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MORTALITY RISK IN LIFE ANNUITIES

ROBERT T. McCRORY

ABSTRACT

The life annuity mortality risk is the variation in actual financial experience of a group of annuitants from that expected, attributable to the mortality experience of the group.

This paper attempts to quantify this mortality risk in various circumstances. The standard deviation, coefficient of dispersion, and certain confidence intervals of the distribution of interest-discounted annuity payments are used as measures of the mortality risk. The effect on the mortality risk of the age of the annuitants, their sex, the assumed interest rate, the assumed level of mortality, the form of annuity, and the number of annuitants in the portfolio is investigated.

Finally, there is a summary of the results and a discussion of their implications in some practical situations.

I. INTRODUCTION

An actuarial present value represents an average. In the case of annuity payments made to a group of annuitants, it represents the average interest-discounted value of payments that will be received by members of the group.

Even if the actuarial interest assumption is met exactly, there will be variation in the value of payments received due to the mortality experience of the group. This variation of actual experience from expected due to mortality is known as mortality risk.

A number of papers have appeared in actuarial literature attempting to quantify the mortality risk in life annuities. Generally, these papers have been concerned with small groups of annuitants, have used approximate techniques, and have been concerned with the likelihood that the cost of annuity payments actually made to the group will exceed the expected value plus a contingency reserve. Setting the amount of a prudent contingency reserve has been a primary concern.

In 1933, Piper [6] calculated the standard deviation of a single life annuity based on the Combined Annuity Table at 4 percent and used a normal approximation to derive contingency reserves for groups of annuitants with the same age, sex, and annuity amount.

Approximate techniques for groups of annuitants have been presented by several authors. Taylor [10] used a Pearson Type III approximation to compute a contingency reserve for a group of life annuities payable under a pension plan. Bowers [2] used a Cornish-Fisher expansion based on cumulants to determine a contingency reserve against adverse mortality experience for a group of annuitants. Fretwell and Hickman [3] used and compared several approximation formulas to put upper bounds on the cost to provide annuities to a small group.

Other techniques have been presented as well. Stone [9] used probability generating functions to investigate the possible financial impact of self-insuring retirement annuities for retirement plans with varying frequencies of retirement. A Monte Carlo simulation was presented in an appendix to Taylor's paper, and was the primary focus of Boermeester's paper [1] in 1956. Here the portfolio of annuities consisted of ten lives aged 65, but one life received a payment higher than the other lives. This annuity portfolio also was investigated by Fretwell and Hickman.

These papers did not deal with a number of questions. First, more needs to be said about the mortality risk in a portfolio containing only one single life annuity. While not significant from an insurance viewpoint, single life annuity portfolios often are involved when an actuary gives expert testimony in connection with a pension plan in a divorce or in connection with a personal injury tort. The actuary usually presents an actuarial present value calculation, but does not present, and usually does not know, the variation that may occur in the value of payments actually received, as a result of the combination of mortality and investment risks.

In these cases, the probability that payments will be substantially lower than the calculated actuarial present value is usually a matter of much more import, dispute, and cross-examination than the probability that payments will substantially exceed the value expected. This is, of course, the opposite of the contingency reserve problem considered by most of the authors mentioned above.

Second, factors affecting the mortality risk need to be investigated. The earlier authors considered annuity portfolios using only one or two mortality tables, interest rates, and ages. Yet the variation in the value of annuity payments due to mortality risk is affected by all of the following factors:

1. Age,
2. Sex,
3. Assumed interest rate,

4. Mortality table,
5. Form of annuity, and
6. Number of annuitants in the portfolio.

This paper will investigate the mortality risk present in a single life annuity and determine the effect of age, sex, assumed interest rate, mortality table, and form of annuity on this risk. Then, it will investigate the mortality risk in various portfolios of annuities. Last, we will draw some conclusions from the results obtained.

Throughout the paper, the "present value" or "discounted value" of a series of payments will mean the value of the payments today, discounted only with interest. The "actuarial present value" of a series of payments will refer to the value of the payments today, discounted for both interest and mortality. Thus, the actuarial present value of a series of payments is the average of the underlying present value distribution. More will be said on this point below.

II. SINGLE LIFE ANNUITY PORTFOLIOS

A. *Methodology*

First consider single life annuities with annual payments of $1 due at the beginning of each year and a single interest rate for discounting. For such a simple annual life annuity due, payable to a person aged x, we are interested in the underlying distribution of present values of annuity payments, of which the actuarial present value is the mean.

Assume, therefore, that we have 10,000 lives at age x. As time goes on and annuity payments are made to each member of this group, some of them will die. Each age at death has a unique present value associated with it—the discounted value of payments received by that life from age x until death. In addition, each age at death has associated with it the number of deaths at that age last birthday from the original group of 10,000 people. Combining these two concepts gives us a distribution of deaths versus present values.

As an example, consider a female aged 25. Assume that the probability of her death at each age is given by the 1971 Group Annuity Mortality Table (1971 GAM Table) for Females. Figure 1 illustrates the distribution of the ages at death for 10,000 such females at age 25.

A unique present value of payments made from age 25 until death would be associated with each age at death. If we were to replace the age at death on the horizontal axis with the present value of the payments associated with that age at death at, say, 6 percent interest, we would have the distribution

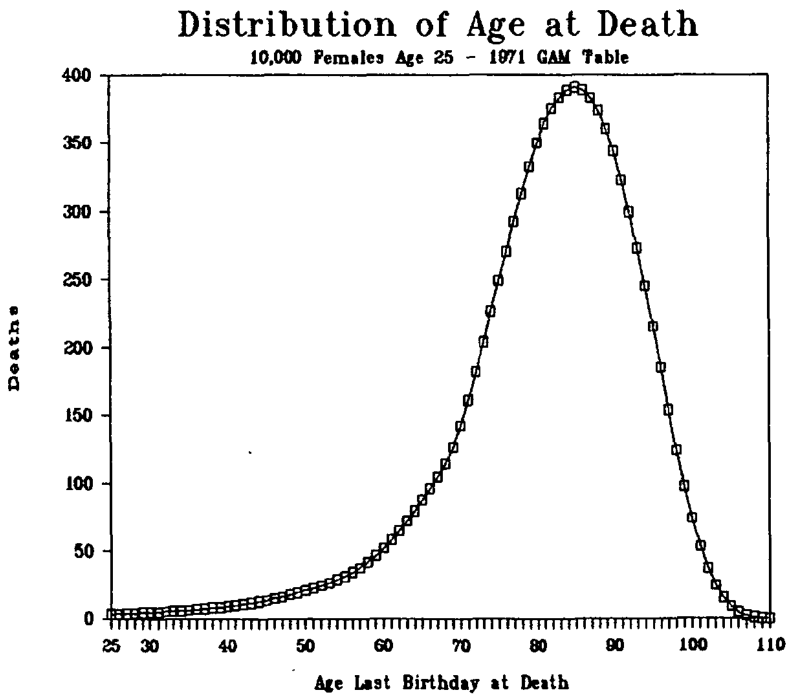

FIG. 1.—Distribution of ages at death for 10,000 females at age 25, based on the 1971 GAM Table.

of present values shown by figure 2. It should be noted that the only difference between figures 1 and 2 is that the age at death on the horizontal axis is replaced by the present value of payments made from age 25 until death. Recall that payments of $1 are made annually at the beginning of each year and are discounted at the rate of 6 percent interest.

It should come as no surprise that the mean of this distribution of present values is precisely the actuarial present value calculated in the usual fashion using commutation functions. The actuarial present value is merely the mean of a distribution of present values; a great deal more information can be obtained.

We can measure the standard deviation of the present value distribution. A more meaningful measure might be the coefficient of dispersion (see [8]), which is the standard deviation divided by the mean, and expressed as a percentage. The coefficient of dispersion gives a numerical indication of whether the underlying present value distribution is spread out or highly peaked.

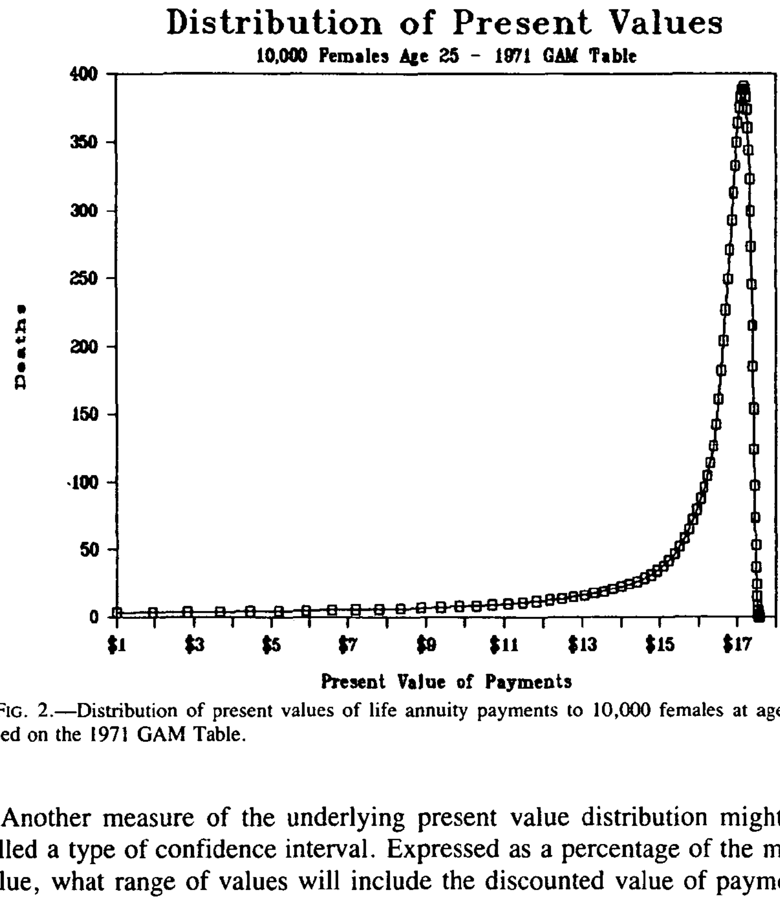

FIG. 2.—Distribution of present values of life annuity payments to 10,000 females at age 25, based on the 1971 GAM Table.

