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A BALLISTIC APPROACH TO ACTUARIAL PROBLEMS

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ABSTRACT

Pricing, in its broadest sense, lies at the heart of virtually all actuarial work. This paper shows how actuarial pricing based on the mathematical concept of present value can be formulated and solved using Markov chains and an approach borrowed from engineering mathematics called ballistic control theory. General discussions of a Markov chain model of insurance and the ballistic approach to pricing are illustrated by examples. A concluding summary briefly reviews the conceptual and practical advantages of such an approach and speculates that rounding errors should not be a problem with the matrix manipulations required for actuarial systems.

I. INTRODUCTION

Pricing is the determination of a schedule of deposits to pay for expected benefits and expenses. This definition is sufficiently broad to include, as special cases, reserve valuation, pension valuation, and premium setting. When the expected incidence of deposits—based on pattern, frequency, and payment period—differs from the expected incidence of benefits and expenses, fund accumulations result. Thus, gross level premiums give rise to asset shares, net level premiums to reserves, level normal costs to accrued liabilities, and so on.

Although the actuarial literature has tended to deal with these various special cases of pricing as separate topics, the mathematical attack on any of these problems is usually a variant of the formula method, the accumulation method, or the pay-as-you-go method. The formula method is used in determining gross premiums [7] and net valuation premiums [3, 13] for individual products, and also level or unit credit normal costs for pensions [16]. The accumulation method is used to determine gross premiums [7] and GAAP reserve premiums [15]. The pay-as-you-go method is used whenever premiums must match current benefit costs, as for renewable term, social insurance plans, and so forth. Under the formula and accumulation methods, a desired incidence of premium is specified in advance, such as level amount, step rate, or level percentage of pay, and this determines the fund accumulations. Even the pay-as-you-go method is a special case of the formula method with the pricing horizon confined to a period during which the premium does not change.

These three basic methods work well when the benefits and expenses are not functionally related to each other or to the premiums. For example, in pricing an individual life coverage, the death benefits often can be determined independently, the cash values can be determined from the death benefits, and gross premiums and asset shares can be determined from the cash values and death benefits. When the benefits are interrelated, however, as when the death benefit is a return of gross premiums that reflect withdrawal assumptions, the traditional methods tend to break down.

This paper discusses an adaptation of an approach called ballistic control theory that is used for such diverse engineering problems as determining the rocket thrust input to maneuver a space vehicle and determining the voltage input to charge a capacitor [2, 5]. Premiums play the role of input. Asset share, cash value, death benefit, and the like play the role of state variables, which describe an actuarial coverage in the same way that position and velocity coordinates describe a space vehicle and that capacitor charge and current describe an electrical circuit. The approach results in (1) a theory of pricing broad enough to include premium setting, reserve valuation, and pension costing as special cases; (2) a solution in closed form to certain actuarial problems for which the functional interrelationships between benefits and/or premiums are difficult to handle by the standard methods; and (3) a computational technique that can be computerized readily.

The remaining sections in this paper discuss and provide examples of a Markov model of insurance, the pricing of actuarial coverages using a ballistic approach based on this model, and the scope and implications of ballistic control theory in actuarial science.

II. A MARKOV MODEL OF INSURANCE

The life insured by an actuarial coverage, whether it is life insurance, health insurance, or annuity coverage, can be viewed as occupying one of a number of states (active) in which it remains or from which it exits to another state (death, withdrawal, or disability). This view of insurance suggests a Markov chain [6, 14] as a mathematical model for an actuarial coverage. The usefulness and generality of this concept are discussed in [12], which also provides a number of references to specific applications. In fact, Markov chains have been used to study mortality classes [17], working life tables [9], present values of annuity and insurance coverages [10], and other problems. One paper [1] even used the Markov chain concept to express collective risk theory in terms of two states, nonruin and ruin, for an insurance company analogous to survival and death for a single life, and in doing so established a common model for life, sickness, property, and casualty insurance and collective risk theory.

A Markov chain not only provides a conceptually useful view of insurance but also renders many actuarial problems tractable by means of linear algebra. Linear algebra can provide elegant formulations of actuarial problems [4, 17] and also is capable of providing solutions that can be computerized readily. This paper exploits both advantages of Markov chains. The model that will be discussed is similar to one described in [12, pp. 17–20] for pension and health coverages. For the first time, however, benefits and fund accumulations will be viewed as state variables, in the engineering sense [2, 5], and premium or normal cost will be determined as the input component of a ballistic control problem.

The life insured by an actuarial coverage is assumed to occupy at each time between issue and maturity any one of a number of states such as active, terminated, disabled, or dead. Benefits may be paid as the result of continued presence in a state, such as an annuity benefit, a disability income, or a waiver benefit, or as the result of a transition between states, such as a lump-sum death benefit or a cash surrender value. Premiums or normal costs are usually paid only while the active state is occupied. A state may be an *absorbing state,* such as death, from which there is assumed to be no return, or it may be possible to leave a state after entering it, for example, recovery from disability or reinstatement from lapse.

Associated with each state are state variables, which include the benefits payable upon the condition of leaving or remaining in that state and one or more measures of the financial significance of those benefits, such as asset share and/or reserve. Since pricing is always based on a simplified model of reality, the total number of state variables that are significant in the pricing process should be quite small. As illustrated by the examples to follow, at any time t the significant state variables from all of the states can be listed in a state vector, $\boldsymbol{x}(t)$, which obeys a recursive equation of balance of the form

$$\begin{aligned} &\mathrm{A}(t,\, t-1)\boldsymbol{x}(t-1) + \mathrm{M}(t,\, t-1)\boldsymbol{u} \\ &\qquad = \mathrm{N}(t,\, t-1)\boldsymbol{x}(t-1) + \mathrm{P}(t,\, t-1)\boldsymbol{x}(t) + \mathrm{Q}(t,\, t-1)\boldsymbol{u}, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is an input vector containing premium variables and $\mathrm{A}(t,\, t-1)$, $\mathrm{M}(t,\, t-1)$, $\mathrm{N}(t,\, t-1)$, $\mathrm{P}(t,\, t-1)$, and $\mathrm{Q}(t,\, t-1)$ are matrices whose elements describe the coverage and contain the actuarial assumptions for the period $t-1$ to t.

