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INTEREST RATE SCENARIOS

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ABSTRACT

This paper describes several methods of creating interest rate scenarios that the actuary can use for pricing analysis or asset-liability testing. The characteristics of the methods as well as possible variations and refinements are discussed. In addition, some aids for using one method are offered.

INTRODUCTION

Recent years have witnessed the increased awareness by actuaries of the risks of changes in interest rates. The proliferation of interest-sensitive products, such as single premium deferred annuities, has accompanied this phenomenon.

The concepts of duration and convexity are excellent tools for measuring the risks of interest rate changes. Strategies like immunization have been developed to deal with such risks. Nevertheless, the abstract nature of such concepts often makes communication difficult, especially with nonactuaries. There are various practical difficulties to implementing or using strategies like immunization. These difficulties have led many actuaries to search for and use other means of analyzing interest rate risk.

Interest rate scenarios are often used to address these hurdles. Interest rate scenarios are readily understood by nonactuaries, and basing actuarial analysis on scenarios facilitates communication of such analysis. Interest rate scenarios can be used to answer "what if" questions about both sides of the asset-liability equation and to develop an easily comprehended distribution of profit (or loss) results under a wide range of possibilities.

Interest rate scenarios have even appeared in the legal arena. Recently the state of New York passed a regulation that incorporates the use of scenarios to aid in formulating an actuarial opinion for the financial statements of insurers who have sold certain kinds of interest-sensitive products. It would not be surprising to see more of this type of legislation in the future, or to see more detailed scenarios.

Although interest rate scenarios have been used by more than a few actuaries, there is little discussion of them in the actuarial literature. This paper addresses that dearth.

PRESET SCENARIOS

A rudimentary way of creating interest rate scenarios is to specify the rates, one by one, for as many scenarios as the creator wants to test. The span of time between different rates (for example, three months or a year), the projection period (for example, ten years), and the kind of rates (for example, 91-day Treasury bills, ten-year high-quality corporate bonds) are chosen to suit the purpose. This is an excellent way to create particular scenarios, such as the "worst case" scenario or a scenario that duplicates a particular historical period. But for many purposes this approach poses several problems:

- Are the chosen scenarios appropriate?
- How many scenarios are enough?
- Specifying all the desired rates can be time-consuming. For example, one might want for each scenario all maturities up to 20 years for Treasury debt, different grades of corporate debt, and mortgage-backed securities.
- Quite often the starting point chosen for each scenario is the current or recent interest rate. In such cases, the scenarios will likely have to be created again when repeat analysis or more analysis is needed at a later date.

Specifying individual rates for particular scenarios is a method that should not be ignored, but leaves much to be desired. Therefore, we consider methods that are more systematic and efficient.

The random generation of scenarios responds to all the problems listed above, particularly efficiency. The user will likely use enough scenarios to cover a wide and representative range of possibilities. Others will be less inclined to question the objectivity of the user's analysis if the scenarios are randomly generated ("random" is often taken to mean "unbiased," and sometimes "objective"). Because of these advantages, the rest of this paper is devoted to random interest rate generation.

COMMENTS ON TERMINOLOGY

The term "bias" is used without being well-defined. It is used in reference to properties of an interest rate generation method that may be considered undesirable, but different methods may have different kinds of biases.

Bias seems to be unavoidable with any probabilistic or stochastic method of interest rate generation. If the method is not biased in one sense, it is biased in some other sense. The term bias can, of course, have many meanings besides those specifically mentioned here. Also, with regard to future interest rates, one person's bias may be another person's idea of what is rational. The way out of this conundrum is to choose the bias that is the least bothersome and to judge the method on the credibility of the scenarios it produces.

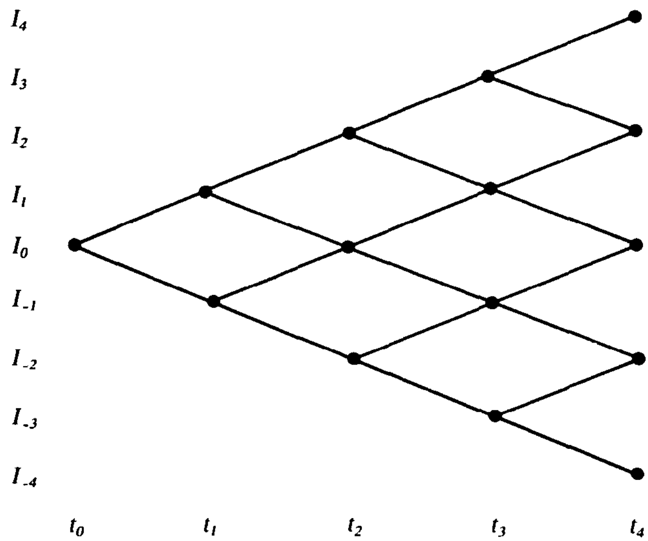
The words "expected" and "expectation" are used only with their strict statistical meaning: the sum of the products of possible outcomes times their respective probabilities. The words "anticipated" and "anticipation" are used when the ordinary sense of "expected" or "expectation" is meant.

BINOMIAL LATTICE [1]

The binomial lattice method of creating interest rate scenarios involves a decision tree and random decisions on which branch of the tree to follow in setting the interest rate at the next point in time. The tree or lattice can be depicted as shown in Figure 1.

FIGURE 1

BINOMIAL LATTICE METHOD FOR GENERATING INTEREST RATE SCENARIOS



Suppose $I_k = 10 + 0.5k$. In other words, the interest rates are evenly spaced 0.5 percent apart with $I_o = 10$ percent. Also, suppose that at any time t_k the probabilities that the interest rate is 0.5 percent higher or lower at t_{k+1} are both ½. Then a random technique (for example, a flip of a coin, or whether a random integer is even or odd) is used to determine whether the next interest rate is on the higher or lower step of the lattice.

This binomial lattice method might be criticized because the set of possibilities for the next interest rate has only two members. It even precludes the possibility of the rate not changing. Such criticism is valid, but not fatal. The lattice could obviously be made trinomial, or even multinomial. Of course, the random technique would need to be modified. The binomial lattice method also becomes problematic as the number of time periods per scenario is increased. This becomes clear later in the discussion.

One might also suggest setting the interest rates at $I_k = 10\,(1.05)^k$ on the grounds that changes in interest rates should be proportionate to their level. The suggestion is reasonable, but one should be fully aware of the effect. Without a change in the probabilities, the change would introduce an upward bias in the interest rate scenarios created. This is because when at I_k the expected interest rate at time $k+1$ is higher than I_k. For example, at 10 percent it is ½(10.5 percent) + ½(9.52 percent) = 10.01 percent. This is not much bias and is insignificant over a small number of time periods, but the effect is compounded with each additional time period.

Such bias could be offset by changing the probabilities so that at interest level I_k the expected rate for the next time period equals I_k. Changing the probabilities, of course, invites the criticism that the lattice is biased in the sense that the next interest rate at any time is anticipated to decrease more often than increase.

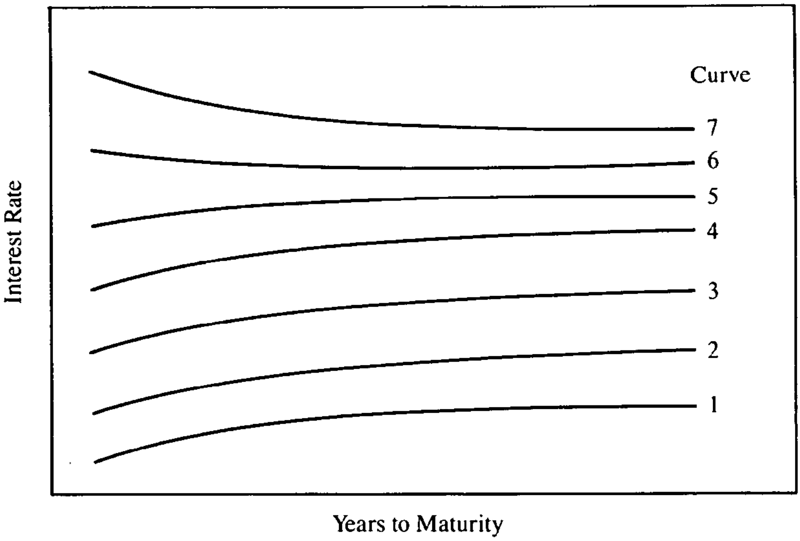
YIELD CURVE JUMPING [3]

A third method is a probabilistic approach to scenario generation. The technique involves specifying a set of interest rate curves (for example, A-rated corporate bonds) and the probabilities of jumping from one curve to any other in a fixed interval of time (for example, one year). Then, using a random technique and the curves and probabilities specified, one can generate as many scenarios for as long a projection period as desired.