Another measure of the underlying present value distribution might be called a type of confidence interval. Expressed as a percentage of the mean value, what range of values will include the discounted value of payments received, with 90 percent confidence? In order to measure the low and high point of this 90 percent confidence interval, we proceed as follows.

First of all, we find the present value that excludes the lowest 5 percent of the observed discounted values. Since this is a discrete distribution, we come as close as we can to 5 percent, without excluding more than that. This represents the low endpoint of our 90 percent confidence interval, and it is divided by the mean and expressed as a percentage. Next, we find the present value that excludes the highest 5 percent of observed discounted values to determine the high point of our 90 percent confidence interval.

As a result of these calculations, we know that any individual's discounted value of payments received must lie between the low point and the high point of the confidence interval with at least 90 percent probability.

For our purposes, the 90 percent confidence interval often will prove too broad. Therefore, 50 percent and 70 percent confidence intervals have been derived using a technique similar to that above. For each distribution of present values, we will measure the mean, the coefficient of dispersion, and 50, 70, and 90 percent confidence intervals to obtain information about the distribution of the present values of payments made to the underlying population.

(We should note in passing that our use of the term "confidence interval" does not correspond to the standard usage in statistics texts. Usually, a statistician will refer to a confidence interval as an interval of real numbers within which some estimated population parameter must lie with some probability. The confidence intervals in this paper are intervals of real numbers within which an individual annuitant's discounted value of actual annuity payments must fall with a specified probability. All our population parameters are assumed; we wish to quantify the variation that will be exhibited by individual samples from the population.)

B. *Changes in the Present Value Distribution with Age*

Let us investigate how the distributions of present values vary with age. Figure 3 presents the distributions of present values of $1 annual annuities due payable to 10,000 females at ages 25, 45, 65, and 85, with interest at 6 percent and mortality based on the 1971 GAM Table for Females.

Figure 3 indicates that the distribution of present values spreads out as the age of annuity commencement increases. The coefficients of dispersion and the confidence intervals presented in table 1 confirm this intuitive judgment. The coefficient of dispersion increases rapidly from 7.2 percent at age 25 to almost 53 percent at age 85. It almost doubles with every twenty-year increment in the starting age of the annuity.

The 90 percent confidence interval spreads out rapidly with increasing age. At age 25, we can be at least 90 percent certain that the present value of payments made to an annuitant will lie between 91.5 percent and 103.7 percent of the mean. At age 85, however, we can be 90 percent certain that the present value of payments will be between 18.0 percent and 192.3 percent of the mean. Clearly, at the higher ages we can have very little confidence in the mean as an estimate of the value of the annuity; there is simply too much variation.

Table 1 implies that the capacity to accurately predict the present value of payments to be received by an annuitant decreases sharply with increasing age as the underlying present value distribution spreads out. This pattern of decreasing predictability with increasing age will be repeated throughout the

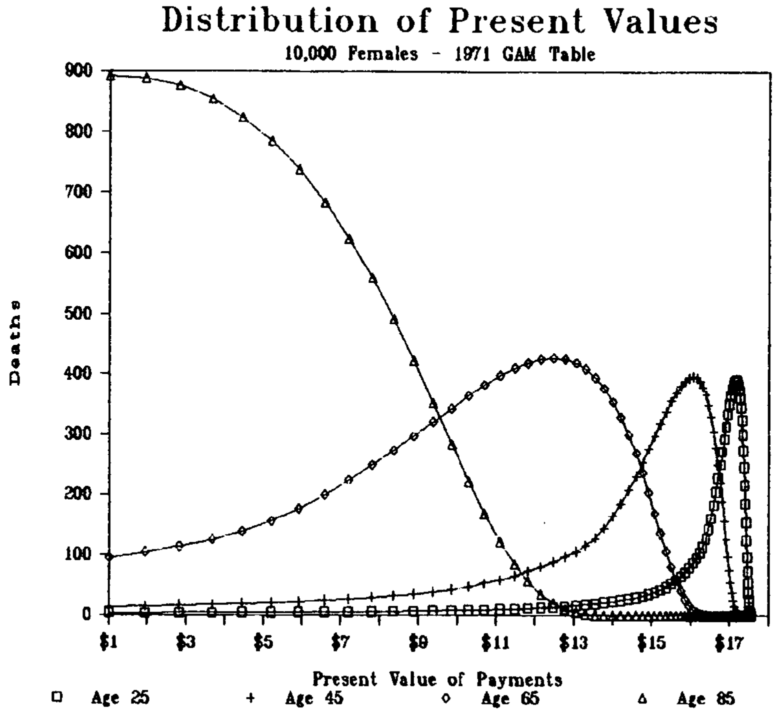

FIG. 3.—Distributions of present values of life annuity payments to 10,000 females, based on the 1971 GAM Table. □ = age 25; + = age 45; ◇ = age 65; △ = age 85.

paper. From these data, we also might infer that the dispersion increases with increasing rates of mortality. This conclusion will be confirmed later.

C. *Changes in the Present Value Distribution with Sex*

Now let us turn from females to males. We consider the present value distributions of males with interest at 6 percent and mortality based on the 1971 GAM Table for Males. Once again, we have assumed 10,000 lives at beginning ages 25, 45, 65, and 85, using a $1 annual life annuity due.

Figure 4 illustrates the present value distributions for males and females at age 45. The male graph has the same general shape as that for females the same age, but appears a bit more spread out. The coefficients of dispersion and the confidence intervals presented in table 2 confirm this analysis. The coefficient of dispersion increases with age, from 9.4 percent at age 25 to 59.0 percent at age 85. At each age, the coefficient of dispersion

TABLE 1

Coefficients of Dispersion and Confidence Intervals for a Life Annuity Based on the 1971 GAM Table for Females at 6 Percent Interest

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	16.79	1.21	7.20%	99.8%	102.8%	98.0%	103.3%	91.5%	103.7%
45........	15.21	2.12	13.91	98.1	108.2	92.0	109.4	73.0	110.8
65........	11.34	3.26	28.79	86.9	121.6	68.8	127.1	39.4	133.1
85........	5.57	2.95	52.92	50.9	140.1	34.9	159.6	18.0	192.3

* Expressed as a percentage of the mean.

TABLE 2

Coefficients of Dispersion and Confidence Intervals for a Life Annuity Based on the 1971 GAM Table for Males at 6 Percent Interest

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	16.38	1.55	9.44%	99.5%	104.6%	96.0%	105.1%	85.5%	105.9%
45........	14.16	2.78	19.62	95.7	112.6	83.5	115.1	55.1	117.6
65........	9.73	3.64	37.43	74.1	128.2	53.6	139.3	29.1	150.0
85........	4.75	2.80	59.00	40.9	138.6	40.9	164.3	21.1	207.5

* Expressed as a percentage of the mean.

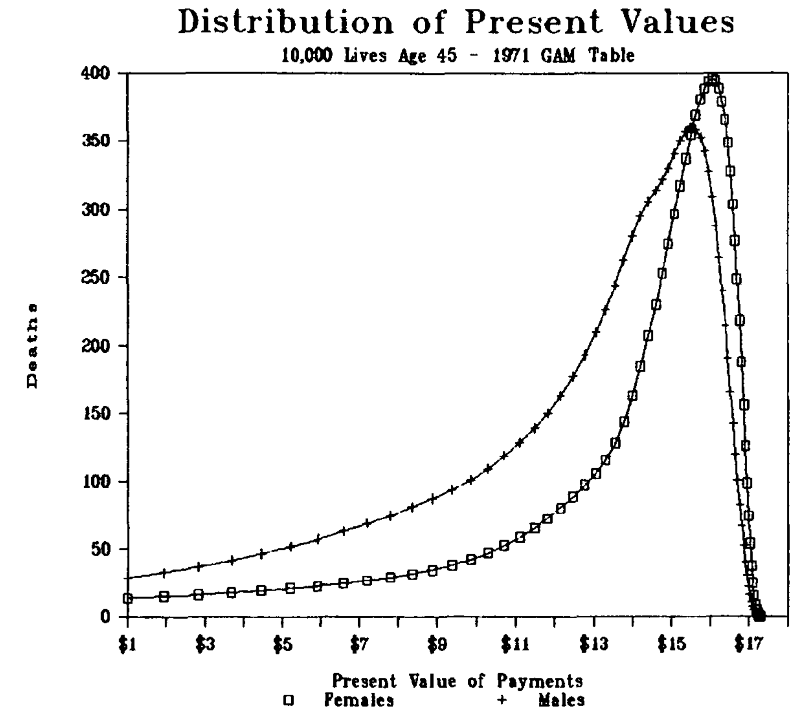

FIG. 4.—Distributions of present values of life annuity payments to 10,000 lives at age 45, based on the 1971 GAM Table. □ = females; + = males.

is greater for males than for females. As before, the coefficient of dispersion almost doubles with each twenty-year increment in the beginning age of the annuity.

Figure 5 presents the present value distributions for males at ages 25, 45, 65, and 85.

Again, the 90 percent confidence interval widens with increasing age. We are 90 percent certain that the present value of payments received by a male annuitant aged 25 will be between 85.5 percent and 105.9 percent of the mean. However, our confidence deteriorates until at age 85 we are certain only that the present value has a 90 percent chance of lying between 21.1 percent and 207.5 percent of the mean; we have very little confidence in our estimate of the present value at the higher ages.