Equation (1) treats each component of $\boldsymbol{x}(t-1)$ as a fund at time $t-1$ that grows with interest and net deposits, as represented by the left side, to its value at time t after paying benefits that can be linear combinations of components of $\boldsymbol{x}(t-1)$, $\boldsymbol{x}(t)$, and $\boldsymbol{u}$, as represented by the right side. This interpretation is clear enough for asset shares, reserves, and individual life cash values. That this same interpretation can describe benefits as components of $\boldsymbol{x}(t)$ may not be so obvious and will be illustrated in the examples that follow. In the examples, interest (i), survival probability (p), death probability (q), withdrawal probability (w), disability probability (d), net-to-gross premium ratio (l), the ratio of premium to the initial premium (ρ), recovery probability (r), accumulation factor for return of premium death benefit (b), and death benefit interpolation factor (γ) will be shown without subscripts to indicate select and/or ultimate attained-age dependence (i.e., which t they belong to). They will, however, have superscripts to identify them with their associated state variables, which are identified by subscripts.

Example 1: Endowment, Term, or Deferred Annuity

The recursive equation of balance, equation (1), for this example is

$$\begin{bmatrix} 1+i^1 & 0 & 0 & 0 \\ 0 & 1+i^2 & 0 & 0 \\ 0 & 0 & 1+i^3 & 0 \\ 0 & 0 & 0 & 1+i^4 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \\ x_4(t-1) \end{bmatrix}$$

$$+ \begin{bmatrix} (1+i^1)l^1\rho^1 & 0 & 0 & 0 \\ 0 & (1+i^2)l^2\rho^2 & 0 & 0 \\ 0 & 0 & (1+i^3)l^3\rho^3 & 0 \\ b^4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & (1-\gamma^1)q^1 \\ 0 & 0 & 0 & (1-\gamma^2)q^2 \\ 0 & 0 & 0 & (1-\gamma^3)q^3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \\ x_4(t-1) \end{bmatrix} + \begin{bmatrix} p^1 & 0 & w^1 & \gamma^1 q^1 \\ 0 & p^2 & w^2 & \gamma^2 q^2 \\ 0 & 0 & p^3 & \gamma^3 q^3 \\ 0 & 0 & 0 & p^4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l^4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

where

$$x(t) = \begin{bmatrix} \text{Asset share at time } t \\ \text{Reserve at time } t \\ \text{Cash value at time } t \\ \text{Death benefit at time } t \end{bmatrix}$$

and

$$u = \begin{bmatrix} \text{Gross premium} \\ \text{Valuation premium} \\ \text{Adjusted premium} \\ \text{Level death benefit increment} \end{bmatrix}.$$

Four separate equations are represented, one for each row of the matrices. The off-diagonal matrix entries reflect couplings among the four equations. The fourth row illustrates that death benefits can be represented in the same general format as the fund accumulations of the first three rows. Table 1 provides more detail on the variety of death benefits that can be handled in the manner of this example.

Notice that the term $-l^4$ was included in the fourth diagonal element of $Q(t, t-1)$, instead of l^4 being included in the fourth diagonal element of $M(t, t-1)$, to emphasize that plan benefits can be indeed functions of input.

The cash value shown in this recursive equation is in the form suitable for calculating the cash values commonly found in individual policies in the United States and Canada; however, termination benefits of the sort commonly found in pension plans also can be set up in the same format as the death benefits of type 3 or 7 in Table 1, assuming that the plan member is the survivor.

Expenses also can be reflected. The probabilities p, q, and w can be loaded for claim expenses and overhead; q and w can be loaded further for interest to the end of the year. All expenses that are proportional to premiums can be reflected in l. Acquisition costs per unit issued and surplus margins at maturity are handled by setting initial and final conditions as shown in the next section.

Some individual products have "overlapping" benefits. For example, the cash value may be payable on death in addition to the level basic benefit; for such a case w^1, w^2, and p^3 would be replaced by $w^1 + q^1$, $w^2 + q^2$, and $p^3 + q^3$, respectively, and $i^4 = 0$, $p^4 = 1$, $b^4 = 0$. Another

TABLE 1

TYPES OF DEATH BENEFIT CONFORMING TO EXAMPLE 1

Type of Death Benefit	Recursive Equation (Fourth Row of Equation [1]); General Form: $(1+i^4)x_4(t-1)+b^4$	$u_1 =$	$p^4 x_4(t) - l^4$	u_4	Interpolation Factor γ
1. Level	$x_4(t-1)$	$=$	$x_4(t)$		1
2. Level plus return of gross premiums $[x_4(t) = x_4(0) + tu_1]$	$x_4(t-1)+$	$u_1 =$	$x_4(t)$		1
3. Return of gross premiums with $100i^4$% interest $[x_4(t) = u_1\ddot{s}_{\overline{t}\rvert}]$	$(1+i^4)x_4(t-1)+$	$(1+i^4)u_1 =$	$x_4(t)$		1
4. $m$thly increasing (u_4 per year) $[x_4(t) = x_4(0) - tu_4]$	$x_4(t-1)$	$=$	$x_4(t)-$	u_4	$(m-1)/2m$
5. $m$thly decreasing (u_4 per year) $[x_4(t) = x_4(0) - tu_4]$	$x_4(t-1)$	$=$	$x_4(t)-(-1)$	u_4	$(m-1)/2m$
6. $100i^4$% T-year mortgage loan balance unpaid (beginning of year) $[x_4(t) = x_4(0)(a_{\overline{T-t}\rvert}/a_{\overline{T}\rvert}) = u_4 a_{\overline{T-t}\rvert}]$	$(1+i^4)x_4(t-1)$	$=$	$x_4(t)-(-1)$	u_4	0
7. $m$thly life annuity to survivor at attained age $x+t$ at time t $[x_4(t) = u_4\ddot{a}^{(m)}_{x+t}]$	$(1+i^4)x_4(t-1)$	$=$	$p_{x+t-1}x_4(t) - (1+i^4)(-\ddot{a}^{(m)}_{x+t-1:\overline{1}\rvert})$	u_4	½

variation would be to pay the reserve in addition to the level death benefit, in which case $P(t, t-1)$ would become

$$\begin{bmatrix} p^1 & q^1 & w^1 & q^1 \\ 0 & p^2+q^2 & w^2 & q^2 \\ 0 & q^3 & p^3 & q^3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This paper does not deal specifically with income tax, but most treatments (e.g., [11]) represent the income tax generated by a policy as a linear combination of such state variables as asset share and reserve. These formulations can be accommodated in equation (1) by means of loading in the appropriate matrix elements.

