{(m)}}\bar{A}^{\,1}_{x:\overline{n}\rceil}$	$t\,v^{1/m-t}$	$a^{(m)}_{x:\overline{n}\rceil}+\frac{1}{2m}\frac{\delta}{i^{(m)}}\bar{A}^{\,1}_{x:\overline{n}\rceil}$

Now, changing from compound interest to simple interest, equation (4) can be restated as

$$\frac{1}{m}(1+mtj)-\frac{1+j}{mj}\,mtj=\frac{1}{m}-t. \qquad (5)$$

Equation (5) leads us to Definition II, which is the definition suggested by the paper.

Therefore, one way of looking at the apportionable annuity due is as a life income from a principal of $1/d^{(m)}$, with Definition I being based on compound interest and Definition II being based on simple interest.

Another way of looking at the apportionable annuity due is as an annuity due with each payment being proportional to the period lived until the next payment. Definition II recognizes that the period lived between the last payment and death cannot be ascertained until death occurs, so that is when the adjustment is made. Definition III theorizes that the last payment should have been t instead of $1/m$ and assesses interest on the overpayment.

All three definitions have merit. Furthermore, other definitions are possible. A reading of the discussion by Cook and Johnson suggests definitions involving adjustments at death based on the interval from the end of the month in which death occurs to the date the next payment would have been made or adjustments based on a multiple of the periodic payment equal to the ratio between gross premiums and net premiums. However, Definition I has several theoretical and practical advantages which make it seem, at least to me, to be the best definition.

As several discussers have pointed out, under Definition I

$$\ddot{a}^{\{m\}}_{x:\overline{n}|}=\frac{\delta}{d^{(m)}}\,\bar{a}_{x:\overline{n}|}, \qquad (6)$$

so that

$${}_nP^{\{1\}}={}_n\bar{P}^{a} \qquad (7)$$

and

$${}^{n}_{t}V^{\{m\}}={}^{n}_{t}\bar{V}\,. \qquad (8)$$

Equations (6), (7), and (8) are exact relationships under Definition I but are only approximations under the other definitions. There are practical advantages if functions on the apportionable basis are exactly equal to functions on the continuous basis rather than approximately equal. The following relationships also hold under Definition I:

$${}_tV^{\{m\}}(\bar{A}_{x:\overline{n}|})=\frac{\ddot{a}^{\{1\}}_{x:\overline{n}|}-\ddot{a}^{\{1\}}_{x+t:\overline{n-t}|}}{\ddot{a}^{\{1\}}_{x:\overline{n}|}}=\frac{\bar{A}_{x+t:\overline{n-t}|}-\bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}}; \qquad (9)$$

$$\ddot{a}^{\{1\}}_{x:\overline{n}|} = (1+i)^{1/m}\mathring{a}^{(m)}_{x:\overline{n}|}; \qquad (10)$$

$$\bar{A}_{x:\overline{n}|} = 1 - d^{(m)}\ddot{a}^{\{m\}}_{x:\overline{n}|}. \qquad (11)$$

Equation (9) (which was suggested by Cook and Johnson), equation (10) (which was suggested by Rosser), and equation (11) are all exact for Definition I but are only approximations under the other definitions. Equations (10) and (11) are easily derived from Nesbitt's defining relation.

Isen and Rosser suggest the relation

$$\ddot{a}_{x:\overline{n}|} = \mathring{a}^{(m)}_{x:\overline{n}|} + \frac{1}{m}\,\delta\,\bar{a}_{x:\overline{n}|}. \qquad (12)$$

Isen then goes on to draw an interesting analogy between this formula and

$$\ddot{a}^{(m)}_{x:\overline{n}|} = a^{(m)}_{x:\overline{n}|} + \frac{1}{m}(1 - {}_nE_x). \qquad (13)$$

It is interesting to note that formula (12) is exact under both Definitions I and II.

The objective of the paper is not a search for greater theoretical accuracy in reserve valuation but rather a search for a simpler and more logical approach. The concept of continuous premium payments is fine for industrial business, but people (particularly nonactuaries) in many companies find the concept of annual premium payments much more logical and easier to understand. On the continuous functions basis, terminal reserves are calculated by use of the *continuous yearly premium*, while mean reserves are calculated as one-half the sum of two consecutive terminal reserves and the *discounted continuous yearly premium*. On the apportionable basis, the same net premium is used to obtain mean reserves as that used to obtain terminal reserves.

Cook and Johnson think that the apportionable basis might be difficult to explain to federal income tax auditors. I do not feel that this would be a major problem, particularly if the amount of reserves is the same as on the continuous basis. On the other hand, the use of the apportionable basis might make it clearer that the amounts refunded at death are premium refunds, deductible for premium tax purposes, rather than additional death benefits.

Cook and Johnson apparently misunderstood my intentions. I have not recommended that the apportionable basis be adopted the next time a new mortality standard is published. Rather, I have suggested that consideration be given to publishing functions on the apportionable basis so that those companies that wish to use the apportionable basis can do so more conveniently. As White implies, the apportionable and continuous bases can be said to be the same thing looked at in two different ways. Personal preference is likely to dictate which will be used or whether either will be used in any given case.

Isen and Nesbitt refer to the problem of calculating premiums and reserves based on different values of m. The derivations in the paper are generalized for academic reasons. As a practical matter, $m = 1$ is the most important case; most companies are likely to continue to base mean reserves on annual premiums and then set up a deferred premium asset.

The Appendix entitled "Analysis of Nonuniform Distribution of Deaths" by Cook and Johnson is a worthwhile addition to our literature.

Finally, I wish to thank Messrs. Mereu, White, Nesbitt, Isen, Cook, Johnson, Cain, and Rosser for discussing my paper. If the paper has any lasting value, it will be due in large part to these discussions.