In general, the confidence intervals for males are broader than those for females of the same age, except at age 85. Our confidence in our ability to estimate the value of payments to be received by males is somewhat lower

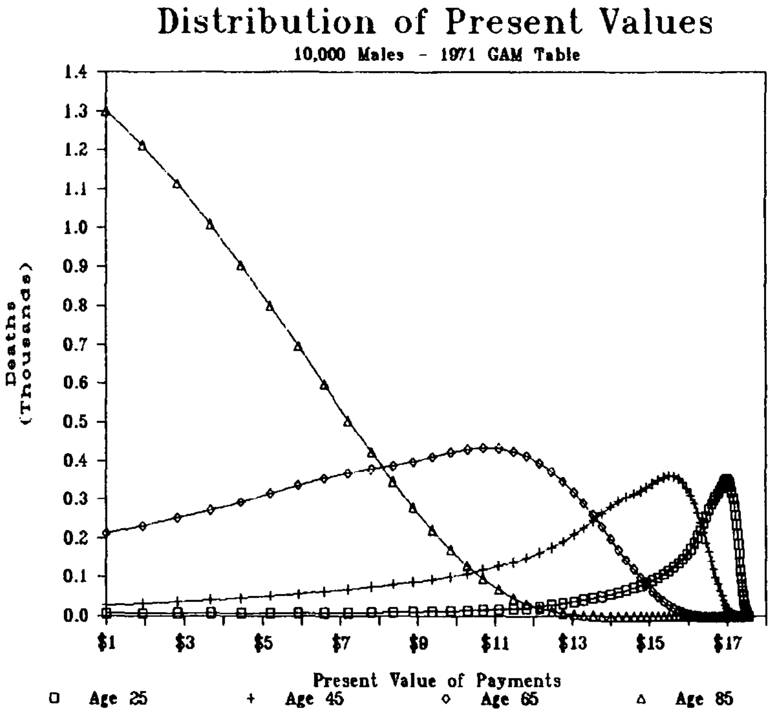

FIG. 5.—Distributions of present values of life annuity payments to 10,000 males, based on the 1971 GAM Table. □ = age 25; + = age 45; ◇ = age 65; △ = age 85.

than for females. This will be seen throughout the paper; the underlying present value distributions for males are consistently broader than those for females.

Again, the higher mortality group, here males, has broader dispersion of present values than the lower mortality group.

D. *Changes in the Present Value Distribution with Interest Rate*

Now consider how the present value distribution changes with the rate of assumed interest. As before, we consider a $1 annual life annuity due payable to 10,000 females at age 45 with mortality based on the 1971 GAM Table for Females.

Figure 6 displays the distribution of present values for rates of interest of 2, 6, and 10 percent. It is clear from the graphs that the distribution is much

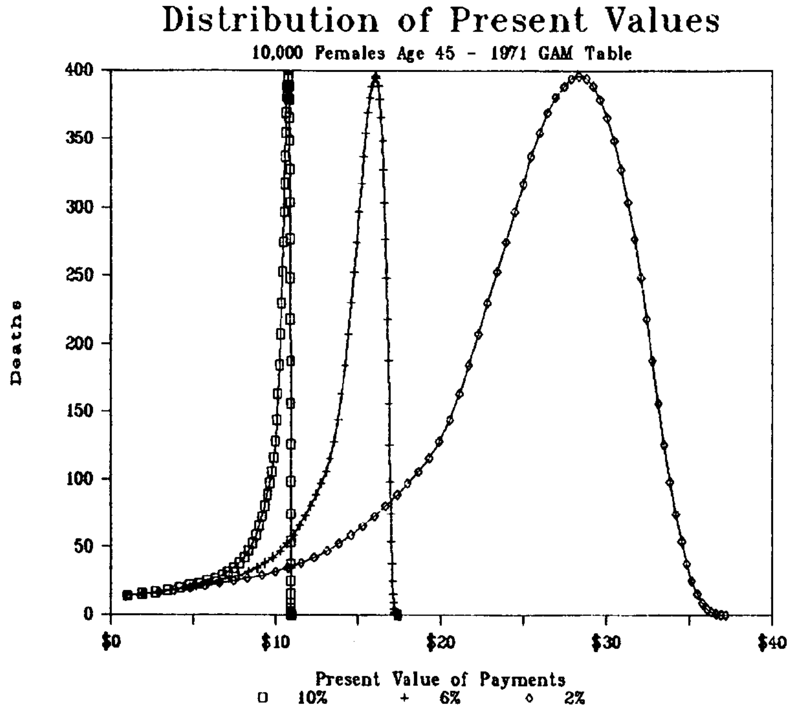

FIG. 6.—Distributions of present values of life annuity payments to 10,000 females at age 45, based on the 1971 GAM Table. □ = 10%; + = 6%; ◇ = 2%.

broader at the lower interest rate. The data presented in tables 3–6 confirm this observation.

Restricting our attention to females for the moment, at 2 percent interest the coefficient of dispersion shows the usual increasing pattern with age. However, it increases from 13.9 percent at age 25 to almost 60 percent at age 85. Throughout its range it is consistently higher than the coefficients of dispersion observed for females at 6 percent interest. The confidence intervals in table 3 are consistently wider than those at 6 percent and exhibit the characteristic pattern of broadening with increasing age.

At 10 percent interest, however, the distribution is much more peaked. The coefficient of dispersion is considerably lower than that observed for 6 percent interest, ranging from 4.86 percent at age 25 to 47.43 percent at age 85. The confidence intervals are consistently narrower than those observed at either 6 percent or 2 percent interest.

These data suggest that we can have considerably more confidence in our

TABLE 3

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A LIFE ANNUITY BASED ON THE 1971 GAM TABLE FOR FEMALES AT 2 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	34.05	4.74	13.93%	95.2%	108.4%	89.5%	111.6%	74.9%	114.5%
45........	26.29	5.77	21.93	91.0	116.0	80.3	120.5	55.4	126.1
65........	16.00	5.98	37.37	77.2	128.2	57.3	139.2	30.0	152.9
85........	6.48	3.89	59.97	45.4	141.3	30.5	166.4	15.4	213.6

* Expressed as a percentage of the mean.

TABLE 4

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A LIFE ANNUITY BASED ON THE 1971 GAM TABLE FOR MALES AT 2 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	31.73	5.53	17.43%	93.5%	111.7%	85.0%	114.6%	66.6%	119.7%
45........	23.13	6.65	28.75	86.1	120.6	69.1	128.2	39.6	136.9
65........	13.07	6.09	46.58	63.7	132.8	43.7	152.4	22.5	174.8
85........	5.42	3.61	66.56	36.6	137.9	36.6	169.2	18.5	228.0

* Expressed as a percentage of the mean.

TABLE 5

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A LIFE ANNUITY BASED ON THE 1971 GAM TABLE FOR FEMALES AT 10 PERCENT INTEREST

AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	10.86	0.53	4.86%	100.5%	101.1%	100.0%	101.1%	97.7%	101.2%
45........	10.42	1.04	10.01	100.6	104.2	97.5	104.6	84.7	104.9
65........	8.67	2.01	23.23	93.4	116.2	77.9	118.8	48.1	121.4
85........	4.89	2.32	47.43	55.9	138.1	39.0	153.1	20.4	175.8

* Expressed as a percentage of the mean.

TABLE 6

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A LIFE ANNUITY BASED ON THE 1971 GAM TABLE FOR MALES AT 10 PERCENT INTEREST

AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	10.76	0.69	6.42%	100.7%	101.9%	99.5%	102.0%	94.4%	102.1%
45........	9.99	1.47	14.69	99.9	107.7	92.1	108.4	67.6	109.1
65........	7.71	2.41	31.29	82.2	123.4	62.2	129.6	35.5	134.6
85........	4.24	2.25	53.13	45.0	138.3	45.0	159.3	23.6	191.0

* Expressed as a percentage of the mean.

present value calculations at higher rates of interest than at lower rates. One way of thinking about this phenomenon is to consider the discounting of future annuity payments for interest to be a geometric transformation applied to the horizontal axis of the above graphs. We start with age-at-death data, as we did in figure 1, by plotting the number of deaths at each age for our 10,000 lives. For each age at death, there is a unique total of annuity payments, and we can replace the values on our horizontal axis by this sum. At this point we apply a transformation to the horizontal axis by discounting the aggregate payments for interest. The effect of this transformation is to squeeze the plotted points to the left, compressing the graph and reducing the coefficient of dispersion. The higher the rate of discount used, the greater the compression of points to the left and the lower the coefficient of dispersion.

Another way of regarding the phenomenon of decreased dispersion with increasing interest rates is to note that as the interest rate increases, the effect on the present value of living longer than average is reduced by the greater discount applied to future payments. This also tends to compress the distribution and reduce the coefficient of dispersion.

The compression due to interest becomes more significant farther from the annuitant's current age. This explains the skewed shape of the graphs and the fact that the observed confidence intervals are asymmetrical, extending farther below the mean than above.

E. *Changes in the Present Value Distribution with Mortality Classification*

The pattern and dispersion of the present value distribution change radically for impaired mortality. Consider the distribution of present values for females at ages 25, 45, 65, and 85 at 6 percent interest and the mortality of the Pension Benefit Guaranty Corporation Mortality Table for Disabled Females Receiving Social Security Benefits (PBGC-SS Table for Females). Once again a $1 annual annuity due is assumed.

The resulting distributions of present values are given in figure 7.

The shapes of the graphs for the disabled distributions changed dramatically from those for healthy lives. These changes occur for males as well as females. There is no obvious increase in the dispersion of the graph, which is confirmed by the data in tables 7 and 8 showing coefficients of dispersion and confidence intervals. For females at age 25, the coefficient of dispersion is 39.4 percent, which is five times its value for a healthy female of the same age. The increase in the coefficient of dispersion with age is less dramatic than for healthy females. In fact, the coefficient of dispersion re-

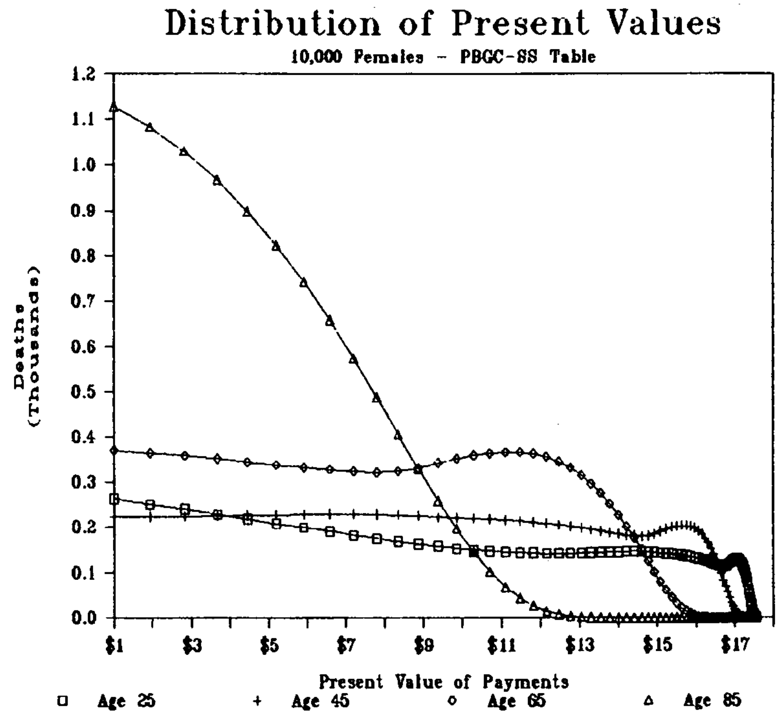

FIG. 7.—Distributions of present values of life annuity payments to 10,000 females, based on PBGC-SS Table. □ = age 25; + = age 45; ◇ = age 65; △ = age 85.

mains approximately level until age 85, but at all ages it is significantly higher than for healthy females.