In a similar fashion, dividend and experience-rating credits commonly are based on formulas that are linear combinations of state variables, such as reserve, and input variables, such as gross premium, and, if applicable, experience premium [8, 11]. Such formulas can be accommodated by equation (1) with suitable loading of the appropriate matrix elements.

This first general example covers a wide variety of insurance and deferred annuity coverages for which nonactive states are assumed to be absorbing states.

Example 2: Three-Year Endowment, Term, or Deferred Annuity with a Disability Waiver

This next example will be more specific than the first one because it will serve as the basis for the numerical solution to an illustrative pricing problem. The coverage will be assumed to provide cash values and \$1,000 of level death benefit, and to contain a disability waiver that continues the insurance coverage and pays to the insurer all expenses that are proportional to premiums. The state vector is

$$\boldsymbol{x}(t) = \begin{bmatrix} \text{Active life asset share at time } t \\ \text{Cash value at time } t \\ \text{Death benefit at time } t \\ \text{Disabled life asset share at time } t \end{bmatrix},$$

and the input vector is

$$\boldsymbol{u} = \begin{bmatrix} \text{Gross premium} \\ \text{Adjusted cash-value premium} \\ \text{Present value at time 0 of all disabled life benefits} \end{bmatrix}.$$

The recursive equation is

$$
\begin{bmatrix} 1+i^1 & 0 & 0 & 0 \\ 0 & 1+i^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+i^4 \end{bmatrix}
\begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \\ x_4(t-1) \end{bmatrix}
$$

$$
+\begin{bmatrix} (1+i^1)l^1\rho^1 & 0 & 0 \\ 0 & (1+i^2)l^2\rho^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1+i^4)l^4\rho^4 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix}
=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \\ x_4(t-1) \end{bmatrix}
$$

$$
+\begin{bmatrix} p^1 & w^1(1+{}^we^1) & q^1(1+{}^qe^1)(1+i^1/2) & d^1(1+{}^de^1) \\ 0 & p^2 & q^2 & 0 \\ 0 & 0 & 1 & 0 \\ r^4 & 0 & q^4 & p^4 \end{bmatrix}
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ (1+i^4)(1+l^1)\rho^1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix},
$$

where the notation and the numerical values are given in Tables 2, 3, and 4. Also, in Tables 2, 3, and 4, the acquisition cost assumptions are represented as negative asset share, reserve, and cash value at time 0 (see, e.g., [3]). Notice that the third component of $\boldsymbol{u}$ is labeled u_4 to identify it with the disabled life asset share.

In pension applications, ρ^1 and ρ^2 could incorporate a salary scale, the decremental expense factors (${}^we^1$, ${}^qe^1$, ${}^de^1$) would usually be zero, and l^1 and l^2 would usually be 1. For a termination benefit, $x_2(t)$, equal to a return of employee contributions u_2, $q^2 = 0$, $p^2 = 1$, $l^2 = 1$, i^2 = interest on contributions; this special case of the cash-value formula, represented in the second row of the matrix-vector recursive equation, is equivalent to the type 3 death benefit in Table 1.

The most significant aspect of this example is the explicit recognition of disability and recovery by means of a state variable associated with the disabled state. An examination of the "fourth row" of the recursive equation,

$$
(1+i^4)x_4(t-1)+(1+i^4)l^4\rho^4u_4=r^4x_1(t)+q^4x_3(t)+p^4x_4(t)+(1+i^4)(1-l^1)\rho^1u_1,
$$

TABLE 2

NUMERICAL VALUES FOR EXAMPLE 2

(Active Life Asset Share Basis)

VARIABLE	VALUES		
	Year Starting at Time 0	Year Starting at Time 1	Year Starting at Time 2
Interest, i^1	.1000	.0900	.0800
Mortality probability, q^1	.0200	.0970	.0400
Withdrawal probability, w^1	.1470	.0300	.0000
Disability probability, d^1	.0330	.0230	.0200
Survival probability, p^1	.8000	.8500	.9400
Premium expense ratio, $1-l^1$	.4000	.3000	.1000
Death claim expense ratio, ${}^qe^1$	.0100	.0100	.0100
Surrender expense ratio, ${}^we^1$	.0200	.0200	.0200
Disability expense ratio, ${}^de^1$	.0150	.0150	.0150
Acquisition cost, $-x_1(0)$	120	N/A	N/A
Maturity asset share, $x_1(3)$	N/A	N/A	800
Premium incidence factor, ρ^1	1	1	1

TABLE 3

NUMERICAL VALUES FOR EXAMPLE 2

(Active Life Cash Value Basis)

VARIABLE	VALUES		
	Year Starting at Time 0	Year Starting at Time 1	Year Starting at Time 2
Interest, i^2	.0350	.0400	.0300
Mortality probability, q^2	.0500	.0400	.0300
Survival probability, p^2	.9500	.9600	.9700
Acquisition cost, $-x_2(0)$	100	N/A	N/A
Maturity cash value, $x_2(3)$	N/A	N/A	600
Premium incidence factor, ρ^2	1	1	1
Premium expense factor, $1-l^2$	0	0	0

TABLE 4

NUMERICAL VALUES FOR EXAMPLE 2

(Disabled Life Asset Share Basis)

VARIABLE	VALUES		
	Year Starting at Time 0	Year Starting at Time 1	Year Starting at Time 2
Interest, i^4	.0800	.0900	.0700
Mortality probability, q^4	N/A	.1000	.1000
Recovery probability, r^4	N/A	.3000	.3000
Disabled survival probability, p^4	1.0000	.6000	.6000
Maturity disabled life asset share, $x_4(3)$	N/A	N/A	800
Premium incidence factor, ρ^4	1	0	0
Premium expense factor, $1-l^4$	0	N/A	N/A

reveals that $x_4(t)$ is a single premium ($\rho = 1$ in first year, 0 thereafter) asset share or reserve providing the following benefits:

1. A recovery benefit $x_1(t)$ (the active life asset share) with recovery probability r^4;
2. The level death benefit, $x_3(t)$, with mortality probability q^4;
3. The disabled life asset share with disabled survival probability p^4; and
4. The expense component of gross premium at beginning of year, $(1 - l^1)\rho^1 u_1$, with interest to the end of the year.