A fairly simple example illustrates the method. Figure 2 shows a limited range of possible yield curves.

FIGURE 2

YIELD CURVE JUMPING METHOD OF SCENARIO GENERATION



The probabilities of moving from any one curve in Figure 2 to any other curve in Figure 2 are given below:

From Curve	To Curve						
	1	2	3	4	5	6	7
1	P_{11}	P_{12}	P_{13}	—	—	—	P_{17}
2	P_{21}	P_{22}	. . .	. . .	. . .	. . .	. . .
3	P_{31}	. . .	. . .	. . .	. . .	. . .	. . .
4	. . .	. . .	. . .	. . .	. . .	. . .	. . .
5	. . .	. . .	. . .	. . .	. . .	. . .	. . .
6	. . .	. . .	. . .	. . .	. . .	. . .	. . .
7	P_{71}	—	—	—	—	—	P_{77}

The sum of the probabilities in each row is 100 percent. Some individual probabilities could, of course, be zero. Interest rate scenarios are generated by using some random choice technique, for example, random numbers that are "mapped" to the probabilities in the table.

The random technique can be visualized as a roulette wheel, or as a spinner for a board game, with 100 possible outcomes. For example, when at curve 3, the 100 possible outcomes are P_{31} 1's, P_{32} 2's, P_{33} 3's, and so on. A spin of the wheel or spinner determines the curve number at the next point in time in the interest rate scenario. Subsequent spins determine the curve for each subsequent point in time in the scenario, and this technique is used for as many scenarios as desired.

When using this method, as well as any other, one should be aware of the possible biases. For example, if the probabilities in the fourth row of the table are not normally distributed around curve 4, there will be a bias, at least in some sense. The expectation could still be upward or downward and would depend on the interest rates on the curves. Of course, this expectation could vary depending on which curve one is on. When on either the top or bottom curve of Figure 2, it is impossible to eliminate bias in the sense of equal probabilities of rates going up or down.

FIRST STOCHASTIC MODEL

Another way to generate interest rates is to use a recursive algebraic formula in which the interest rate for the next period is determined from the current period interest rate, a random variable, and one or more parameters or constraints. The formula that comes readily to mind is the following:

$$I_{t+1} = I_t \,(1 + Z \times VF)$$

where I_t is the current interest rate; I_{t+1} is the interest rate one period later; Z is a random variable that is normally distributed with a mean of 0 and standard deviation of 1 (Appendix A describes means of developing these random variables, which can be easily programmed on a computer); and VF is a volatility factor appropriate for the time period from t to $t+1$ and for the particular interest rate (for example, the one-year Treasury rate or the ten-year Treasury rate).

This model has an advantage over the previous models in that the number of possible outcomes is limited only by the number of possible Z's. It also invites looking at actual historical interest rates to set the volatility factor.

Not much is said about this model because the next model described is quite similar, yet better than this one. The above formula, without a constraint added, would have the flaw that it could produce a zero or negative interest rate. All it would take is for $Z \times VF$ to be less than or equal to -1. Therefore, the formula should only be used with a lower bound on I_{t+1} or Z.

SECOND STOCHASTIC MODEL

Development

Let us alter the previous model by assuming instead that ln $[(I+\Delta I)\div I]$ is normally distributed. Now we can generate interest rates by using the following formula:

$$I_{t+1} = I_t e^{Z \times VF}$$

where the symbols have the same meaning as in the previous formula.

This author has found that changes in historical interest rates are approximately normally distributed, whether measured simply or logarithmically.

Suppose we have a history of one-year Treasury rates over many years at our disposal and find that the standard deviation over one-year periods of ln $[(I+\Delta I)\div I]$ equals 0.27. Further, suppose we set $VF = 0.27$ in the above formula and generate scenarios.

Unfortunately, this approach only *appears* to produce interest rates consistent with history. Actually, the distribution of the randomly generated interest rates is much more varied than the historical rates over multiple time periods, even though the volatility over one-year periods is fine. This is clear from the following formula:

$$I_{t+n} = I_t \, e^{VF(Z_1 + Z_2 + \ldots Z_n)}.$$

For example, suppose $I_t = 8$ percent, and $\Sigma_{k=1}^{n} Z_k = 5$. Then $I_{t+n} = 30.86$ percent.

How might the model be changed to avoid such unrealistic rates? Three possibilities are:

1. Lower the volatility factor.
2. Put bounds on how high or how low the rates can go.
3. Both of the above.

The first solution may be appropriate if short-term rate fluctuations are not important to the application; however, they often are. The second solution is more practical; it may even be suitable for the application. But it will cause the interest rates in a number of scenarios to "stick" at or near the upper or lower bound. Also, it invites the criticism of a bias against rates outside of the bounds. Interest rates may also "stick" with the yield curve jumping model, unless the probabilities of moving from the extreme curves are very skewed.

Comparison with Prior Models

Suppose we were to construct a series of lattices, L_3, L_5, L_7, . . . , the subscript indicating the number of branches at each node on the lattice. Further suppose that for each lattice the interest rates were geometrically spaced and the probabilities were "normally distributed" as much as possible. Then the limit of the series would be the lognormal model just described (with no bounds).

We saw earlier that a lattice with the interest rates geometrically spaced resulted in an expected interest rate at time $t+1$ higher than I_t, but that this bias could be removed by changing the probabilities. This becomes difficult as the number of branches at each node is increased, but would be practical with this model. It could be done by shifting the mean of the variable Z downward from zero to a small negative number.

Let us assume that the interest rates on the curves in the yield curve jumping model are geometrically spaced and that the probabilities at each curve are "normally distributed" as much as possible. That would be analogous to this second stochastic model with upper and lower bounds.

THIRD STOCHASTIC MODEL

The third stochastic model addresses some of the shortcomings of the last model and extends it to generate yield curves, rather than just one kind of interest rate. The model assumes a starting one-year Treasury rate, $T_1(0)$, and a starting twenty-year Treasury rate, $T_{20}(0)$. The future interest rates are generated in steps: first the one-year rate, then the twenty-year rate, and finally the intermediate rates.

One-Year Treasury Rate

To maintain the short-run volatility shown by history yet temper the long-run volatility, this model incorporates a mean-reverting formula, rather than imposing an upper and lower bound. The one-year Treasury rate one year later, $T_1(t+1)$, is calculated as:

$$T_1(t+1) = [T_1(t) + f(t)]e^{Z \times VF}$$

where

$f(t)$ = minimum $\{.015[T_1(\infty) - T_1(t)]^3; .5[T_1(\infty) - T_1(t)]\}$ if $T_1(t) < T_1(\infty)$

= maximum $\{.015[T_1(\infty) - T_1(t)]^3; .5[T_1(\infty) - T_1(t)]\}$ if $T_1(t) \geq T_1(\infty)$;

$T_1(\infty)$ is a parameter signifying the "long-run" T_1; and

Z and VF are defined as in the last model.

The function $f(t)$ defined above is not derived from a detailed analysis of historical data. It does, however, reflect a commonly held view about real-world interest rates. Its varying effect is consistent with the economic law of supply and demand. Clearly, it has very little effect if the difference between $T_1(t)$ and $T_1(\infty)$ is small and a greater effect as the difference increases. When current interest rates are above or below what interested parties perceive to be "normal," then market forces exert their effect to bring them back into line. And the further from "normal" rates are, the stronger the forces.

A mean reverting function might be criticized from the standpoint that at any given time no one really knows where future interest rates will be; yet the model says that the anticipated rate, $T_1(t) + f(t)$, is toward $T_1(\infty)$. The criticism has a certain amount of validity, but perhaps the only alternative is a less realistic model.

One more point is worth discussing. Where does one set $T_1(\infty)$? It would typically be set at $T_1(0)$, but that is not obligatory. One may set it above or below to effect an upward or downward bias on the interest rate scenarios, if it is thought that the starting interest rates are "low" or "high."

Let us now compare this model with the prior one. The prior model, with the upper and lower bounds, also has a sort of market force as just described. But its behavior is quite different. It has no effect until the interest rate is far from normal. It does not allow the interest rate to go past the bound. It does not pull the interest rate back toward normal.