The 90 percent confidence intervals for disabled females are almost comical. At age 25, the best that we can say with 90 percent probability is that the present value of annuity payments will be between 15.5 percent and 138.1 percent of the mean. As age increases, the low end of the confidence interval remains relatively stable, whereas the high end increases to 197.4 percent at age 85.

For a 45-year-old female, for whom we have about the most certainty among the lives sampled, we can be only about 50 percent certain that the present value of annuity payments received will lie between 76 percent and 132 percent of the mean.

Similar patterns are observed in table 8 for disabled male mortality; in most cases the coefficient of dispersion is higher and the confidence interval is broader than for similarly situated females. This verifies our earlier spec-

TABLE 7

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS FOR A LIFE ANNUITY BASED ON THE PBGC-SS TABLE FOR FEMALES AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25	12.52	4.93	39.41%	75.0%	133.0%	47.3%	136.0%	15.5%	138.1%
45	11.70	4.50	38.44	76.0	132.5	50.6	137.9	24.2	141.8
65	9.43	4.06	43.03	69.8	135.4	47.4	143.7	20.6	154.8
85	4.99	2.79	55.86	56.8	144.5	38.9	167.5	20.0	197.4

* Expressed as a percentage of the mean.

TABLE 8

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS FOR A LIFE ANNUITY BASED ON THE PBGC-SS TABLE FOR MALES AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25	10.85	5.26	48.45%	54.6%	142.9%	33.9%	150.3%	17.9%	156.7%
45	10.06	4.38	43.58	65.5	137.1	44.4	148.5	19.3	159.6
65	7.52	3.97	52.86	59.4	147.7	37.7	161.7	13.3	180.2
85	3.92	2.32	59.09	49.6	132.9	25.5	167.9	25.5	213.2

* Expressed as a percentage of the mean.

ulation that the dispersion of the present value distribution increases with increasing rates of mortality. In fact, the data indicate that, at best, the actuarial present value of annuity payments made to a disabled life is a very tentative estimate of the present value of payments that actually will be made.

F. *Changes in the Present Value Distribution with the Type of Annuity*

Next we consider how the present value distribution changes for temporary and deferred annuities.

First, consider a temporary life annuity due of $1 annually at the beginning of each year until age 65. Assume a female, aged 45, with mortality based on the 1971 GAM Table for Females and interest at 6 percent. Since all deaths after age 65 receive payments of the same present value, we would expect a reduction in the dispersion of the present value distribution. The actual distribution of present values is shown in figure 8.

The data in table 9 show that, in fact, the coefficient of dispersion is

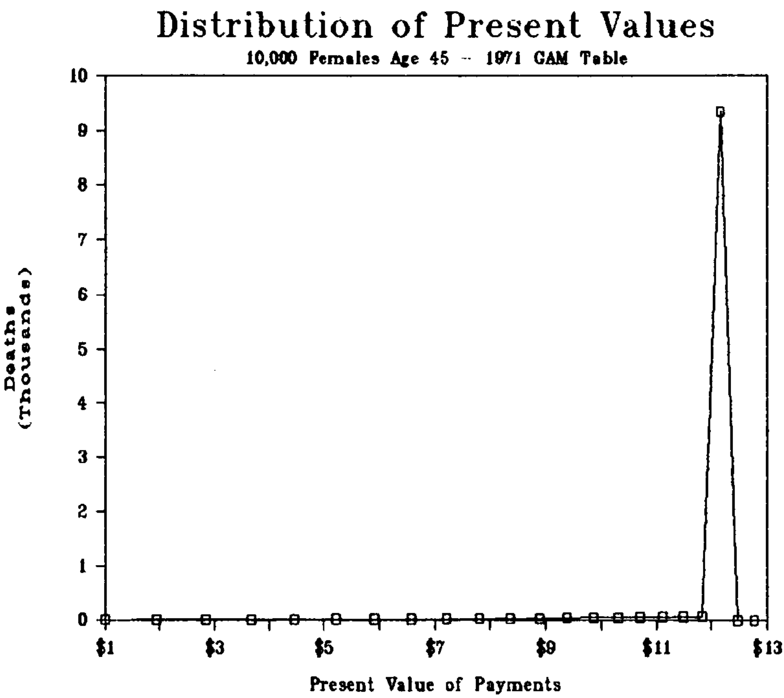

FIG. 8.—Distribution of present values of temporary life annuity payments to 10,000 females at age 45, based on the 1971 GAM Table.

TABLE 9

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS FOR A TEMPORARY LIFE ANNUITY TO AGE 65 BASED ON THE 1971 GAM TABLE FOR FEMALES AT 6 PERCENT INTEREST

				CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	Low	High	Low	High	Low	High
25........	15.78	0.97	6.13%	101.1%	101.1%	101.1%	101.1%	97.4%	101.1%
35........	14.38	1.11	7.69	101.5	101.5	101.5	101.5	95.8	101.5
45........	11.94	1.11	9.32	101.8	101.8	101.8	101.8	93.0	101.8
55........	7.67	0.74	9.67	101.8	101.8	101.8	101.8	101.8	101.8

* Expressed as a percentage of the mean.

TABLE 10

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS FOR A TEMPORARY LIFE ANNUITY TO AGE 65 BASED ON THE 1971 GAM TABLE FOR MALES AT 6 PERCENT INTEREST

				CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	Low	High	Low	High	Low	High
25........	15.62	1.28	8.21%	102.1%	102.1%	100.8%	102.1%	89.7%	102.1%
35........	14.15	1.51	10.64	103.1	103.1	100.4	103.1	81.1	103.1
45........	11.64	1.66	14.22	104.5	104.5	101.6	104.5	67.0	104.5
55........	7.47	1.15	15.33	104.4	104.4	104.4	104.4	69.7	104.4

* Expressed as a percentage of the mean.

reduced. At age 25, the coefficient of dispersion for the temporary annuity is 6.1 percent compared with 7.2 percent for a comparable life annuity. At age 45, the coefficient of dispersion is 9.3 percent for the temporary annuity, versus 13.9 percent for the life annuity.

Similar reductions in the coefficient of dispersion are observed for males, although, once again, the coefficient of dispersion for males exceeds that for females of the same age (see table 10).

The confidence intervals are narrower for temporary annuities than for life annuities, with the reductions occurring most dramatically at the higher ages. For a 25-year-old female, the 90 percent confidence interval for a life annuity runs between 91.5 percent and 103.7 percent of the mean. The comparable figures for a 90 percent confidence interval for a temporary annuity are 97.4 percent to 101.1 percent.

These results should come as no surprise. The truncation of payments under a temporary annuity eliminates the variation in the distribution attributable to the longest-lived members of the population. However, in the previous graphs, most of the variation in the distribution has been due to the short-lived members, since the payments to the long-lived members in the distant future are heavily discounted. Thus, while the variation decreases somewhat for a temporary annuity, the decrease is not as great as one might expect.

Now consider one of the mainstays of the pension industry, the deferred life annuity. Specifically, consider a $1 annual life annuity payable at the beginning of each year, deferred until age 65, with mortality based on the 1971 GAM Table and interest at 6 percent. As has been our practice, we look at the females first.

While the least dispersion for females occurs at age 25 for the other forms of annuity, this is not the case for a deferred annuity. The effect of the deferral of payments is to accentuate the variation in the distribution. This occurs in two ways.

1. A present value of zero applies to all population members who die before the annuity commencement date. This exacerbates the variation caused by the short-lived lives.
2. The mean present value decreases, causing our measure of relative dispersion, the coefficient of dispersion, to increase.

As the attained age approaches the annuity commencement date, the effect of the deferral diminishes. Fewer population members die before the annuity commencement date, and the mean present value increases. Therefore, the coefficient of dispersion decreases slightly with increasing age, which runs counter to the pattern previously observed.

The coefficients of dispersion and confidence intervals are shown in tables 11 and 12. The distribution of present values for a female aged 45 is shown in figure 9.

There is considerable variation in the present value of a deferred annuity by age and sex. For a female aged 25, the coefficient of dispersion is 43 percent, which is almost six times the coefficient of dispersion for a life annuity. The coefficient of dispersion decreases slightly with increasing age to approximately 38 percent at age 55. For all ages, the 90 percent confidence interval for a female deferred annuity ranges between zero at the low end to approximately 140 percent at the high end. Even the 50 percent confidence interval ranges from about 80 percent to 130 percent.

For male annuitants, the pattern is approximately the same although—as usual—the dispersion is greater for males than for females. The coefficient of dispersion starts at about 64 percent at age 25 and decreases to about 55 percent at age 55. The 90 percent confidence interval ranges from zero to 170 percent or so at all ages, while even the 50 percent confidence interval ranges from about 50 percent to 150 percent at all ages.

In all cases, the dispersion of the present value distribution is much greater for deferred annuities than for life annuities. In fact, the dispersion is even longer for deferred annuities than for disabled life annuities. A similar pattern is observed for the confidence intervals. Thus, the credibility of present value estimates of single life deferred annuities must be very poor indeed.