Since the active life asset share is the recovery benefit, recovery actually restores the policy to the surplus position it would have been in had no disability occurred.

The effects of excess mortality in the first few years after recovery or disablement could be handled very simply by means of explicit loading in the recovery and disability probabilities r^4 and d^1, respectively.

The cash values provided are similar to the prospective cash values found in individual life insurance coverages:

$$x_2(t) = A_{x+t} - u_2\ddot{a}_{x+t},$$

where

$$x_2(t) = {}_tCV, \qquad u_2 = \beta.$$

However, a specific first-year acquisition cost was used in Table 3 to define the cash values, rather than the more traditional formula

$$x_2(0) = {}_0CV = -(\beta - \alpha).$$

Thus the first-year expense allowance is represented by a negative cash value at time 0, and the first-year premium in this formulation would be β, not α. This approach is equivalent to a zero first-year expense allowance and a first-year premium of α, the approach used in [13].

Other types of termination benefit, such as return of premium or employee contributions, which are more common in pension coverages, can be accommodated in the same fashion as the death benefits illustrated in Table 1. In pension costing situations, the withdrawal or termination probability could incorporate a vesting factor.

These two examples demonstrate the wide range of coverages to which equation (1) applies. In general, any coverage priced by means of present-value methods can be reduced to equation (1) because present values, whether reserves, accrued liabilities, asset shares, or whatever, represent idealized fund accumulations, according to specific actuarial assumptions, which can be represented by recursive equations. Also, the benefits provided are generally expressible as recursive equations along the lines shown for death benefits in example 1. Even an arbitrary string of benefits,

$$x_k(0),\ x_k(1),\ \ldots,\ x_k(t-1),\ x_k(t),\ \ldots,$$

can be expressed recursively, as

$$x_k(t) = (1+i^k)x_k(t-1),$$

where i^k for the period $t - 1$ to t is given by

$$1+i^k = \frac{x_k(t)}{x_k(t-1)}.$$

III. THE BALLISTIC APPROACH TO PRICING

Equation (1) can be solved for $\boldsymbol{x}(t)$ in terms of $\boldsymbol{x}(t - 1)$ and $\boldsymbol{u}$:

$$\boldsymbol{x}(t) = \Phi(t,\ t-1)\boldsymbol{x}(t-1) + \mathrm{B}(t,\ t-1)\boldsymbol{u}, \tag{2}$$

where

$$\Phi(t,\ t-1) = \mathrm{P}(t,\ t-1)^{-1}[\mathrm{A}(t,\ t-1) - \mathrm{N}(t,\ t-1)]$$

and

$$\mathrm{B}(t,\ t-1) = \mathrm{P}(t,\ t-1)^{-1}[\mathrm{M}(t,\ t-1) - \mathrm{Q}(t,\ t-1)].$$

The invertibility of $\mathrm{P}(t, t - 1)$ follows from the fact that a well-defined actuarial pricing problem would *uniquely* define all the components of $\boldsymbol{x}(t)$, which are benefits and idealized fund accumulations, (such as asset shares and reserves) at time t, in terms of their values at time $t - 1$, (the components of $\boldsymbol{x}(t - 1)$) and their associated premiums (the components of $\boldsymbol{u}$).

On the other hand, equation 2 can be solved for $\boldsymbol{x}(t - 1)$:

$$\boldsymbol{x}(t-1) = \Phi(t,\ t-1)^{-1}[\boldsymbol{x}(t) - \mathrm{B}(t,\ t-1)\boldsymbol{u}] = \Phi(t-1,\ t)[\boldsymbol{x}(t) - \mathrm{B}(t,\ t-1)\boldsymbol{u}],$$

where $\Phi(t - 1, t)$ is defined to be the inverse of $\Phi(t, t - 1)$. That $\Phi(t, t - 1)$, and therefore $[\mathrm{A}(t, t - 1) - \mathrm{N}(t, t - 1)]$, should be invertible follows from the idea that the net present value of benefits and premiums, which is what a reserve, asset share, or accrued liability is, should be unique. Death benefits expressed recursively also can be given a present-value interpretation that uniquely defines them, as illustrated in Section II.

It is convenient to adopt the following definitions:

$\Phi(t, k) = \Phi(t, t-1)\Phi(t-1, t-2). . .\Phi(k+1, k),$ for $t>k$,

$\Phi(k, t) = \Phi(t, k)^{-1},$ for $t>k$,

$\Phi(t, t) =$ The identity matrix,

$B(0, -1) =$ A zero matrix, and

$\sum_{k=1}^{0} . . . = 0.$

The trajectory of a system, for example, the progress of the expected financial position of a surviving life insurance policy, which begins at state $\boldsymbol{\alpha}$ at time 0 and receives input defined by $\boldsymbol{u}$, such as initial premiums for asset shares, reserves, cash values, and so on, and which obeys equation (2) is given by

$$\boldsymbol{x}(t) = \phi(t, 0)\boldsymbol{\alpha} + \left[\sum_{k=1}^{t} \Phi(t, k)B(k, k-1)\right]\boldsymbol{u}. \tag{3}$$

This readily can be seen to satisfy equation (2):

$$\boldsymbol{x}(t) = \Phi(t, t-1)\left[\Phi(t-1, 0)\boldsymbol{\alpha} + \sum_{k=1}^{t-1}\Phi(t-1, k)B(k, k-1)\boldsymbol{u}\right] + B(t, t-1)\boldsymbol{u}$$

$$= \Phi(t, t-1)\boldsymbol{x}(t-1) + B(t, t-1)\boldsymbol{u},$$

and to satisfy the initial condition

$$\boldsymbol{x}(0) = \Phi(0, 0)\boldsymbol{\alpha} = \boldsymbol{\alpha}.$$

The ballistic state control problem, for the purposes of this paper, is to determine $\boldsymbol{u}$ such that

$$\boldsymbol{x}(0) = \boldsymbol{\alpha},$$

$$\boldsymbol{x}(t) = \Phi(t, t-1)\boldsymbol{x}(t-1) + B(t, t-1)\boldsymbol{u} \quad \text{for } 0<t\leq T, \tag{4}$$

and

$$\boldsymbol{x}(T) = \boldsymbol{\omega},$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are given. It is often convenient to formulate the pricing problem in terms of another variable, $y(t)$, called output, which is related to $\boldsymbol{x}(t)$ by

$$y(t) = C(t)\boldsymbol{x}(t).$$

For the purposes of this paper, the ballistic output control problem is to determine $\boldsymbol{u}$ such that

$$\begin{aligned} \boldsymbol{x}(0) &= \boldsymbol{\alpha}, \\ \boldsymbol{x}(t) &= \Phi(t,\ t-1)\boldsymbol{x}(t-1) + \mathrm{B}(t,\ t-1)\boldsymbol{u}, \\ y(T) &= \boldsymbol{\omega}'. \end{aligned} \tag{5}$$