Twenty-Year Treasury Rate

We noted that the one-year Treasury rate is a function of its anticipated rate and a random term. A similar approach is taken for the twenty-year Treasury rate. The formula is as follows:

$$T_{20}(t+1) = [a \times T_1(t+1) + b] + Z\sigma_{20}.$$

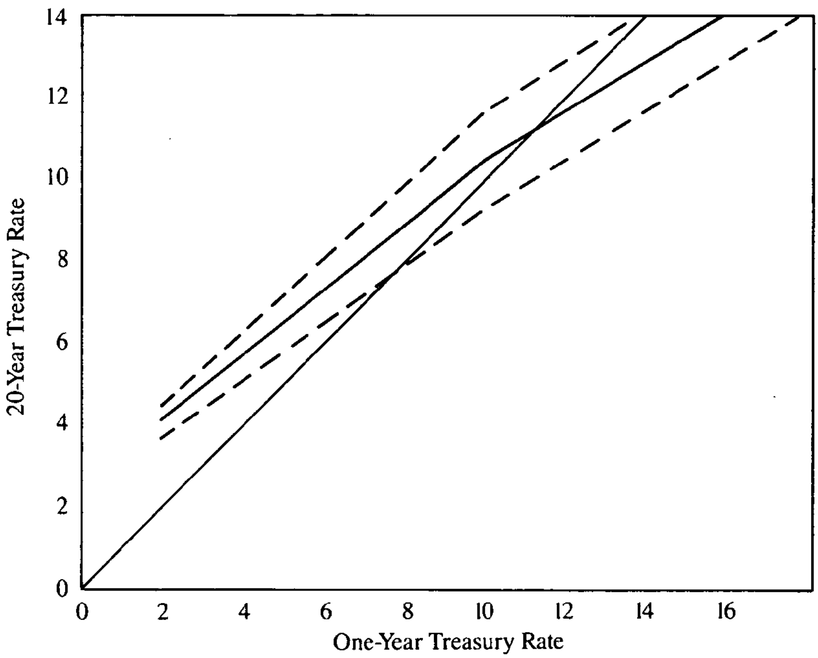
The variable Z is random, normally distributed with a mean of 0 and a standard deviation of 1. The term σ_{20} is the standard deviation of the spreads, presumably derived from historical data, between one-year and twenty-year Treasury rates. Suggested values for a, b, and σ_{20} are given in Appendix B.

The bracketed part of the formula is the anticipated rate and recognizes an important constraint on the "random" generation of simultaneous interest rates: There must be a reasonable relationship between any one rate and another.

The above formula, using the values suggested in Appendix B, is graphically depicted in Figure 3. The solid curve shows the anticipated twenty-year rate. The broken curves above and below the solid curve indicate the anticipated twenty-year rate $\pm$ σ_{20}. Thus, about 68 percent of the time $T_{20}(t+1)$ would be anticipated to fall between the broken curves. Points on the graph that are below the 45-degree line would give inverted yield curves.

FIGURE 3

TWENTY-YEAR TREASURY RATE AS A FUNCTION OF THE ONE-YEAR TREASURY RATE AND A RANDOM TERM (VALUES FROM APPENDIX B)



A historical graph of twenty-year rates versus one-year rates would not look as simple as the one in Figure 3. For example:

- The middle curve, representing average twenty-year rates, would be bumpier.
- The "band" would be a little wider in the middle and narrower at the high end, probably because of the high number of occurrences of interest rates in the midrange and low number of occurrences of interest rates above 12 percent.

History cannot be formulated so easily. But Figure 3, and hence the values in Appendix B, will give good results for almost any practical application.

Other Treasury Rates

Linear combinations of $T_1(t)$ and $T_{20}(t+1)$ for the intermediate maturities would probably give satisfactory results for many practical applications. The formula could be expressed as:

$$T_m(t+1) = [W_1(m) \times T_1(t+1)] + [W_{20}(m) \times T_{20}(t+1)] \qquad (2 \leq m \leq 19)$$

in which W_1 and W_{20} are weights for the one-year and twenty-year rates. Alternatively, one could set $T_m(t+1)$ for two to four key values of m, for example, $m = 2, 5, 7$, and 10, and interpolate to find the remaining $T_m(t+1)$.

This approach makes no attempt to simulate the multitude of shapes that the Treasury yield curve may have in the real world. The user is invited to search for a formula or formulas, or even add one or more random elements, to more closely represent the real world.

The Starting Curve

If one were to select a starting curve from the real world, only by pure coincidence would it match the curve that would result from starting with $T_1(0)$ and then calculating $T_m(0)$ $(m > 1)$ with $Z = 0$, using the above formulas. Rather, the real-world starting curve would be to some degree inconsistent with the later randomly generated curves. One could ignore the inconsistency, bend the real-world curve to fit the model, or adjust the formulas. The last option might involve minor adjustments initially that later diminish and grade into the unadjusted formulas over time. For example, one might use two different formulas for deriving the anticipated twenty-year rate from the one-year rate. The first formula might be for $t = 0$, the second for $t \geq 5$, and a weighted average of the two formulas for $t = 1$ to 4.

MULTIPLE CURVES

The last model described a method of generating yield curves for Treasury rates. What if the user also wanted other simultaneously occurring interest rates, such as corporate debt or mortgages?

Interest rates for other kinds of debt can be determined by simple formulas using the Treasury rates for the same maturity. Two possibilities readily come to mind. The first is an addition to the Treasury curve, that is, a so-called quality spread. The second is a multiple of the Treasury rate, which would give a "quality spread" that varies with the level of the Treasury rate. The latter is probably more realistic. Either method would probably yield satisfactory results for any practical application. Perhaps time studies of quality spreads have been made and published by someone, but the author is not aware of any.

Whichever formula is chosen, all the following pertain:

- Quality spreads vary significantly with the number of years to maturity.
- If the application does not model defaults, then the calculation of the interest rates should incorporate a credit risk deduction to indirectly allow for defaults.
- In the real world, quality spreads are dismayingly volatile. One might specify quality spreads that are higher (or lower) than normal at the starting point for each scenario and grade into more normal spreads later in the scenario. However, doing so would build in a bias.

The volatility of quality spreads in the real world might suggest two kinds of refinements:

- Quality spreads that vary with the economic cycle, narrowing during good times and widening during bad times, however the user might determine those times by interest rates.
- Adding a random element to the calculation of the interest rates for the other kinds of debt.

However, the effort that would be required to measure, program, test, and monitor the refinements probably would far exceed the benefits derived from the refinements.

CONCLUSION

This paper has described methods of creating interest rate scenarios, which are a valuable tool for analyzing interest rate risk and facilitate communication of analysis of such risk. The methods described are skeletal in form. The use of variations, combinations, refinements, and different parameters is encouraged, with one precaution: Analyze the practical results. Are they reasonable in relation to historical rates? What is the short-run volatility? The long-run volatility? How frequent are the highs and lows? How high and low are they?

The thrust of the first and second stochastic model was toward randomly generated interest rates that are closer to historical reality. This goal was pursued more vigorously with the third stochastic model. Without a comprehensive theory of the behavior of interest rates, such a pursuit lacks scientific precision. That does not mean that the pursuit should not be undertaken. The more realistic we actuaries can make our scenarios, the more credible will be the results.

The use of interest rate scenarios by actuaries is of fairly recent origin, as is the systematic development of such scenarios. The methods described in this paper eventually may be viewed as unsophisticated. But this author hopes that this paper will contribute to the evolution.

How might the model be further improved? Two ideas that seem worthy of consideration are the cyclical nature of interest rates and changes in interest rates between different eras, such as before and after October 1979.

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APPENDIX A

The stochastic models require random variables, Z, which are normally distributed with a mean of 0 ($\mu = 0$) and a standard deviation of 1 ($\sigma = 1$). Most statistics textbooks tell us that a normal distribution closely approximates a binomial distribution if certain conditions are met. This fact is used because binomial distributions are easily done on a computer, which will give random numbers.

Let $X_1, X_2, \ldots, X_N$ be a sequence of 1's and 0's each indicating a "success" or "failure" where the probability of a success is ½ ($P = ½$). Let X equal the sum of the sequence. If N, the number of trials, is sufficiently large, then $\mu = NP$, $\sigma = \sqrt{NP(1-P)}$, and $(X - \mu) \div \sigma$ is normally distributed.

The sequence of 1's and 0's can be produced on a computer by random selection on two possibilities, for example, determining whether a random integer from a range of integers (the range even-numbered or very large) is even or odd. Therefore, it takes little more than N random numbers to produce $Z = (X - 0.5N) \div 0.25N$, which is normally distributed with $\mu = 0$ and $\sigma = 1$. $N = 30$ is suggested as a minimum. It is reasonable to discard Z if $|Z| > 3$.