III. MULTIPLE LIFE ANNUITY PORTFOLIOS

A. *Methodology*

A change in methodology is required to investigate the mortality risk present in multiple life annuity portfolios. For single life portfolios, a direct and simple application of the mortality table was sufficient to determine the distribution of annuity present values. For multiple life portfolios, we are concerned with the average present value of payments made to the members of the portfolio. Therefore, the problem is one of averaging random samples drawn from a population; for such a problem, simulation using Monte Carlo methods is often appropriate.

As before, the calculations are based on single life annuities with payments of $1 due at the beginning of each year and an interest rate of 6 percent. The variable we will measure is the average for all members of each portfolio of the interest-discounted present value of payments received before death.

The results are based on 10,000 trials of a portfolio in n lives. In each trial, an age at death was determined for each of the n lives, using the

TABLE 11

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A DEFERRED LIFE ANNUITY COMMENCING AT AGE 65 BASED ON THE 1971 GAM TABLE FOR FEMALES AT 6 PERCENT INTEREST

AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	1.01	0.43	43.01%	80.7%	130.8%	50.3%	139.1%	0.0%	145.6%
35........	1.81	0.77	42.37	80.3	130.2	56.9	138.4	0.0	144.9
45........	3.28	1.34	41.06	84.6	131.2	56.3	137.2	0.0	143.6
55........	5.99	2.27	37.97	82.8	128.5	61.4	134.3	0.0	140.6

* Expressed as a percentage of the mean.

TABLE 12

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A DEFERRED LIFE ANNUITY COMMENCING AT AGE 65 BASED ON THE 1971 GAM TABLE FOR MALES AT 6 PERCENT INTEREST

AGE	MEAN	STANDARD DEVIATION	COEFFICIENT OF DISPERSION	CONFIDENCE INTERVAL*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
25........	0.77	0.49	63.55%	46.5%	153.9%	0.0%	165.1%	0.0%	182.4%
35........	1.39	0.87	62.65	46.1	152.7	0.0	163.8	0.0	180.9
45........	2.53	1.53	60.78	55.1	150.1	0.0	164.3	0.0	177.9
55........	4.76	2.61	54.81	61.1	146.2	22.8	156.0	0.0	168.9

* Expressed as a percentage of the mean.

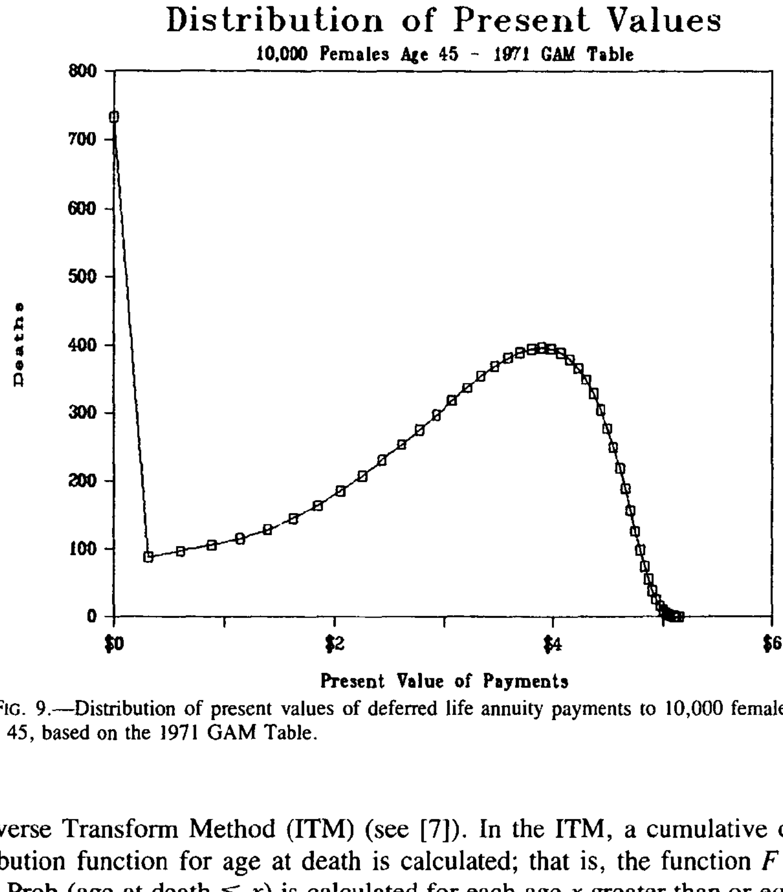

FIG. 9.—Distribution of present values of deferred life annuity payments to 10,000 females at age 45, based on the 1971 GAM Table.

Inverse Transform Method (ITM) (see [7]). In the ITM, a cumulative distribution function for age at death is calculated; that is, the function $F(x) = \text{Prob}(\text{age at death} \leq x)$ is calculated for each age x greater than or equal to the annuitant's age. A random number z between 0 and 1 is generated by the computer, and the greatest age x such that $F(x) \leq z$ is determined. This becomes the age at death. Having determined the age at death for each member of the portfolio on the given trial, the present value of the annuity payments made to each life and the average present value of payments made to all members of the portfolio are calculated.

Some changes are necessary in the graphic representation of results. Single life portfolios may be represented by discrete distributions. As the number of lives in a portfolio increases, the number of possible average values expands rapidly, causing the distribution to approach continuity. Therefore, the range between zero and the maximum possible present value has been divided into 100 equal units. The number of the 10,000 trial portfolios with an average present value falling within a given unit is plotted against the central value.

This methodology is applied to an annuity portfolio of one life in figure 10.

The solid line in figure 10 is the distribution of annuity present values determined for a single life portfolio in the last section. The plotted +'s show the number of the 10,000 portfolios simulated with average present values as indicated.

The simulation produces a distribution of present values that agrees well with the single life calculations of the previous section. For a large number of the present values on the x-axis, no portfolios were observed, since these present values could not be realized by a portfolio of one life.

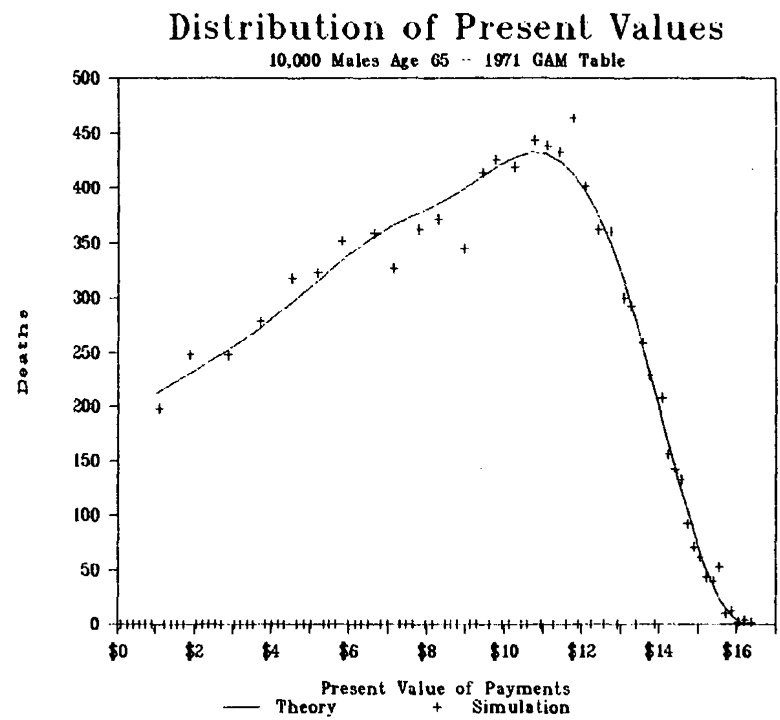

FIG. 10.—Distributions of present values of life annuity payments to 10,000 males at age 65, based on the 1971 GAM Table. ——— = theory; + = simulation.

B. *Changes in the Present Value Distribution with the Number of Lives in the Annuity Portfolio*

As the number of lives in our portfolio is increased, the central limit theorem (see [4], p 125) predicts that the distribution of the average present value should approach normality, and that the standard deviation of this average present value should decrease in proportion to the square root of the number of lives in the portfolio. Figure 11 and table 13 demonstrate that these predictions are correct.

The solid line in figure 11 represents the distribution for the corresponding single life portfolio discussed in the last section. As the number of lives in the portfolio increases, the distribution of present values becomes more symmetrical and more peaked, becoming a closer approximation to a normal curve as predicted by the central limit theorem.

The values in table 13 support this interpretation, with the standard deviation of the present value distribution decreasing as predicted with the square root of the number of lives in the portfolio. The confidence intervals narrow and become more symmetrical as the number of lives in the portfolio increases.

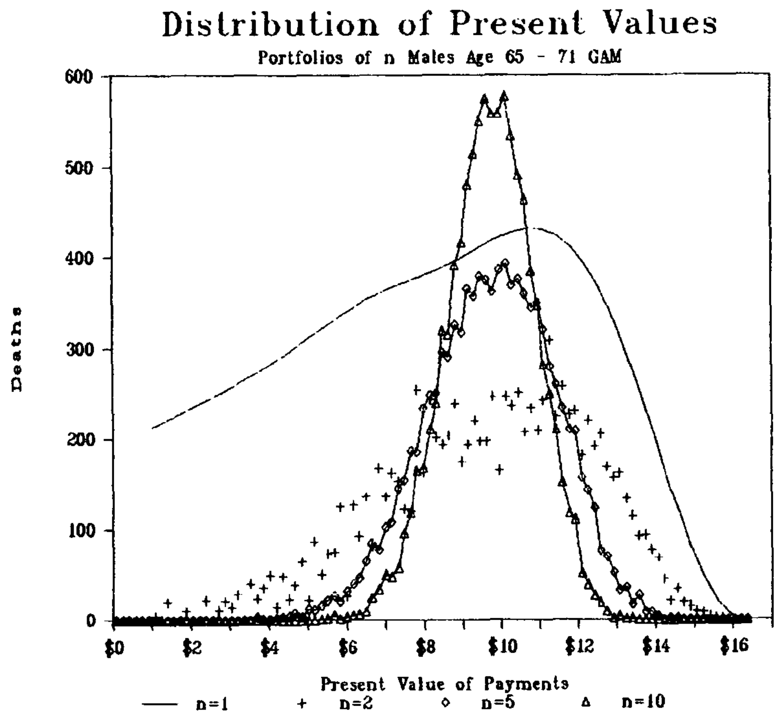

FIG. 11.—Distributions of present values of life annuity payments for portfolios of n males at age 65, based on the 1971 GAM Table. ———, $n = 1$; +, $n = 2$; ◇, $n = 5$; △, $n = 10$.