The relevance of these two ballistic control problems to actuarial pricing is evident from a reconsideration of example 1. The pricing problem for example 1 can be defined by the following:

$$\boldsymbol{x}(0) = \begin{bmatrix} -(\text{Acquisition cost according to pricing assumptions}) \\ -(\text{Acquisition cost according to reserve assumptions}) \\ -(\text{Acquisition cost according to nonforfeiture assumptions}) \\ \text{Initial death benefit} \end{bmatrix} = \boldsymbol{\alpha}$$

and

$$\boldsymbol{x}(T) = \begin{bmatrix} \text{Maturity asset share} \\ \text{Maturity reserve} \\ \text{Maturity cash value} \\ \text{Death benefit at maturity} \end{bmatrix} = \boldsymbol{\omega},$$

where the input $\boldsymbol{u}$ is to be determined as the solution to a ballistic state control problem.

Table 1 shows that sometimes u_4 is redundant in the pricing problem. In such cases it is convenient to delete the redundant element and to redefine $\boldsymbol{u}$, $\mathrm{M}(t,\ t-1)$, and $\mathrm{Q}(t,\ t-1)$ and to make use of the output, $y(t)$:

$$\boldsymbol{u} = \begin{bmatrix} \text{Gross premium} \\ \text{Reserve premium} \\ \text{Adjusted premium} \end{bmatrix},$$

$$\mathrm{M}(t,\ t-1) = \begin{bmatrix} (1+i^1)l^1\rho^1 & 0 & 0 \\ 0 & (1+i^2)l^2\rho^2 & 0 \\ 0 & 0 & (1+i^3)l^3\rho^3 \\ b^4 & 0 & 0 \end{bmatrix},$$

$$Q(t, t-1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$y(t) = C(t)x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \text{Asset share at time } t \\ \text{Reserve at time } t \\ \text{Cash value at time } t \\ \text{Death benefit at time } t \end{bmatrix}$$

$$= \begin{bmatrix} \text{Asset share at time } t \\ \text{Reserve at time } t \\ \text{Cash value at time } t \end{bmatrix}.$$

The maturity condition $x(T) = \omega$ then would be replaced by

$$y(T) = \omega' = \begin{bmatrix} \text{Maturity asset share} \\ \text{Maturity reserve} \\ \text{Maturity cash value} \end{bmatrix},$$

and u would be the solution to a ballistic output control problem.

Alternatively, it sometimes is desirable to formulate the pricing problem in the form of a surplus objective:

$$y(t) = C(t)x(t) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \text{Asset share at time } t \\ \text{Reserve at time } t \\ \text{Cash value at time } t \\ \text{Death benefit at time } t \end{bmatrix};$$

$$y(T) = \omega' = \begin{bmatrix} \text{Maturity surplus} \\ \text{Maturity reserve} \\ \text{Maturity cash value} \end{bmatrix}.$$

Again, u would be determined by solving a ballistic output control problem. A profit objective expressed as a present value of book profits also can be incorporated into a ballistic output control problem:

$$y(t) = C(t)x(t) = \begin{bmatrix} v^t\, {}_tp_x & -v^t\, {}_tp_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

$$= \begin{bmatrix} \text{Present value at issue of book profits in first } t \text{ years} \\ \text{Reserve at time } t \\ \text{Cash value at time } t \end{bmatrix};$$

$$y(T) = \boldsymbol{\omega}' = \begin{bmatrix} \text{Present value at issue of book profits} \\ \text{Maturity reserve} \\ \text{Maturity cash value} \end{bmatrix}.$$

Of the two, the ballistic output control problem is more general and reduces to the ballistic state control problem when $\mathrm{C}(t)$ is the identity matrix and $\boldsymbol{\omega}' = \boldsymbol{\omega}$.

The relevance of the ballistic control problems to pricing established, it now remains to solve equation (5), which includes equation (4) as a special case. The maturity condition, $y(T) = \boldsymbol{\omega}'$, can be combined with equation (3) and solved for $\boldsymbol{u}$:

$$\boldsymbol{u} = \left[\sum_{k=1}^{T} \mathrm{C}(T)\Phi(T, k)\mathrm{B}(k, k-1) \right]^{-1} [\boldsymbol{\omega}' - \mathrm{C}(T)\phi(T, 0)\boldsymbol{\alpha}]. \qquad (6)$$

This solution can be substituted repeatedly in equation (2) to generate the state and output vectors:

$$\begin{aligned}
&\boldsymbol{x}(0) = \boldsymbol{\alpha}, && y(0) = \mathrm{C}(0)\boldsymbol{x}(0);\\
&\boldsymbol{x}(1) = \Phi(1, 0)\boldsymbol{x}(0) + \mathrm{B}(1, 0)\boldsymbol{u}, && y(1) = \mathrm{C}(1)\boldsymbol{x}(1);\\
&\boldsymbol{x}(2) = \Phi(2, 1)\boldsymbol{x}(1) + \mathrm{B}(2, 1)\boldsymbol{u}, && y(2) = \mathrm{C}(2)\boldsymbol{x}(2);\\
&\cdots\cdots\cdots && \cdots\cdots\\
&\boldsymbol{x}(T) = \Phi(T, T-1)\boldsymbol{x}(T-1) + \mathrm{B}(T, T-1)\boldsymbol{u}, && y(T) = \mathrm{C}(T)\boldsymbol{x}(T).
\end{aligned}$$

This repeated substitution is self-checking, because it should produce $y(T) = \boldsymbol{\omega}'$. The degree to which $y(T)$ differs from $\boldsymbol{\omega}'$ is a measure of rounding error.