There are more esoteric and efficient methods for generating values of Z that are normally distributed with $\mu = 0$ and $\sigma = 1$. For example, there is the Box-Muller method [2]. Let U_1 and U_2 be random numbers, both from the unit interval (0,1). Then two random numbers Z_1 and Z_2 can be calculated as:

$$Z_1 = (-2 \ln U_1)^{1/2} \cos(2\pi U_2)$$

$$Z_2 = (-2 \ln U_1)^{1/2} \sin(2\pi U_2).$$

APPENDIX B

The volatility factors in all stochastic models and the σ_{20} in the third stochastic model are important. An effort was made in both models to allow parameters based on historical data. Of course, the numerical value implied by the data will vary with the period of history studied.

For the second stochastic model (the lognormal formula) a reasonable value for the volatility factor to model one-year Treasury rates at one-year intervals is about 0.27.

For the third stochastic model a reasonable value for the volatility factor to model one-year Treasury rates at one-year intervals is about 0.23. It is lower than the 0.27 because the mean-reverting function adds some short-run volatility.

Suggested parameters for calculating the twenty-year Treasury rate are as follow:

$a = 0.8$	$b = 2.5$	if $T_1 \le 10\%$	
$a = 0.6$	$b = 4.5$	if $T_1 > 10\%$	
$\sigma_{20} = 0.2 + 0.1T'_{20}$		if $T'_{20} \le 10\%$	T'_{20} is the
$\sigma_{20} = 1.2$		if $T'_{20} > 10\%$	anticipated rate.

Suggested weights for calculating the in-between Treasury rates are as follow:

m	$W_1(m)$	$W_{20}(m)$
2	0.64	0.36
5	0.39	0.61
7	0.24	0.76
10	0.16	0.84

DISCUSSION OF PRECEDING PAPER

STEPHEN J. STROMMEN:

Mr. Jetton has provided an excellent and thorough introduction to the topic of interest rate scenarios. His paper is a welcome addition to the actuarial literature.

This discussion describes two possible modifications to the third stochastic model described in the paper. These modifications have proven useful in practice, and one of them is supported by historical data. Throughout this discussion, the notation follows that used by Mr. Jetton in describing his third stochastic model.

A Different Mean-Reverting Formula

One of the assumptions in the third stochastic model is the long-run normal interest rate on one-year Treasuries $T_1(\infty)$. When interest rates are higher than $T_1(\infty)$, they tend to fall, and when they are lower than $T_1(\infty)$, they tend to rise. This is accomplished in the model by using a function, $f(t)$, which is negative when interest rates are higher than $T_1(\infty)$ and positive when interest rates are lower than $T_1(\infty)$. The assumption is that next year's interest rate, $T_1(t+1)$, is randomly distributed around $T_1(t)+f(t)$.

The difficulty with this procedure is in choosing an appropriate value for $T_1(\infty)$. As Mr. Jetton noted, if the chosen value is different from $T_1(0)$, an upward or downward bias is built into every randomly generated scenario. One could consider that to be undesirable, because it is very difficult to build a strong argument for any particular choice of $T_1(\infty)$.

An alternative approach involves an assumption that there is a range of normal values of $T_1(\infty)$ rather than a single value. For example, assume that $T_1(\infty) = T_1(t)$ as long as $T_1(t)$ lies between 4 percent and 10 percent. $T_1(\infty)$ is then never lower than 4 percent nor higher than 10 percent. Mr. Jetton's formulas for $f(t)$ can still be used, but this change in $T_1(\infty)$ results in $f(t)$ being zero whenever the one-year Treasury rate lies between 4 percent and 10 percent.

This approach eliminates the need to choose a single normal level for interest rates and replaces it with the need to choose a normal range. Aside from the effect on the resulting scenarios, the use of a range of normal values can make it easier to get agreement from a large number of people that a given assumption is realistic. It also makes it less likely that the assumption will need to change from time to time.

A Different Approach to the Slope of the Yield Curve

Throughout the paper, Mr. Jetton tacitly assumes that the slope of the yield curve is directly connected to the level of interest rates. The higher the interest rates, the smaller the slope of the yield curve.

This shows up in Figure 2 for the "yield curve jumping" method. In Figure 2, all the yield curves for the high interest rates are inverted, while all of those for low interest rates are positively sloping.

It shows up again in Figure 3 for the third stochastic model. In Figure 3 the difference between the one-year Treasury yield and the 20-year yield is randomly distributed around a figure that gets smaller as interest rates rise. By using this model, it is very unlikely that one could randomly generate a positively sloping yield curve when interest rates are at 14 percent, nor could one generate an inverted yield curve when interest rates are at 6 percent.

An alternative approach to the slope of the yield curve is based on a fundamentally different assumption:

The slope of the yield curve depends more on whether interest rates previously went up or down than on the current absolute level of interest rates.

This assumption can be implemented by using the procedure described below. In addition, it will be shown to have some support from historical data.

To aid in the discussion, let us define the "slope" of the yield curve as:

$$S(t) = [T_{20}(t) - T_1(t)] \,/\, T_1(t)$$

so that

$$T_{20}(t) = T_1(t)\cdot[1 + S(t)].$$

Under this definition, the slope is zero when the yield curve is flat. Let us also assume that there is some "normal" value of $S(t)$; call it $S(\infty)$.

To implement our assumption, let us make $S(t)$ follow a recursive rule. In particular,

$$S(t + 1) = (1 - a) \cdot S(\infty) + a\cdot S(t) + b\cdot[T_1(t + 1) - T_1(t)]/T_1(t) \quad [1]$$

In words, this means that next year's yield curve slope is equal to a weighted average of this year's slope and the normal slope, plus a shock factor based on the change in interest rates. Assuming that b is less than zero, any sudden increase in interest rates tends to decrease the slope (perhaps creating an inverted yield curve), but if interest rates remain level thereafter, the slope returns to normal.

It can be useful to put limits on the projected value of $S(t)$; one might assume that it must fall between -0.25 and $+0.50$. This prevents the slope from getting unreasonable in a scenario in which interest rates move in the same direction in sizable amounts for a few years in a row.

Given these definitions, historical data can be used to arrive at reasonable values for the constants $S(\infty)$, b, and a. If we make the transformation $c = S(\infty)\cdot(1 - a)$, we can express [1] in a form suitable for a linear regression on two independent variables, as shown in [2] below.

$$[2] \qquad Y = c + aX_1 + bX_2$$

where $Y = S(t + 1)$
$X_1 = S(t)$
$X_2 = [T_1(t + 1) - T_1(t)]/T_1(t)$

Use of historical data on government securities for the period 1960–84 leads to the following regression results:

Parameter	Estimate
c	0.064
a	0.718
b	−0.587
$S(\infty) = c/(1-a)$	0.227
R − squared	0.934

Using these values, the fit between actual and projected slopes is quite good, as is shown in Chart 1 (the data are given in Table 1). In Chart 1, the estimated slope $S(t+1)$ is calculated from the previous estimated value of $S(t)$ rather than the previous actual value. The fact that the estimated slopes follow the actual ones so closely indicates that the recursive rule works quite well when using parameters estimated from the regression.

When using this alternative approach to the slope of the yield curve, the procedure generating an interest rate scenario for Treasury rates is as follows:

1. Calculate the one-year Treasury rates $T_1(t)$ for all future years.
2. Calculate the yield curve slope $S(t)$ recursively for each future year, starting with the current actual slope.
3. Calculate the 20-year Treasury rates from the one-year rates and the slopes: $T_{20}(t) = T_1(t)\,[1 + S(t)]$.
4. Calculate other points on each year's yield curve in the manner suggested by Mr. Jetton.

Chart 1

Yield Curve Slope 1960–84

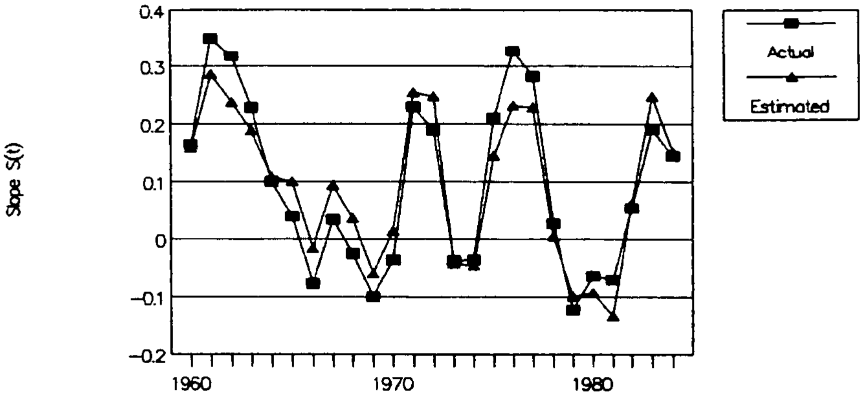


TABLE 1

Year	Slope	
	Actual	Estimated
1960	0.1634	0.1571
1961	0.3495	0.2859
1962	0.3180	0.2367
1963	0.2280	0.1877
1964	0.1000	0.1078
1965	0.0393	0.0997
1966	−0.0781	−0.0159
1967	0.0335	0.0927
1968	−0.0253	0.0358
1969	−0.1007	−0.0597
1970	−0.0369	0.0127
1971	0.2290	0.2529
1972	0.1885	0.2467
1973	−0.0373	−0.0427
1974	−0.0365	−0.0469
1975	0.2090	0.1430
1976	0.3277	0.2310
1977	0.2828	0.2278
1978	0.0268	0.0042
1979	−0.1233	−0.1004
1980	−0.0646	−0.0933
1981	−0.0706	−0.1324
1982	0.0540	0.0618
1983	0.1901	0.2470
1984	0.1438	0.1515

Random numbers are only used in Step 1. The rest of the procedure is completely deterministic.