TABLE 13

COEFFICIENTS OF DISPERSION AND CONFIDENCE INTERVALS
FOR A LIFE ANNUITY DUE COMMENCING AT AGE 65 BASED ON THE 1971 GAM TABLE FOR MALES AT 6 PERCENT INTEREST

Number of Lives	Mean	Standard Deviation	Coefficient of Dispersion	Confidence Interval*					
				50 percent		70 percent		90 percent	
				Low	High	Low	High	Low	High
1.........	9.73	3.65	37.50%	73.6%	127.8%	53.3%	139.6%	29.6%	149.7%
2.........	9.69	2.60	26.80	82.4	119.7	70.5	128.2	53.5	138.4
5.........	9.73	1.64	16.82	88.8	112.4	82.0	117.5	71.9	126.0
10........	9.73	1.15	11.78	92.2	107.4	87.1	112.5	80.4	119.3

* Expressed as a percentage of the mean.

In a portfolio of ten lives, for example, we can be 90 percent certain that the average present value of annuity payments will be within 20 percent of the mean. This is a significant improvement over the range of 29.6 percent to 149.7 percent in a portfolio of one life.

Changes in the age, sex, assumed interest rate, mortality table, or form of annuity would produce corresponding changes in the mean and distribution of the present values, as discussed in the last section. Under any circumstances, however, the distribution will become more normal and more highly peaked as the number of lives in the portfolio increases.

IV. DISCUSSION

A. *Qualitative Results*

From the data presented above, we can draw some qualitative conclusions about the mortality risk in life annuities.

1. Generally there is less variation in present values of annuities on female lives than of those on male lives.
2. Variation in present values tends to increase with the age of the annuitant, except for deferred annuities.
3. Variation in present values tends to decrease with increasing interest rates.
4. Variation is much greater for annuities on disabled lives than for those on healthy lives.
5. Variation in present values is somewhat lower for temporary annuities than for comparable life annuities.
6. There is wide variation in the present values of deferred annuities.
7. As the number of lives in a portfolio increases, the variation in average present value decreases and the distribution of the average approaches a normal distribution.

For annuities, the mortality and investment risks arise in a reciprocal fashion. For example, when the mortality risk is low, as for a female aged 25 being paid an immediate annuity, the investment risk is quite high because of the uncertainties involved in a long-duration investment—57 years in this case. When the variation due to mortality is high, as for a male aged 85, the investment uncertainty is much lower because of the shorter expected duration of the investment.

Similarly, the mortality-risk decreases as the assumed interest rate rises. However, in recent years we have observed that a high rate of return on fixed-income investments is accompanied by great volatility, which tends to increase the investment risk.

B. *Implications for Expert Testimony*

Actuaries involved in expert testimony should find these results disquieting.

As an example, assume a dissolution involving a healthy male aged 45 who is entitled to an accrued pension benefit of $200 monthly commencing at age 65 and payable for the rest of his life. In many jurisdictions, this annuity would be included in the assets available for division between the parties involved in the divorce. Usually an actuary is called upon to calculate the value of the accrued pension benefit.

Generally, the actuary would use standard commutation function tables based on a mortality table and a reasonable rate of interest to calculate an actuarial present value. The 1971 GAM Table for Males and a 6 percent assumed interest rate would produce an actuarial present value of $6,072. However, the results presented above show clearly that the actuarial present value calculated is only the mean of a distribution of possible present values. In fact, table 12 indicates that the range of possible present values is very broad in this case. A 90 percent confidence interval for the discounted value of the actual pension payments extends from $0 to $10,802. We can be only 50 percent certain that the value of the actual pension payments will be between $3,346 and $9,114.

Furthermore, these ranges do not reflect the variation in the value of payments as a result of investment returns that differ from the assumed rate.

Seen in this light, how much credence should the court put in our estimate of the pension's value? Should the actuarial present value be given the same weight in the settlement as the appraised value of the family home, backed as it usually is by the market values of comparable homes? In addition, a number of related questions arise:

1. Should we continue to present an actuarial present value alone, or should we also present information about the variation of the present value distribution?
2. As an alternative to a simple present value presentation, should we present some confidence intervals?
3. Will the presentation of both mean and variance, or of mean and confidence intervals, enhance our position in court and improve the quality of the decision the court makes, or will it merely confuse the issue?
4. Under any circumstances, do we have a professional obligation to inform our clients of the range of possible error in our estimates?

These concerns are applicable to dissolutions and torts. There are no easy answers.

The author has elected expediency for the time being. At present, we present only actuarial present values, although we will provide information about variability and confidence intervals if it is requested. This reflects a belief that the court requires a single figure and, since the actuarial present value is the best available, gratuitous information about variability will only confuse the issue.

C. *Some Implications for Retirement Plans*

Many medium-sized and small retirement plans are funded through group deposit administration (DA) or immediate participation guarantee (IPG) contracts with insurance companies. These contracts usually provide for the payment of retirement benefits through the purchase of annuities or by the payment of monthly retirement benefits from the fund as they come due.

Plan sponsors frequently seek advice about which payment mechanism to use. A proper quantitative analysis of the question requires a simulation of retiree annuity experience similar to that used for multiple life annuity portfolios above.

As an example, suppose a plan anticipates about five retirements at age 65 each year. In addition, suppose that the return on plan assets is expected to average about 8.5 percent annually for the foreseeable future. The annuity purchase rate currently available to the plan for a male aged 65 is \$100 per \$1 monthly payment for life. We wish to advise the plan sponsor about the likelihood that paying the retirees monthly from the fund will be a less expensive alternative than buying the annuities from the insurance carrier. This question can be answered by a Monte Carlo simulation; the distribution of average present values resulting from a simulation of each year's retirees is shown in figure 12.

The results of the simulation suggest that the average present value of the annuity payments from the fund for each year's crop of retirees will be below the \$100 purchase price about 60 percent of the time. Therefore, in this case, self-insurance of retiree annuities by the plan seems to be a good risk, depending on the plan sponsor's risk tolerance.

This analysis is purely hypothetical; each case must be analyzed individually. The example merely illustrates the use of our knowledge about the mortality risk in life annuities and a simulation tool to derive quantitative results to aid in decision-making.

D. *Some Implications for Insurers*

Recently many insurers have been competing aggressively in the market for "structured settlements." In this context, a "structured settlement" is an annuity arrangement offered to a plaintiff in a liability tort as a substitute

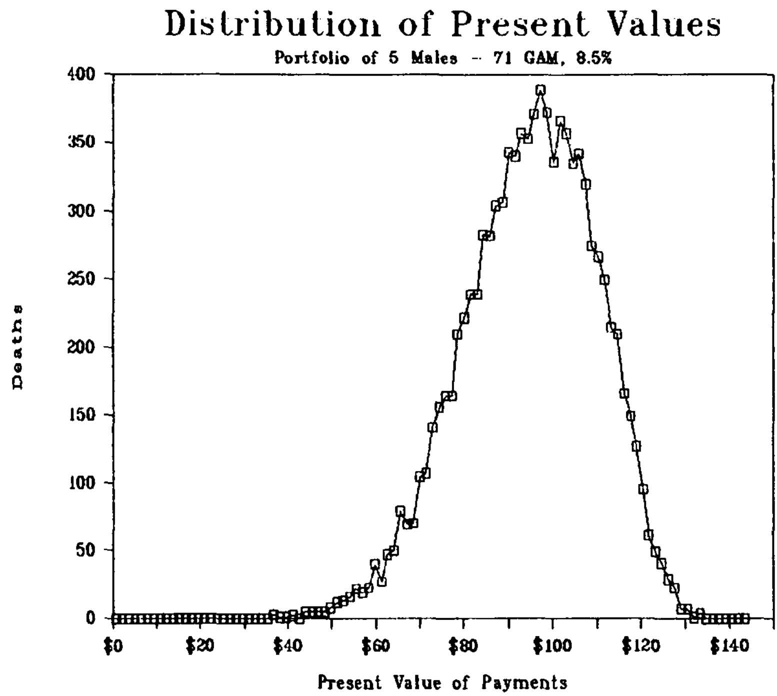

FIG. 12.—Distribution of average present values of life annuity payments for a portfolio of five males, based on the 1971 GAM Table, at 8.5 percent interest.

for a lump-sum settlement. An overview of the field is presented in Mangino's article [5].

Some structured settlements involve lifetime payments to a plaintiff who has been severely injured; often the annuitant has a higher mortality risk as a result of the injuries. Substandard annuity underwriting is necessary in this extremely competitive area.

Insurers should review carefully the results presented above for disabled lives, especially figure 7 and tables 7 and 8. Mortality risk is especially significant for impaired lives. Furthermore, if the structured settlement market is split among a large number of competing insurers, few companies may have annuity portfolios large enough to reduce the mortality risk to acceptable levels. The combination of a highly competitive market, higher mortality risks, a limited number of available cases, and the sizable consideration involved in each case warrants cautious underwriting and pricing if losses are to be avoided.

E. *Conclusion*

The paper presents a systematic investigation of the mortality risk in life annuities. Investment risk has been ignored deliberately; it could be the subject of another paper. A number of questions have been raised; none of them are answered easily. Nonetheless, we hope that the quantification of the variation in the present value calculations due to mortality will help actuaries assess the magnitude of the problem in its various contexts.