Numerical Solution of Example 2

Included in Tables 2, 3, and 4 with the other actuarial assumptions were acquisition costs of \$120 on the active life asset share basis and \$100 on the active life cash-value basis. These are issue costs that are not proportional to premium but that have been expressed on the basis of \$1,000 of insurance. Any issue costs proportional to premium would be included in the premium expense ratio $(1 - l^1)$. Since the policy is sold in the active life state, there is no such acquisition cost assigned to the disabled state. These acquisition-cost assumptions can be treated as negative funds and can be represented along with the initial death benefit in the initial condition:

$$x(0) = \alpha$$

$$= \begin{bmatrix} -(\text{Acquisition cost according to pricing assumptions}) \\ -(\text{Acquisition cost according to cash-value assumptions}) \\ \text{Initial death benefit} \\ -(\text{Acquisition cost according to disabled life assumptions}) \end{bmatrix}$$

$$= \begin{bmatrix} -120.00 \\ -100.00 \\ 1{,}000.00 \\ 0.00 \end{bmatrix}.$$

Also included in Tables 2, 3, and 4 are maturity value conditions: $800 of asset share, $600 of cash value, $1,000 of death benefit at maturity, and $800 of disabled life asset share at maturity. These can be expressed in vector form:

$$x(3) = \omega = \begin{bmatrix} \text{Active life asset share at maturity} \\ \text{Active life cash value at maturity} \\ \text{Death benefit at maturity} \\ \text{Disabled life asset share at maturity} \end{bmatrix} = \begin{bmatrix} 800.00 \\ 600.00 \\ 1{,}000.00 \\ 800.00 \end{bmatrix}.$$

The maturity asset share and cash value specify a profit objective, while the $800 disabled life asset share at maturity represents a maturity benefit for the disabled life (to purchase an immediate annuity, perhaps) and may also include a profit margin.

The input obtained by means of equation (6) with $C(t)$ equal to the identity matrix would be

$$u = \begin{bmatrix} \text{Gross premium} \\ \text{Cash-value premium} \\ \text{Annual death benefit increment} \\ \text{Present value at time 0 of disabled life benefits} \end{bmatrix}.$$

However, since the death benefit is level, it is known in advance that $u_3 = 0$, so it is more convenient to drop the redundancy in the target maturity position by using the output vector and equation (6) as follows:

$$y(3) = \omega' = \begin{bmatrix} \text{Maturity active life asset share} \\ \text{Maturity active life cash value} \\ \text{Maturity disabled life asset share} \end{bmatrix} = \begin{bmatrix} 800 \\ 600 \\ 800 \end{bmatrix},$$

$$u = \begin{bmatrix} \text{Gross premium} \\ \text{Cash-value premium} \\ \text{Present value at time 0 of disabled life benefits} \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By way of illustration, the computational matrices for the latter approach are shown in Table 5 to four decimal places, but in practice as many decimal places as possible should be used in the computations to minimize rounding problems. The input vector u turns out to be

$$u = \begin{bmatrix} 426.63 \\ 248.46 \\ 879.79 \end{bmatrix} = \begin{bmatrix} \text{Gross premium} \\ \text{Cash-value premium} \\ \text{Present value at time 0 of disabled life benefits} \end{bmatrix},$$

and the state vectors obtained by repeated substitution are

$$x(0) = \alpha = \begin{bmatrix} -120.00 \\ -100.00 \\ 1{,}000.00 \\ 0.00 \end{bmatrix}; \quad x(1) = \begin{bmatrix} 107.94 \\ 109.12 \\ 1{,}000.00 \\ 765.87 \end{bmatrix};$$

$$x(2) = \begin{bmatrix} 366.27 \\ 345.71 \\ 1{,}000.00 \\ 809.02 \end{bmatrix}; \quad x(3) = \begin{bmatrix} 800.00 \\ 600.00 \\ 1{,}000.00 \\ 800.00 \end{bmatrix} = \omega.$$

Thus, the cash value at time 2 is 345.71, the asset share at time 1 is 107.94, and so forth.

While the actuarial assumptions may not be realistic, this example does illustrate a workable approach to pricing an actuarial coverage that involves disability and recovery. Only three years were used to allow complete presentation of the numerical details for the reader to check. Al-

TABLE 5

COMPUTATIONAL MATRICES FOR EXAMPLE 2

	$t=1$	$t=2$	$t=3$
A(t, t − 1).........	$\begin{bmatrix} 1.1000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0350 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0800 \end{bmatrix}$	$\begin{bmatrix} 1.0900 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0400 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0900 \end{bmatrix}$	$\begin{bmatrix} 1.0800 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0300 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0700 \end{bmatrix}$
M(t, t − 1)	$\begin{bmatrix} 0.6600 & 0.0000 & 0.0000 \\ 0.0000 & 1.0350 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0800 \end{bmatrix}$	$\begin{bmatrix} 0.7630 & 0.0000 & 0.0000 \\ 0.0000 & 1.0400 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.9720 & 0.0000 & 0.0000 \\ 0.0000 & 1.0300 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$
N(t, t − 1)	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$
P(t, t − 1).........	$\begin{bmatrix} 0.8000 & 0.1499 & 0.0212 & 0.0335 \\ 0.0000 & 0.9500 & 0.0500 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.8500 & 0.0306 & 0.1024 & 0.0233 \\ 0.0000 & 0.9600 & 0.0400 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.3000 & 0.0000 & 0.1000 & 0.6000 \end{bmatrix}$	$\begin{bmatrix} 0.9400 & 0.0000 & 0.0420 & 0.0203 \\ 0.0000 & 0.9700 & 0.0300 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.3000 & 0.0000 & 0.1000 & 0.6000 \end{bmatrix}$