Closing Comment

This discussion has provided two possible modifications to the third stochastic model presented by Mr. Jetton. Given the current state of actuarial knowledge and practice in this area, it is not possible to say whether they represent a definite improvement. The key, as Mr. Jetton pointed out, is to "Analyze the results!"

JEFFREY M. GURSKI:

Mr. Jetton should be commended for his valuable ground-breaking efforts. The projection of plausible yield curves is an essential and timely topic for many actuaries who need to model their business over various future scenarios. This discussion presents an alternative to Mr. Jetton's third stochastic model, as well as some comments on the author's version. The alternative version involves first the projection of the ten-year rate in a manner similar to the author's projection of the one-year rate. Then conditional distributions are used to generate movements of the one- and thirty-year rates in a manner that correlates them to the ten-year move.

I. ANALYSIS OF HISTORICAL DATA—RESULTING ASSUMPTIONS

Historical Data

Salomon Brothers' Analytical Record of Yields and Yield Spreads provides an historical data base of month-to-month Treasury yields ($i^{(2)}$ basis) for various maturities back to 1954. This data base was used to:

(1) Test whether the natural logarithms of the changes in yields are indeed normally distributed
(2) Determine what are reasonable levels and slopes of the yield curve
(3) Study the volatility patterns of the one-, ten- and thirty-year yields
(4) Study the correlation between the movements of the one- and thirty-year yields to those of the ten-year.

It could be argued, especially in today's rapidly changing financial world, that history should not be used to predict future results. However, we are not predicting any one future. Rather, we are trying to determine a set of plausible futures. Actual historical results will likely provide more reliable guidelines for this purpose than our intuition.

Definition of Random Variables

Because the alternative version involves the projection of the one-, ten- and thirty-year Treasury yields, we focus only on these. The rest of the curve is filled in by using an interpolation that is detailed in Appendix B. Three rates are projected rather than two in order to better model the various shapes that the yield curve can take on. The thirty-year rate is important when modeling annuity or other long-duration business. When analyzing business having little or no long cash flows, projection of just the one- and thirty-year rates would be sufficient.

First, we define the random variables X_1, X_{10}, and X_{30} for the natural logarithms of the ratios of yields from one month to the next for each of the respective maturities. That is,

$$X_i = \ln [T_i(t + 1)/T_i(t)] \text{ for } i = 1, 10, \text{ and } 30,$$

where $T_i(t)$ represents the yield on the i-year Treasury bond at month t.

Tests of Normality

It is indeed reasonable to assume that each of the random variables X_1, X_{10}, and X_{30} are normally distributed. Chi-square tests for normality (at the 5 percent level of significance) over recent subsets of the historical data show that this hypothesis cannot be rejected. For example, normality is confirmed for each of the three Treasuries when taking samples over the last two and one-half years, over the last ten years, and over the last twenty years. However, note that samples over the last thirty years or over the entire data base would lead us to reject the normality assumption for each of three Treasuries. This could be due to the lack of 'normal' type changes in rates seen in samples of the early data.

Volatility of Rates

The measure of interest rate volatility that we are interested in is the standard deviation of each of the random variables X_1, X_{10}, and X_{30}. Let these standard deviations be denoted by S_1, S_{10}, and S_{30}, respectively.

Based on the premise that the true variance of the underlying distributions can change over time, one might analyze sample standard deviations over various time periods to see how this measure is "trending." In deciding on an appropriate level of volatility for our purposes, the projection of *future* yield curves, it is reasonable to weigh the volatility of recent periods more heavily. With this in mind, reasonable estimates for S_1, S_{10}, and S_{30} (annualized basis) turned out to be 22.0, 18.5 and 15.0 percent, respectively.

These choices are somewhat subjective and could justifiably be influenced somewhat by one's view of the direction of future volatility levels. Although the magnitude of these numbers is not important to this discussion, the process of estimating volatility is a worthy topic of discussion in itself.

II. THE METHOD

Background: Conditional Distributions

We will be considering the joint normal distributions $f(x_1, x_{10})$ and $f(x_{30}, x_{10})$, rather than the multivariate form $f(x_1, x_{10}, x_{30})$. For simplicity, I have chosen to ignore the relationship between x_1 and x_{30}, but invite those who delight in the more advanced statistics to expand the formulas below to the multivariate case.

It can be shown that the conditional distribution $f(x_1 \mid x_{10})$ is itself normal with mean

$$\mu_1 + R_{1,10}\,(S_1/S_{10})(x_{10} - \mu_{10})$$

and standard deviation

$$S_1\,(1 - R^2_{1,10})^{1/2}$$

where x_{10} is a fixed observation of X_{10} (not a random variable)
$R_{1,10}$ is the correlation coefficient for the random variables x_1 and x_{10}
μ_1 and μ_{10} are the means of the random variables x_1 and x_{10}.

Note that the means are assumed to be zero (although the entire historical data do reflect an upward trend in rates, this mean is negligible over the more recent data).

The mean and standard deviation of $f(x_{30} \mid x_{10})$ are analogous to those for $f(x_1 \mid x_{10})$ given above. Using $\mu_1 = \mu_{10} = \mu_{30} = 0$, we have, in summary, the following normal distributions of interest:

Distribution	Mean	Standard Deviation
$f(x_{10})$	0	S_{10}
$f(x_1 \mid x_{10})$	$R_{1,10} \cdot (S_1/S_{10}) \cdot x_{10}$	$S_1 \cdot (1 - R^2_{1,10})^{1/2}$
$f(x_{30} \mid x_{10})$	$R_{30,10} \cdot (S_{30}/S_{10}) \cdot x_{10}$	$S_{30} \cdot (1 - R^2_{30,10})^{1/2}$

Development of Projection Formulas

We want to generate random observations x_{10}, x_1, and x_{30} from the distributions $f(x_{10})$, $f(x_1 \mid x_{10})$, and $f(x_{30} \mid x_{10})$, respectively. This is done by making random generations z from the standard normal, one for each of the three observations needed. In order to keep these three generations from the standard normal distinct, they are labeled z_{10}, z_1, and z_{30}. Now we need to write the x_i's in terms of z_i's. Recall that a normal random variable X with mean μ_x and standard deviation S_x is defined in terms of the standard normal random variable Z by using the following relation: $Z = (X - \mu_x)/S_x$ or $X = \mu_x + S_x Z$. Thus,

To Sample from	Formula in Terms of Standard Normal
$f(x_{10})$	$x_{10} = S_{10} z_{10}$
$f(x_1 \mid x_{10})$	$x_1 = S_1 [z_1 (1 - R^2_{1,10})^{1/2} + z_{10} R_{1,10}]$
$f(x_{30} \mid x_{10})$	$x_{30} = S_{30} [z_{30} (1 - R^2_{30,10})^{1/2} + z_{10} R_{30,10}]$

Finally, we can write the formulas for the Treasury rates we are seeking:

$$T_{10}(t + 1) = T_{10}(t) \cdot \exp[S_{10} \cdot z_{10}(t)]$$

$$T_1(t + 1) = T_1(t) \cdot \exp\{S_1 \cdot [z_1(t) \cdot (1 - R^2_{1,10})^{1/2} + z_{10}(t) \cdot R_{1,10}]\}$$

$$T_{30}(t+1) = T_{30}(t) \cdot \exp\{S_{30} \cdot [z_{30}(t) \cdot (1 - R^2_{30,10})^{1/2} + z_{10}(t) \cdot R_{30,10}]\}$$

Changes in T_1 and T_{30} are influenced by two factors:

(1) The change in T_{10}, determined by $z_{10}(t)$
(2) An independent random movement, determined by $z_1(t)$ or $z_{30}(t)$.