REFERENCES

1. BOERMEESTER, J. M. "Frequency Distribution of Mortality Costs," *TSA*, VIII (1956), 1–9.
2. BOWERS, N. L. "An Approximation to the Distribution of Annuity Costs," *TSA*, XIX (1967), 295–309.
3. FRETWELL, R. L., and HICKMAN, J. C. "Approximate Probability Statements about Life Annuity Costs," *TSA*, XVI (1964), 55–60.
4. HOEL P. G. *Introduction to Mathematical Statistics*. New York: Wiley, 1971.
5. MANGINO, R. M. "Structured Settlements," *CLU Journal* (October 1982), 28–34.
6. PIPER, K. B. "Contingency Reserves for Life Annuities," *TASA*, XXXIV, Part 2, No. 98 (October 1933), 240–49.
7. RUBENSTEIN, R. Y. *Simulation and the Monte Carlo Method*. New York: Wiley, 1981.
8. SPIEGEL, M. E. *Theory and Problems of Statistics*. New York: McGraw-Hill, 1961.
9. STONE, D. G. "Mortality Fluctuations in Small Self-Insured Pension Plans," *TASA*, XLIX, Part 1, No. 119 (May 1948), 81–91.
10. TAYLOR, R. H. "The Probability Distribution of Life Annuity Reserves and its Application to a Pension System," *PCAPP*, II (1952), 100–150.

DISCUSSION OF PRECEDING PAPER

WARREN LUCKNER:

Mr. McCrory has made a significant contribution to the actuarial literature with this paper both theoretically and practically. His analysis adds a discussion of the variability of the present value distribution for annuities. He also points out the implication of the magnitude of this variability on an actuary's expert testimony in court cases.

This discussion extends Mr. McCrory's analysis (standard derivations and coefficients of dispersion only) to the case of Unisex mortality assumptions and the present value distribution for whole life insurance.

For analyzing the impact of assuming Unisex mortality, the 1969-71 U.S. Life Tables are used since that data was the most readily available. 6 percent interest is assumed.. Tables A-C summarize the results.

Comparing tables A-B above to tables 1-2 of the paper yields the following observations:

TABLE A

COEFFICIENTS OF DISPERSION FOR A LIFE ANNUITY
BASED ON THE 1969-71 U.S. LIFE TABLES WHITE FEMALE
AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25.......	16.47	1.60	9.69%
45.......	14.52	2.73	18.77
65.......	10.49	3.48	33.20
85.......	4.89	2.77	56.55

*Expressed as a percentage of the mean.

TABLE B

COEFFICIENTS OF DISPERSION FOR A LIFE ANNUITY
BASED ON THE 1969-71 U.S. LIFE TABLE WHITE MALE
AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25.......	15.92	2.08	13.05%
45.......	13.29	3.23	24.29
65.......	8.80	3.76	42.68
85.......	4.28	2.57	59.98

*Expressed as a percentage of the mean.

TABLE C

COEFFICIENTS OF DISPERSION FOR A LIFE ANNUITY BASED ON THE 1969-71 U.S. LIFE TABLE TOTAL AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25	16.19	1.87	11.55%
45	13.91	3.04	21.86
65	9.70	3.70	38.11
85	4.66	2.70	58.02

*Expressed as a percentage of the mean.

1. As might be expected, the use of population mortality generally increases the variability as measured both by standard derivations and coefficients of dispersion. The one exception is the lower standard derivations for the population mortality at age 85 (but the means are small enough that the coefficients of dispersion are greater).
2. The patterns by age are the same for measures of variability for both sets of tables.
3. The ratios (1969-71 U.S. Life Tables to 1971 GAM) of the magnitudes of the two measures decreases significantly by age as shown in the following illustrations.

FEMALE RATIOS

Age	Standard Deviation	Coefficients of Dispersion
25	1.32	1.35
45	1.29	1.35
65	1.07	1.15
85	.94	1.07

MALE RATIOS

Age	Standard Deviation	Coefficients of Dispersion
25	1.34	1.38
45	1.16	1.24
65	1.03	1.14
85	.92	1.02

These patterns are somewhat surprising since they seem to indicate that the variability does not increase with age as significantly for the population mortality as it does for the group annuitant mortality.

As might be expected the variability using total mortality (table C) lies between that for female mortality (table A) and that for male mortality (table B). The variability for total mortality obviously depends on the underlying population mix by sex, but the female and male values provide bounds.

For analyzing the variability of the present value distribution for whole life insurance, the same three mortality options and 6 percent interest were used. Tables D-F summarize the results.

It is interesting to compare the coefficients of dispersion for the whole life insurance tables with those for the life annuity tables as shown in the following illustrations.

FEMALE RATIOS

Age	Coefficients of Dispersion
25	13.82
45	4.61
65	1.46
85	.38

MALE RATIOS

Age	Coefficients of Dispersion
25	9.10
45	3.04
65	.99
85	.32

TOTAL RATIOS

Age	Coefficients of Dispersion
25	11.00
45	3.70
65	1.22
85	.36

The ratios indicate a much greater variability for the whole life insurance present value distribution than for the life annuity present value distributions at the younger ages. The difference in variability decreases with age until

TABLE D

COEFFICIENTS OF DISPERSION FOR WHOLE LIFE INSURANCE BASED ON THE 1969-71 U.S. LIFE TABLES WHITE FEMALE AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25	.0675	.0904	133.87%
45	.1781	.1543	86.60
65	.4063	.1971	48.52
85	.7229	.1567	21.67

*Expressed as a percentage of the mean.

TABLE E

COEFFICIENTS OF DISPERSION FOR WHOLE LIFE INSURANCE
BASED ON THE 1969-71 U.S. LIFE TABLE WHITE MALE
AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25.......	.0991	.1176	118.70%
45.......	.2477	.1827	73.77
65.......	.5017	.2127	42.39
85.......	.7576	.1454	19.20

*Expressed as a percentage of the mean.

TABLE F

COEFFICIENTS OF DISPERSION FOR WHOLE LIFE INSURANCE
BASED ON THE 1969-71 U.S. LIFE TABLE TOTAL
AT 6 PERCENT INTEREST

Age	Mean	Standard Deviation	Coefficient of Dispersion*
25.......	.0833	.1059	127.04%
45.......	.2128	.1721	80.85
65.......	.4511	.2092	46.38
85.......	.7361	.1531	20.80

*Expressed as a percentage of the mean.

the life annuity distributions show greater variability at the older ages. This makes sense analytically considering the present value random variables $\ddot{a}_{\overline{K+1}|}$ and v^{K+1} where K is the number of complete years lived (and the whole life insurance benefit is paid at the end of the year of death). At the young ages, the differences in value of $\ddot{a}_{\overline{K+1}|}$ will be relatively less than the difference in values of v^{K+1} since most of the deaths occur at higher ages, which are far from the age of valuation. At the higher ages the reverse occurs.

For whole life insurance, there may not be as direct a practical impact of the variability of the present value distribution as the one Mr. McCrory pointed out for the life annuity. Moreover, perhaps a more important random variable to analyze for life insurance is the loss random variable L, which is expressed as the difference of a random variable for the present value of benefits paid, less a random variable for the present value of premiums received. A complete analysis of that random variable would seem to be a subject worthy of a separate paper.

LARRY M. GORSKI:

This paper is timely when considering the recommendation of the Joint Committee on the Role of the Valuation Actuary in the United States concerning valuation reserves and contingency reserves. In particular, I see this paper contributing to the pool of techniques used by actuaries to measure the adequacy of valuation reserves when they are to be set at a level to withstand random or statistical fluctuations. While the methodology discussed in the report of the Joint Committee considers investment and insurance cash flows, actuaries involved in the regulatory process might use this paper to establish statutory reserve requirements which are set at a lower limit compared to the valuation reserves as defined in the report of the Joint Committee. In order to demonstrate this paper's usefulness to such actuaries, I tested the margin in the 1983 Individual Annuity Mortality (IAM) table which was recently adopted by the NAIC. The mortality tables in the 1983 valuation table were set equal to 90 percent of the rates from the corresponding basic table.

The methodology that I used to test the adequacy of the margins in the 1983 Individual Annuity Mortality table was to use the underlying Basic Table to develop the distribution of annuity costs for a single life portfolio, assuming a 9 percent interest rate. By utilizing Mr. McCrory's conclusion that, as the number of lives in the portfolio increases, the distribution of annuity costs for the portfolio approaches that of a normal distribution, I tested the margins in the statutory reserves based on the 1983 IAM table and 9 percent for different portfolio sizes. For different values of N, the size of the portfolio, I compared the annuity factor based on the valuation mortality table and 9 percent, and the upper limit of a confidence interval of 80 percent centered on the annuity value derived from the Basic Table. Table 1 summarizes the results of the analysis. $\overline{X}$ is the annuity value based on the Basic Table mortality and 9 percent, while $\ddot{a}_x$ is based on the valuation mortality table and 9 percent. The analysis leads me to conclude that the 1983 IAM table does not have sufficient margins to cover random fluctuations in mortality at the 80 percent confidence level.

My conclusion prompted me to review the work done by the "Committee to Recommend a New Mortality Basis for Individual Annuity Valuation." The report submitted by the committee and the 1979 Reports of Mortality and Morbidity Experience, which initially published the annuity experience used in the construction of the 1983 table, indicated that only the larger companies were supplying experience. Since these were the companies that were used in the tests of the margins, a 10 percent margin might be appropriate for them but might not be adequate for companies with less individual annuity business.

My analysis ignores interest as does the author's. Whether valuation interest rates are sufficiently conservative to cover what I consider are deficiencies in the mortality margins is another matter that needs additional research.

TABLE 1

MALE AGE 60		
(1)	(2)	(3)
N	$\bar{X} + Z \cdot \sigma_N$	$(2)/\ddot{a}_{60}$
10,000	9.9743	1.024
7,500	10.0346	1.031
5,000	10.1360	1.041
2,500	10.3649	1.065
MALE AGE 65		
(1)	(2)	(3)
N	$\bar{X} + Z \cdot \sigma_N$	$(2)/\ddot{a}_{65}$
10,000	9.2485	1.029
7,500	9.3161	1.036
5,000	9.44294	1.049
2,500	9.6851	1.077
MALE AGE 70		
(1)	(2)	(3)
N	$\bar{X} + Z \cdot \sigma_N$	$(2)/\ddot{a}_{70}$
10,000	8.3625	1.030
7,500	8.4351	1.039
5,000	8.5568	1.0544
2,500	8.8316	1.0883

HSIEN-MING KEH:

Mr. McCrory's interesting paper is an excellent example of applying probability and statistical theories to a practical insurance problem. It is easy to read and understand. After studying the paper, I have some comments and questions.