TABLE 5—*Continued*

	$t=1$	$t=2$	$t=3$
$Q(t, t-1)$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.4320 & 0.0000 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.3270 & 0.0000 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.1070 & 0.0000 & 0.0000 \end{bmatrix}$
$\Phi(t, t-1)$	$\begin{bmatrix} 1.3750 & -0.2042 & -0.0166 & -0.0452 \\ 0.0000 & 1.0895 & -0.0526 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0800 \end{bmatrix}$	$\begin{bmatrix} 1.3002 & -0.0395 & -0.1160 & -0.0506 \\ 0.0000 & 1.0833 & -0.0417 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -0.6501 & 0.0198 & -0.1087 & 1.8420 \end{bmatrix}$	$\begin{bmatrix} 1.1615 & 0.0000 & -0.0415 & -0.0389 \\ 0.0000 & 1.0619 & -0.0309 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -0.5807 & 0.0000 & -0.1459 & 1.8028 \end{bmatrix}$
$B(t, t-1)$	$\begin{bmatrix} 0.8431 & -0.2042 & -0.0452 \\ 0.0000 & 1.0895 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.4320 & 0.0000 & 1.0800 \end{bmatrix}$	$\begin{bmatrix} 0.9253 & -0.0395 & 0.0000 \\ 0.0000 & 1.0833 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -1.0077 & 0.0198 & 0.0000 \end{bmatrix}$	$\begin{bmatrix} 1.0492 & 0.0000 & 0.0000 \\ 0.0000 & 1.0619 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.7029 & 0.0000 & 0.0000 \end{bmatrix}$

$$\sum_{k=1}^{3} C(3)\Phi(3, k)B(k, k-1) = \begin{bmatrix} 3.5141 & -0.4111 & -0.2103 \\ 0.0000 & 3.4655 & 0.0000 \\ -6.1289 & 0.5160 & 3.7052 \end{bmatrix}$$

though individual product terminology was used, this example could have represented the coverage on a pension plan member where

$$x(t) = \begin{bmatrix} \text{Active life accrued liability at time } t \\ \text{Termination benefit at time } t \\ \text{Death benefit at time } t \\ \text{Disabled life accrued liability at time } t \end{bmatrix}$$

and where the gross premium would represent normal cost or current service cost.

IV. CONCLUSIONS

The control theory discussed in this paper differs somewhat from the control theory in engineering. First of all, engineering systems are usually assumed to obey the laws of classical mechanics and/or electromagnetism, which are often easily formulated as differential equations:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t); \\ y(t) &= C(t)x(t). \end{aligned} \tag{2a}$$

System (2a) is much more difficult to solve and leads to theoretical difficulties which equation (2) sidesteps. Also, actuaries are more accustomed to working with probabilities and regular intervals, such as one year, than with forces of interest and decrement in conjunction with continuous assumptions; however, an actuarial coverage based on a continuous Markov process [14] would lead to a formulation of the form of system (2a). Second, system (2a) also allows the input, $u(t)$, to be a function of time, thus leaving the incidence of input unspecified. This can result in a multiplicity of solutions from which an optimal one can be selected [2]. Yet the similarities between a large class of engineering problems and a large class of actuarial pricing problems are very striking and suggest the possibility of fresh insights as well as new and powerful tools for actuaries.

As outlined in the introduction, the ballistic approach to pricing based on a Markov model of insurance provides a general theory of actuarial pricing, a straightforward solution to certain actuarial coverages that are not readily tractable when the conventional formula and accumulation methods are used, and a readily computerized algorithm based on simple linear algebra.

This paper has not dealt with the rounding problem associated with matrix inversions other than to point out that the recursive formulation provides a good check on its own rounding error. Heuristically, it seems that as long as survival probabilities are not very small relative to probabilities of decrement, then the diagonal elements of $P(t, t - 1)$ should dominate, and the necessary inversions should not cause a rounding problem. This is a conjecture, however, which has not been tested beyond a small number of examples.

VI. ACKNOWLEDGMENTS

Mr. R. W. Ziock, F.S.A., provided encouragement to publish the ideas contained in this paper after an informal presentation made by the author at an actuarial workshop. Several colleagues, especially Ms. Wafaa Babcock, F.S.A., reviewed various drafts and made a number of useful comments. The members of the Papers Committee who reviewed an earlier version of this paper offered a number of comments and criticisms that led to improvements in the paper. The author wishes to express sincere thanks and appreciation to these people.

REFERENCES

1. AMSLER, M. H. "Vers un modèle mathématique commun à l'assurance-vie et à l'assurance non-vie," in *Transactions of the Twenty-first International Congress of Actuaries*. Vol. I: *Generalized Models of the Insurance Business (Life and/or Non-Life Insurance)*, 1980, p. 1.
2. BROCKETT, R. W. *Finite Dimensional Linear Systems*. New York, London, Sydney, Toronto: John Wiley & Sons, 1970.
3. CHAMBERS, M. W., and VAN BOLHUIS, C. "Double Decrement Reserves," *Proceedings of the Canadian Institute of Actuaries*, VII, No. 3 (1976), 346.
4. COLLETT, R. L. "A Consistent Description of Actuarial Financial Projections Using Matrix Notation and Terminology," *ARCH*, 1972.4.
5. D'ANGELO, H. *Linear Time Varying Systems: Analysis and Synthesis*. Boston: Allyn & Bacon, 1970.
6. FELLER, W. *An Introduction to Probability Theory and Its Applications*. Vol. I. 3d ed. New York, London, Sydney: John Wiley & Sons, 1970.
7. *Gross Premiums for Individual Life Insurance and Annuities*. Part 7 Study Note. Chicago: Society of Actuaries, 1971.
8. *Group Pension Dividends*. Part 7 Study Note. Chicago: Society of Actuaries, 1971.
9. HOEM, J. M. "A Markov Chain Model of Working Life Tables," *Scandinavian Actuarial Journal*, 1977, p. 1.
10. HOEM, J. M., and AALEN, O. O. "Actuarial Values of Payment Streams," *Scandinavian Actuarial Journal*, 1978, p. 38.
11. HUNTINGTON, H. S. *Distribution of Surplus*. Part 7 Study Note. Chicago: Society of Actuaries, 1979.