The higher the correlation, the greater the weight given to factor (1), and the lesser the weight given to factor (2). Analysis of the historical data shows that $R_{1,10}$ has been about 0.85 recently, while a value of 0.95 for $R_{30,10}$ shows the strong relationship between the month-to-month changes in T_{10} and T_{30}.

III. ADJUSTMENTS TO PROJECTED RATES

Mean Reverting Process

I agree with Mr. Jetton that it is desirable to adjust the random projections to keep the rate levels reasonable in relation to historical precedents. In fact, the mean reverting adjustment Mr. Jetton suggests is quite similar to the adjustment I have used in my scheme. I will just note my variation:

$$T_i^{adj}(t + 1) = T_i(t + 1) + C \cdot [T_i(\infty) - T_i(t+1)]$$

where C is the "central tendency" factor and $T_i(\infty)$ is the long-term mean for the i year rate. $T_i(\infty)$ could also be used as an assumed level for T_i under "normal" economic conditions. A value of 0.01 for C produces reasonable results when projecting rates month to month.

This mean reverting adjustment has the desirable property of creating a cyclical pattern of interest rates. For example, as rates get high relative to their historical means, there is a fairly strong influence bringing the rates down (because the adjustment is larger as the rates get farther from their mean.)

Constraints on Spreads

The recognition of the historical correlations between rate movements in the methodology helps to keep the projected rates reasonable in relation to one another. However, if these relationships get out of line with historical standards, it is desirable to make adjustments. Analysis of the historical data shows that the spread between T_1 and T_{10} is rarely greater than ± 225 basis points (bp). Only 10 out of 411 observations had greater spreads. These observations were: -308, -259, -237, 226, 228, 230, 235, 240, 243, and 250 bp. Note that the amount above or below the 225 points is usually not too great.

Because using a fixed constraint would cause spreads to cluster, unusual spreads are dampened significantly but not eliminated entirely. This is accomplished by using the following process. If the spread between the projected rates T_1 and T_{10} becomes greater than 225 bp, the excess is reduced by 60 percent using an appropriate adjustment to T_1. There is nothing special about the 60 percent figure, except that it helped to constrain the spreads in a manner that produces results similar to the historical experience.

Likewise, the spread between T_{10} and T_{30} was kept in line with historical standards. Only 10 out of 411 historical observations of $(T_{30} - T_{10})$ were outside the range $-$ 70 to 90b: -79, -75, -74, 92, 92, 92, 94, 95, 95, and 121. So, using the same method as above, if the projected $(T_{30} - T_{10})$ is less than -70 bp by an amount x, then T_{30} is increased by 60 percent of x. If $(T_{30} - T_{10})$ is greater than 90 bp by x, then T_{30} is reduced by 60 percent of x.

Results Using the Method

This method was used to project monthly interest rate scenarios over a five-year horizon in order to model our spot-rated group pension business. Thus, 61 yield curves (including the starting curve) were needed for a single trial and 6100 for 100 trials. Using the one-, ten- and thirty-year Treasury yields as of 8/1/88 as a starting point, these rates were projected using the method outlined above. There are 6000 observations in these data which correspond to the random variables X_1, X_{10}, and X_{30}, the natural logarithms of the ratios of yields from one period to the next.

Even though the raw random generations were artificially adjusted using the mean reverting process and for spread considerations, sample statistics from the projected Treasuries were very consistent with the assumptions of the underlying distribution:

						Normally Distributed?		
	S_1	S_{10}	S_{30}	$R_{1,10}$	$R_{10,30}$	X_{10}	$X_1\|X_{10}$	$X_{30}\|X_{10}$
Assumed	22.0	18.5	15.0	0.85	0.95	yes	yes	yes
Projected	21.5	18.2	15.0	0.85	0.95	yes	yes	yes

The patterns of the spreads $(T_{10} - T_1)$, $(T_{30} - T_{10})$ and $(T_{30} - T_1)$ were also analyzed. The average amounts these spreads changed month to month in the projected curves were reasonable in relation to the historical data. A comparison is shown below:

AVERAGE ABSOLUTE MONTHLY CHANGES IN SPREAD (BASIS POINTS)

	$\|\Delta(T_{10} - T_1)\|$	$\|\Delta(T_{30} - T_{10})\|$	$\|\Delta(T_{30} - T_1)\|$
Historical (last 20 years)	30.6	10.7	34.8
Historical (all data)	17.9	8.4	26.0
Projected Curves	22.9	12.6	24.5

IV. COMMENTS ON THE AUTHOR'S APPROACH

The similarities between Mr. Jetton's third stochastic model and the approach outlined above are apparent. The author's method for generating values of T_1 is essentially identical to my generation of values of T_{10}. Our "mean reverting" formulas are similar and are meant for the same purpose.

Where I disagree with the author's method is in his calculation of his second rate, T_{20}, and how he relates it to T_1. With the equation

$$T_{20}(t + 1) = [a \times T_1(t + 1) + b] + Z\sigma_{20}$$

the author is assuming a fixed linear relationship between the rate levels T_{20} and T_1, and then perturbs the projected values away from the line by using the random component $Z\sigma_{20}$. This equation prescribes what the slope of the yield curve will be, on average, based on the level of the one-year rate. Yet the historical data show that there have been flat, inverted and positively sloping yield curves at many different rate levels. A simple linear regression is the statistical method under which we can study the relationship between T_{20} and T_1. Actually, to be consistent with Mr. Jetton's approach, we would break the data into two separate regressions: one set with $T_1 \leq 10$ percent and the other with $T_1 > 10$ percent. Either way, regressions of the historical data do seem to reveal a strong linear relationship between the variables (as measured by the coefficient of determination), ostensibly justifying the use of Mr. Jetton's formula for T_{20}. However, upon closer analysis, the following points become evident.

(1) The use of "the standard deviation of the spreads" σ_{20} is not appropriate for the author's formula. More consistent would be the use of the standard deviation of the error or "residual" random variable e_i from the regression equation $Y_i = a X_i + b + e_i$.

(2) Whether we analyze actual spreads $(T_{20} - T_1)$ or the residual of the regression, one finds that these random variables are not normally distributed. Furthermore, tests show there is a strong serial correlation of these random variables through time. For example, if the spread $(T_{20} - T_1)$ is high this month, it is likely to be high next month also. A similar phenomenon is seen when studying the residual random variables one month to the next. As would be expected, this effect begins to wear off as the observation period is lengthened. For example, the spread $(T_{20} - T_1)$ is less dependent on what it was one year ago than what it was one month ago. The table below shows the serial correlation of the $(T_{20} - T_1)$ spread for various time lags:

Months lag	1	2	3	6	12	24
Correlation	0.93	0.84	0.77	0.61	0.47	0.11

The serial correlations were also very high for the residual variable. These high serial correlations mean that neither the spreads nor the residuals form a stochastic process, because they do not have the element of randomness through time. Thus, it would be inappropriate to use the author's "random term" $Z\sigma_{20}$ in his formula for T_{20} because the residuals are neither normally distributed nor even random in nature.

REFERENCES

Alan Kulig did much of the original work for the method of projecting interest rates described above.

1. ABRAMOWITZ, M. AND STEGUN, I.A. *Handbook of Mathematical Functions.* New York, N.Y.: Dover Publications, Inc., 1972, p. 953.
2. HILLIER, F.S. AND LIEBERMAN, G.J. *Introduction to Operations Research.* San Francisco, Calif.: Holden-Day Inc., 1980, pp. 650–52.
3. HOEL, P.G. *Introduction to Mathematical Statistics.* New York, N.Y.: John Wiley & Sons, Inc., 1971, pp. 149–154, 318–319.
4. PFAFFENBERGER, R.C. AND PATTERSON, J.H., *Statistical Methods.* Homewood, Ill.: Richard D. Irwin, Inc., 1977, pp. 394–413.

APPENDIX A

Another method of generating random observations from the standard normal distribution is described below (using APL).

Store in the variable F values of the cumulative distribution function of the standard normal for the following 10,001 values of Z: -5.000, $-4.999, \ldots, 4.999, 5.000$. This can be done efficiently by using SAS, with the results transferred to an APL environment. Then, random observations of Z can easily be generated by using the APL step:

$$Z \leftarrow (-5) + 0.001 \times +/F > 0.000001 \times (?1000000)$$

See Hillier and Liberman [2] for an explanation of the theory behind this approach.

APPENDIX B

Because only the one-, ten-, and thirty-year Treasury rates are projected, we need to interpolate (and extrapolate for rates shorter than one year) to obtain Treasury rates at other maturities. The method described below was quite effective in reproducing historical yield curves from the historical one-, ten-, and thirty-year rates.