My impression from each table in the paper is that the coefficient of dispersion increases with age. Actually, as the issue age increases, the coefficient of dispersion approaches zero (mean = 1 and standard deviation = 0 at age w). Since the coefficient of dispersion is a measure of the confidence one may have in using the mean of the distribution, it is important to know at what age the coefficient of dispersion is the greatest so that extra caution can be taken in presenting the single actuarial present value around that age.

The distribution in figure 8 appears to be extended too far to the right. At age 45, most of the assumed 10,000 female annuitants will receive the twenty annual payments of $1 per year. The distribution should end at its peak where the present value is the largest for all annuitants. Also at age 45, the most annual payments the annuitant can receive is twenty. The distribution in figure 8 has twenty-two plotted points. It appears that the two points beyond the peak point do not belong.

The confidence interval shown in tables 9 and 10 does not make sense. Some of the lower end points of the confidence intervals appear to be wrong.

According to the selection criteria used in the paper, the lower end point of any positive confidence interval can never be greater than 100 percent of the mean.

MURRAY PROJECTOR:

I congratulate Mr. McCrory for the mathematical development in his paper. However, I find the asserted relevance of single life annuity portfolio theory to actuarial divorce testimony most debatable. Mr. McCrory's paper may be useful to an actuary involved in personal injury litigation, one who is pricing a structured settlement, but this usefulness does not apply when testifying to the value of a *community* (marital) pension asset in *dissolution* (divorce) litigation.

It is true, as Mr. McCrory writes, that the actuary who testifies to the actuarial present value of a pension benefit, "does not present, and usually does not know, the variation that may occur in the value of payments actually received." I have prepared some 5,000 pension plan present value reports for southern California dissolution hearings without once presenting or knowing the variation in the present values which are based on mathematical expectation. This omission is of no consequence because variation information is not relevant to my function as a divorce actuary.

The right to a pension is a community asset which needs appraisal, as do other assets such as the family home, a closely held corporation, oil and gas interests, art and antiques, and so forth. The function of a divorce actuary is to determine a reasonable current (present) value of the asset, not to predict the current value based on the duration of life for one particular individual whose duration of life is unknown. Concern with the variation of the mathematical expectation would be justifiable only if the divorce actuary were hired to predict the present value of the pension benefits that the employee spouse may receive. Dissolution actuaries are not engaged to predict. We are hired to appraise the right to future pension payments. The validity of the appraisal is unrelated to the benefits actually received.

Pension actuaries understand the difference between appraisal and prediction. It is our responsibility to teach this distinction to the other professions involved in divorce litigation.

I have had some success with the following analogy in getting lawyers and judges to understand the importance of the distinction between appraisal and prediction. Suppose a marital asset is a ticket that entitles the bearer to an interest in a single toss of a coin. If the coin comes up heads, the bearer receives $50,000. If tails come up, then the bearer receives nothing. Suppose then that you are asked to appraise the fair value of this ticket, or marital asset.

Most attorneys and judges come up with $25,000 for the fair market value of this asset. Now suppose we perform 1,000 appraisals of this kind, and we then match the actual payouts with our $25,000 appraisals. Since the only possible payouts are $50,000 and zero, we find that in all 1,000 cases the actual payout differs by $25,000 from our appraised value. Does this mean that all 1,000 appraisals are wrong because none of them correctly predicted the actual payout?

This apparent paradox is resolved by the fact that the correctness of any one appraisal is wholly independent of whether or not it is confirmed by that ticket holder's experience. What is obvious for coin tossing should be equally obvious for a pension appraisal in divorce litigation. The validity of a divorce actuary's valuation, based on mathematical expectation not a prediction, cannot be measured by any one employee's experience. Consequently, variation data and confidence bands are right answers for the wrong question in divorce litigation.

The dissolution actuary should not offer more than the best estimate of the employee's pension value derived from mathematical expectation. Thus, Mr. McCrory's questions about the need for presenting present value distributions, means and variances, and confidence intervals are not relevant in our work. These questions are right for annuity companies' pricing actuaries. Divorce actuaries may validate asset values by fair market value, but we are not responsible for producing annuity market prices which reflect confidence intervals, means and variances, and so forth.

Curiously, Mr. McCrory concludes section V.B., with the statement that he presents only actuarial present values, the policy of which "reflects a belief that the court requires a single figure, and since the actuarial present value is the best available, gratuitous information about variability will only confuse the issue." I agree with this statement which suggests that the rest of the section is inapplicable to dissolution litigation.

ROBERT C. TOOKEY:

This discussion of Mr. McCrory's excellent paper is limited to the section on expert testimony. In the pursuit of the truth, the author has removed the lid from a rather large barrel of moray eels that probably would have emerged without his assistance.

To volunteer any of the information suggested by the author would tend to obfuscate the entire issue on the value of an annuity. It would tend to undermine the credibility of the mean itself. One would ask, "Is the mortality table really appropriate, and is the assumed interest rate realistic?" Already there have been cases where the about-to-be divorced husband's attorney has argued that the value of the annuity is overstated because divorced males have a higher mortality rate than married males. He may also seek a reduced value of the annuity on the basis of the husband's present state of health.

The actuary should not present numerical information on variation within various sets of confidence limits, nor does he appear to have an obligation to do so unless specifically instructed. All he need say about variations would be a statement to this effect, "The value we have calculated for the annuity represents an average of values for all annuitants of the same age. If he dies tomorrow, it is worth nothing. If the annuitant lives to be 100, it is worth a lot more than our indicated amount."

To provide all the additional information mentioned by the author in divorce cases could result in an impasse that would prevent immediate cash settlements. The two parties would simply wait it out and divide the annuity payments in accordance with a previously agreed to ratio. In most cases, this is undesirable.

To end this discussion on a positive note, I feel that Mr. McCrory's paper should be of great value in its applications to retirement plans and other areas not complicated by litigation.

(AUTHOR'S REVIEW OF DISCUSSION)

ROBERT T. MCCRORY:

Let me thank Messrs. Luckner, Gorski, Keh, Projector and Tookey for their perceptive remarks.

Messrs. Luckner and Gorski made a valuable contribution by extending the concepts of my paper to life insurance. In particular, the pattern of a decreasing coefficient of dispersion with age for life insurance is notable, since it is the opposite of the generally increasing pattern observed for annuities.

Mr. Luckner's suggestion about the loss function has occurred to others. In the *Journal of the Institute of Actuaries*, Volume 95, pages 79-115, A.H. Pollard and J.H. Pollard wrote an article entitled "A Stochastic Approach to Actuarial Functions." In this paper, they addressed the problem of setting retention and reinsurance limits.

Mr. Keh raises an interesting point: where is the coefficient of dispersion maximized? I did not address this question in the paper because I seldom have to deal with ages much above 85.

With regard to Mr. Keh's other points, the selection of the points graphed in figure 8 was a function of the software used to produce the graphs.

The confidence intervals in tables 9 and 10 are correct. It is important in reviewing these confidence intervals to note that this is a discrete distribution, so finding confidence intervals containing exactly 50, 70 or 90 percent of all cases is not generally possible. It is also important to distinguish between the median and the mean. For a highly skewed distribution, the choice of symmetric confidence intervals may indeed exclude the mean but will, of course, always include the median.

Messrs. Projector and Tookey raise questions about my comments concerning expert testimony. My remarks were meant to invite discussion about the problem; I seem to have succeeded in that regard. In the hope of provoking some additional thought and discussion about the issue, let me make a couple of additional comments on expert testimony, particularly as it regards division of property in a divorce.

My comments are based on two principles:

1. the financial models we build as actuaries must reflect reality; and
2. there is more to life than expectation.

Let us see how these principles affect our analysis.

First of all, actuaries are in the business of building financial models. These financial models must reflect the underlying reality of the systems we attempt to model.

What is the reality in a divorce? Two results may occur:

1. The pension benefit may be split at the source between the employee and nonemployee spouses. This is permissible via the Qualified Domestic Relations Order created by the Retirement Equity Act. In this case, an actuary is not needed to value the pension—the pension payments are simply divided between the husband and wife.
2. The employee spouse will keep his or her pension, while the nonemployee spouse will receive some compensating asset representing his or her interest in the pension benefit. For example, the nonemployee spouse may receive the equity in the house to compensate for the employee spouse retaining his or her full pension rights. In this case, the actuary is called upon to value the pension benefit, thus determining the amount of other assets to be received by the nonemployee spouse.

In a divorce, the reality of the situation is that when an actuary is involved one party will retain the right to the pension while the other party will receive some compensating asset. This leads us to our second principle. It is not merely the expected value of the pension benefit and the expected value of the compensating asset that matter in the division of assets in a divorce. The risk associated with the assets also matters.

It is here that I disagree strongly with Mr. Projector. Mr. Projector uses the example of a ticket with a 50 percent chance of winning $50,000. There is no way that $25,000 in cash and such a $50,000 ticket are equal assets.

While their expected values are equal, the risks associated with each are greatly different.

Certainly the expected value of Mr. Projector's ticket is $25,000. This means that across a large number of dissolutions involving such tickets the equities would balance out. Consequently, if Mr. Projector has valued 5,000 such dissolutions, he can take reasonable satisfaction in knowing that the equities have balanced out on the average.

Under Mr. Projector's scheme, every single valuation is guaranteed to be wrong—and wrong in the most fundamental sense that equities have not been served. Mr. Projector's offsetting of assets with equal expected values but widely different risks guarantees that there will be a loser in every case. Therefore, the more sensible course of action with Mr. Projector's ticket would be to give each party in the divorce an equal interest in the ticket, thus forcing them to share the risk.

As we indicated, in the divorce the underlying reality is that the employee spouse retains his pension rights while the nonemployee spouse receives compensating assets. Under the second principle, the risk of these assets matters. Therefore, if the risk associated with the pension benefit is very different from the risk associated with the compensating assets, then the distribution on divorce is not fair. Instead, each of the assets should be divided. Under the Retirement Equity Act, the pension benefit can be divided at the source; other assets may be divided similarly.

For too long, actuaries have concentrated on expectation and have ignored variation. By examining the risk associated with the pension at divorce, we can assess whether the risk involved is substantially greater for the employee spouse than for the nonemployee spouse who will receive the compensating asset. If the risks differ widely, then some other approach is called for.