12. JEWELL, W.S. "Models in Insurance: Paradigms, Puzzles, Communications, and Revolutions." Report of Introduction, *Volume S, Transactions Twenty-first International Congress of Actuaries*, Zurich and Lausanne, 19-26, June 1980.
13. JORDAN, C. W. *Life Contingencies*. Chicago: Society of Actuaries, 1976.
14. KLEINROCK, L. *Queueing Systems*. Vol. I: *Theory*. New York, London, Sydney, Toronto: John Wiley & Sons, 1975.
15. WARNOCK, R. L. *GAAP Reserves*. Part 7 Study Note. Chicago: Society of Actuaries, 1978.
16. WINKLEVOSS, H. E. *Pension Mathematics, with Numerical Illustrations*. Homewood, Ill.: Richard D. Irwin; Georgetown: Irwin-Dorsey, 1977.
17. ZIOCK, R. W., and LEZAK, E. H. "A Reformulation of the Theory of Mortality Classes in Terms of Markov Transition Matrices," *ARCH*, 1973.5.

DISCUSSION OF PRECEDING PAPER

MICHAEL STRAMAGLIA:

Mr. Smith should be complimented for providing a novel approach to solving an old problem. I was particularly pleased with his treatment of "A Markov Model of Insurance," as it involves an area of probability theory that I feel is worth considerably more attention than it receives from most actuaries.

As the author has shown, the theory of Markov chains is suited nicely to modeling the multidecrement environments that frequently arise in insurance applications. A good illustration of this is given in the paper's second example, where a disability decrement is recognized (with subsequent possible recovery) in addition to the usual lapse and mortality decrements. However, since rates of recovery and mortality vary by both attained age and duration since inception of disability, a select-type approach to the disabled life state should be pursued in practice.

The formulation of the state-space for the Markov chain requires the introduction of a unique state variable for each duration in the select period and one for the ultimate disabled life state. Perhaps this extension of the disabled life state can be clarified by presenting the network diagram of one-step transitions. The nodes represent the various state variables and the directed edges indicate the possible one-step migration routes between adjacent states. The transition probability assigned to each of these edges must be known before the Markov Chain can be run. Also, of course, the sum of the transition probabilities assigned to all edges leaving any node must be one.

Figure 1 illustrates the network diagram for the original state space. Note that a continuous string of disability occurs via repeated loops through the same disabled life state.

Figure 2 displays how the structure of the state is altered when the additional effect of the duration of disability is recognized. For simplicity, a rather short select period (length 2) is shown. In this case, a lengthy string of disability traces a path through a number of different states before looping through the ultimate disabled state. With this mechanism in place, two disabled lives with the same attained age can experience different forces of decrement (in terms of the model) since they can reside in different states of disability. When the multidecrement one-step transition probabilities are calculated through some transformation of the single decrement rates, a

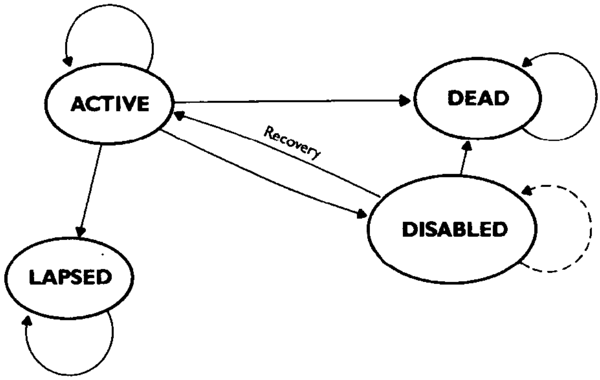

FIG. 1.—Network diagram of one-step transitions without select disabled lives states.

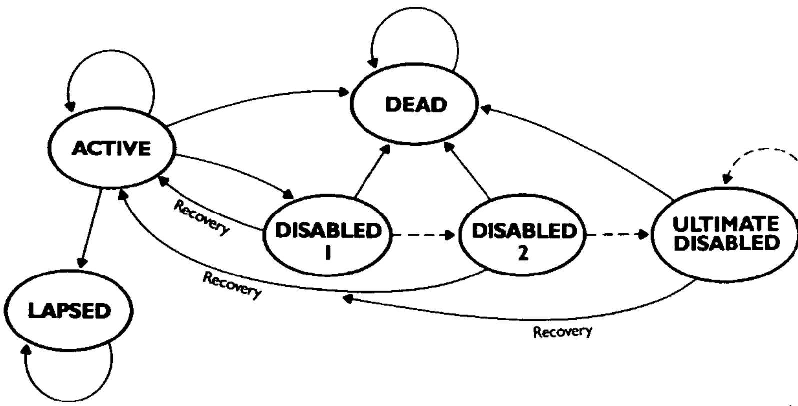

FIG. 2.—Network diagram of one-step transitions with two-year select disabled lives states.

single decrement rate of unity should be assumed for transitions out of a select disabled state into a subsequent state of disability.

In much the same manner, I believe that the effect of risk selection on the active life state easily could be incorporated into Mr. Smith's algorithm, and I would welcome his comments on the details of implementing such a modification.

(AUTHOR'S REVIEW OF DISCUSSION)

J.C. MCKENZIE SMITH:

Mr. Stramaglia's discussion makes two valuable contributions. For one thing, he shows how to incorporate select and ultimate assumptions into a Markov-chain representation of an actuarial coverage. In addition, he makes use of network diagrams to illustrate the essential features of a given Markov chain.

In applying the approach described by Mr. Stramaglia, the disabled state is split into select disabled states and an ultimate disabled state. Each select state is "homogeneous" to the extent that each of its occupants were disabled within the same policy year. This makes it possible to incorporate the effects of selection on recovery and mortality probabilities. When using the ballistic approach, each select state would be represented in the state vector, $x(t)$, by its own asset share or reserve. This asset share or reserve would obey a recursive equation similar to that for the ultimate state except that there would be no probability of remaining in the same state. This is because each occupant of a select state must either die, recover, or "move on" to the next select state (or to the ultimate state). Similarly, the active state could be split into "select recovered active" states and an "ultimate active" state. For each additional select state, there would have to be an additional maturity condition and an additional input component representing the "net single premium" at issue of future benefits (less future premiums in the case of a select recovered active state). This approach would offer more refinement than the method of loading the disability and recovery probabilities as suggested in the paper. However, the price for this extra refinement is the higher dimensionality of the state, input, and ouput vectors.

Mr. Stramaglia's use of network diagrams is very appealing. It would seem that a network diagram would be a useful tool in examining any actuarial coverage whether to price or to value it.

I wish to express my sincere thanks to Mr. Stramaglia for his insights, which have added to the ideas raised in the paper.