We seek to fit two logarithmic splines of the form

$$T_t = (1/K) \cdot \ln(a + bt + ct^2)$$

where t is the maturity of the Treasury or,

$$Y_t = a + bt + ct^2$$

where $Y_t = \exp(K\ T_t)$ and where $K = 70$ proved to be an effective constant, and we need to solve for a, b, and c. One spline is fit between T_1 and T_{10} and the other between T_{10} and T_{30}, with the curves going through the points T_1, T_{10} and T_{30}.

We solve the following two 3×3 systems:

$$a_1 + b_1 + c_1 = Y_1$$

$$a_1 + 10b_1 + 100c_1 = Y_{10}$$

$$b_1 + 20c_1 = M = (Y_{30} - y_{10})/20 = \text{desired slope of } Y_t \text{ @ } t = 10.$$

and

$$a_2 + 10b_2 + 100c_2 = Y_{10}$$

$$a_2 + 30b_2 + 900c_2 = Y_{30}$$

$$b_2 + 20c_2 = M.$$

a_1, b_1, and c_1 define the curve between T_1 and T_{10}, and a_2, b_2, c_2 define the curve between T_{10} and T_{30}.

For rates with a maturity less than one year, the first spline can be used to extrapolate. However, when this method was attempted to reproduce historical three-month rates from the historical T_1, T_{10} and T_{30}, it was found that the reproductions were on average about 25 bp too high. Thus, an adjustment can be made for this, to give the interpolating curve a downward bend more like the actual historical rates.

Occasionally, this method produces some rates that are complex, so the results should be checked for this. When it occurs, quadradic splines can be used instead. The same framework as above is used except that $Y_t = T_t$ (no exponential transformation is made).

JOSEPH J. BUFF AND RICHARD B. LASSOW:

Mr. Jetton has written a useful and interesting paper on an essential aspect of asset/liability management, namely, interest rate scenarios. We would like to offer some references and comments in order to complement the author's work.

References on Scenarios

Mr. Jetton comments that there is little discussion of interest rate scenarios in the actuarial literature. He cites three references in the bibliography of his paper, none published later than 1985. Through the forum of Society of Actuaries meetings and annual Valuation Actuary Symposia, there has been an ongoing exposition of interest rate scenarios.

The model Mr. Jetton calls yield curve jumping is an application of a general mathematical process called a stationary Markov chain. For a general discussion of the mathematics of such random walk processes, the reader can consult:

FELLER, W. *An Introduction to Probability Theory and Its Applications,* 3d ed. New York, N.Y.: John Wiley & Sons, 1968, Chapter XV, p. 372.

The approach that Mr. Jetton calls the second stochastic model is often referred to in the financial literature as the log-normal model. References for the log-normal model include:

ROLL, R. *Behavior of Interest Rates.* New York, N.Y.: Basic Books, 1970.

BRENNAN, M. and SCHWARTZ E. "A Continuous Approach to the Pricing of Bonds," *Journal of Banking and Finance* (1979):133–55.

Some specific numerical case studies using the yield curve jumping approach to yield curve modeling may be found in:

DEAKINS, P.B. AND TULIN, S.B. "C-3 Risk," Chapter III in *The Valuation Actuary Handbook*, by the Committee on Life Insurance Company Valuation Principles. Itasca, Ill.: Society of Actuaries, 1987.

BATTE, M.C. "Corporate Modeling and Forecasting (Practical Aspects of the Valuation Actuary Recommendation)," panel discussion by S.B. Tulin, *RSA* 12, No. 2 (1986):1239.

BUFF, J.J. "Investment Considerations in Product Development," panel discussion by P.B. Deakins, *RSA* 13, No. 2 (1987):939.

Some specific numerical case studies using the log-normal model may be found in:

BUFF, J.J. "Testing Interest-Sensitive Cash Flows," Section 4, Chapter II in *The Valuation Actuary Handbook*, by the Committee on Life Insurance Company Valuation Principles. Itasca, Ill.: Society of Actuaries, 1987.

BATTE, M.C. "Corporate Modeling and Forecasting (Practical Aspects of the Valuation Actuary Recommendation," panel discussion by D. Carr, *RSA* 12, No. 2 (1986):1224.

GRIFFIN, M.W. "Investment Strategy for Life Insurance Products," panel discussion by J.J. Buff, *RSA* 14, No. 2 (1988):821–61.

Comparison of Scenario Generators

Mr. Jetton's discussion of his third stochastic model does not include a numerical example. Some case studies based on the third stochastic model would be useful to answer these questions:

1. How different are the interest rate scenarios based on the third stochastic model compared to those based on the second stochastic model? More importantly, how different is the range of output financial results based on the two models?
2. How important to the output is the choice of the long-term interest rate trend assumption? How critical are the values of the parameters chosen for Mr. Jetton's $f(t)$?

We raise these questions for the following reasons:

1. As with any other financial model, there is probably some point of diminishing returns with regard to the real effective impact, on management decisions, of detail refinements to the model. It is not yet clear whether the differences, in how the second and the third stochastic models try to avoid "unreasonable" interest rates, would lead to material differences in the management information to be derived from the scenario projections. Perhaps the author or other researchers can address this question.
2. The parameterization of the third stochastic model is presumably important to its practical use. Sensitivity testing regarding the choice of critical assumptions is always advisable—even the New York insurance regulation referred to in Mr. Jetton's introduction, Regulation 126, calls for sensitivity testing. Some guidance as to the choice of parameters, and their impact on ultimate management decisions, would be of considerable interest. Again, perhaps Mr. Jetton or someone else can address this question in the future.

Having argued that the users of different stochastic scenario models ought to compare the results of the different models, we would like to draw attention to some published research on this question. The *Proceedings* of the 1987 and 1988 Valuation Actuary Symposia include comparisons of projections output based on two different scenario generators used by some practitioners, namely, what Mr. Jetton has called the yield curve jumping model and what the present writers prefer to call the log-normal model. This ongoing research indicates that it can sometimes be quite important to identify the effect on an insurer's management information system of:

1. Which scenario generator is used
2. How the scenario generator is parameterized

3. How many scenarios are run
4. How the results are summarized.

Critiques of Different Stochastic Models

Every stochastic generator, like every other actuarial or financial model, will have both strengths and weaknesses. The present writers are not aware that a thorough comparison of the strengths, pitfalls, and hidden assumptions behind the different scenario generators yet exists in one place in the literature. We believe that the compilation of a definitive comparison would be a major undertaking. Purely for argument's sake, we offer some basic questions that practitioners ought to ask regarding any scenario projection procedures they might rely upon:

1. What aspects of historical interest rate behavior does the model encompass, and what aspects does it ignore? In what ways does the model recognize that future interest rate behavior may differ from historical experience?
2. What assumptions about yield curve behavior are implicit in the model, such as a finite or infinite number of possible yield curves, one process or another for avoiding "unrealistic" interest rates, and dependence or independence of the direction of interest rate movements from one period to the next.
3. What parameters for the model can be derived from or based upon historical experience, and which ones are more heavily dependent on professional judgment?

Users of interest rate scenarios should bear in mind that there is some controversy among financial theorists with respect to many aspects of the modeling of interest rates. Opinions tend to be divided on such matters as the appropriateness of mean reversion adjustments, expectations hypothesis adjustments, and arbitrage pricing theory. The choice of method will depend on how the interest rate scenario projections are to be applied. It is possible that in some instances there is no single perfect approach nor one "right answer."

Default Risk

Mr. Jetton touches briefly on the subject of asset default risk. To leave out any provision for default risk in the preparation of interest rate scenario projections may be inappropriate in situations in which default risk exposure is not trivial. Again, New York Regulation 126 does include guidelines as to the treatment of default risk.

Readers looking for more reference material on default risk scenarios may wish to consult:

ALBERT, F. "Quantifying the C-1 Risk (Defaults in Fixed Dollar Investments and Market Value Changes in Equity Investments)," panel discussions by I. Vanderhoof and J. Buff, *RSA* 13, No. 3 (1987):1591–1622.

In addition, Dr. Vanderhoof has written a paper for the *Transactions* on the subject of default experience, which will appear in Volume XLI.

How Many Scenarios Are Enough?

Mr. Jetton raises as one problem in stochastic scenario modeling the question of how many scenarios are enough. This subject is discussed at length in the panel discussion "An Approach to the Stochastic Modeling of C-3 Risk" referenced above. In addition, research on this question is presented in the 1988 *Valuation Actuary Symposium Proceedings*. We would like to give one example of the application of such research.

Purely for illustrative purposes, let us consider how stochastic scenario modeling can be used to help determine whether a reserve and accompanying assets are "adequate" to support a block of liabilities. Let us suppose that "adequate" means that the business is solvent at the end of some projection period in 90 percent of the scenarios tested. (Of course other definitions of reserve adequacy could be used instead. Furthermore, many applications of stochastic scenario testing do not pertain to reserve adequacy.)

If the projection assumptions and the scenario model parameters are frozen, the percentage of scenarios tested that actually show a solvent outcome can be regarded as a random variable. The value of the random variable will depend on the particular set of scenarios produced by the scenario generator. Thus the "success rate" among the scenario set is subject to sampling error, unless "enough" scenarios are used.

The variable being sampled is the underlying success rate across the (potentially infinite) universe of scenarios whose distribution is specified by the stochastic model and its parameters. Running a set of scenarios and determining the sample success rate provide an estimate by Monte Carlo techniques of the underlying "population" success rate.

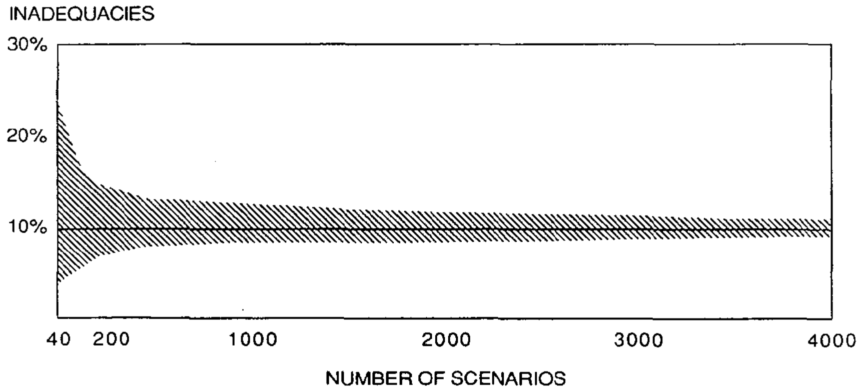
The statistical credibility of a sample "success rate" based on a sample of given size can be quantified by determining confidence intervals using probability theory. Chart 1 graphs the 95 percent confidence interval for the population insolvency rate, in an example in which the observed rate of insolvency among the tested scenarios is 10 percent. The confidence interval size is a function of the number of scenarios run. This calculation, using combinatorial techniques we will document elsewhere in the future, is not dependent on the particular stochastic model used. Note that the X-axis of this graph has a nonlinear scale.

Chart 1 illustrates how sampling credibility increases with sample size. However, there appears to be a point of diminishing returns somewhere around 200 or 500 scenarios, if the user is attempting to test reserve adequacy according to the definition we have chosen for illustration. In general, the "right" number of scenarios to run will depend on the particular problem being analyzed. The matter is subject to considerable professional judgment as well as probability theory.

Chart 1

HOW MANY SCENARIOS ARE ENOUGH?

Observed Reserve Inadequacies = 10%
95% Confidence Interval for True Inadequacy Probability



Conclusion

We have enjoyed reading and discussing Mr. Jetton's paper. We hope that our discussion will help to demonstrate that this area of work is very fertile.

GRAHAM LORD:

Mr. Jetton presents a collection of straightforward methods of developing interest rate scenarios, methods that are accessible not only to actuaries but also to those with whom actuaries must interact. The desirability of such methods has been heightened by the need to adequately analyze and value the plethora of insurance and other financial products that have come into being in recent years and by the need to treat the effect of the interactions between such products (as occurs in asset/liability management, for example.) The complexity of these products has forced on financial analysts (including actuaries) a review of the earlier standard valuation techniques: such techniques prove to be inadequate except in the simplest of applications. A subsequent search for methods that more accurately capture the behavior of these complex products leads directly to models that develop interest rate scenarios.

Mr. Jetton states that there has been little discussion in the American actuarial literature on the topic of modeling interest rates and interest rate scenario generation. However, the topic has been very actively pursued by financial economists, econometricians, and actuaries in Europe. Many of the results of these researchers' efforts are available in the public domain. For example, a steady stream of papers related to this topic has been published by the *Journal of the Institute of Actuaries*. To cite one earlier paper with which many American actuaries are familiar, there is our Society's Halmstad Prize-winning article of Phelim Boyle [3]. Other actuarial publications that contain papers of interest include our own Society's *Record, The Valuation Actuary Handbook*, and the *Proceedings of the Valuation Actuary Symposia*, 1987 and 1988. And there is the *Transactions* paper of Robert Clancy [4] referred to by Mr. Jetton. (Note this latter paper also won the Halmstad Prize.)

Mr. Jetton's article has opened the door through which we can peek into the world of interest rate analysis and forecasting. His models are relatively simple, each with its own shortcomings. Some of these weaknesses have been avoided in more sophisticated extensions not mentioned, whereas other flaws appear to be inherent in the type of model chosen. An example of the latter is in the yield curve jumping method in which the transition rates, for moving from one curve to another, seem as if they can only be determined subjectively. Their values critically affect the outcome of the random generation of the scenarios and so bias the results.

The binomial lattice approach in finance has its origin in the discrete versions of the Black-Scholes-Merton option pricing theory. Clancy [4] developed an adaptation to the pricing of fixed-income instruments but does acknowledge some of his model's weaknesses mentioned by the paper's discussants [7]. An example is the lack of put-call parity in his approach.

A refined model that corrects these problems is presented in [6]. The lattice method is used not only for the pricing of options on equities but also the pricing of contingent cash flows associated with fixed-income instruments and their derivatives and with interest-sensitive insurance products. The method also underlies many of the discretizations of the stochastic differential equation models of interest rates, such as the models of Cox, Ingersoll and Ross [5] and others. Note here that trinomial and multinomial lattices referred to in the paper prove to be unfruitful extensions [1,2].

A desirable criterion for yield curve scenarios mentioned by Mr. Jetton is mean reversion. He presents an example, apparently not supported by detailed evidence, in $f(t)$ for his third stochastic model. If there is reversion, one would think it would be towards an expected forward one-year Treasury rate and not, as is suggested, to the initial rate of $T_1(0)$. To determine the forward rate, account must be taken of (i) the structure of the yield curve and of an acceptable term structure theory—whether it be the expectations hypothesis, the liquidity preference theory or the preferred habitat theory, for example, and (ii) the statistical distribution of the generated scenarios. The link between the one-year and the twenty-year rate and the shape of the yield curve along each interest rate path would then be better specified.

In the context of Mr. Jetton's model, the function $f(t)$ does push successive interest rates, $T_1(t)$, towards $T_1(\infty)$, but can do so excruciatingly slowly. The following table shows the difference between future values of the one-year rate and the long-run rate for a selection of initial rates when there is no volatility, that is, with $VF = 0$. With a large difference between the starting and the long-run value, the next one-year rate is moved sharply towards the limit, but the closer the value becomes to the limit, the slower the convergence. The table indicates how slow that convergence is. (A nonzero volatility could have the effect of making the convergence even slower.) The gradual attenuation is caused by the choice of the constants as well as the form of $f(t)$. How have these been determined? In fact, one wonders what the basis is for all the parameters of his principal, the third, stochastic model.

The brief comments made here hint at the extent of the subject of interest rate modeling and indicate how it can become quite technical as models are improved to better mimic reality. Mr. Jetton's paper gives an introduction to a variety of these approaches and in so doing provides a useful starting point for further exploration.

DEVELOPMENT OF THE DIFFERENCE BETWEEN THE ONE-YEAR RATE FROM THE LONG-RUN RATE IN THE ABSENCE OF VOLATILITY

t	$\lvert T_1(t) - T_1(\infty)\rvert$		
0	10	4	1
1	5	3.04	0.99
2	3.13	2.62	0.97
5	2.18	2	0.93
10	1.64	1.56	0.88
100	0.57	0.56	0.50
1000	0.18	0.18	0.18

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JOHN A. MEREU:

I found Mr. Jetton's paper to be of great interest. His first stochastic model is one in which the interest rate takes a random walk with steps that are normally distributed about a zero mean. The model can be unrealistic because it permits the occurrence of negative interest rates.

In his second stochastic model it is the log of the interest rate that takes a random walk. This model is more plausible in that negative interest rates cannot occur. However, the interest rates in this model can become unbelievably large because there is no mean reverting feature.

In his third stochastic model the log of the interest rate takes a random walk, but with a component returning it to a defined point of central tendency. I believe this is more realistic than the second model and is suitable for an economic environment in which inflation is constant.

Where inflation varies, I would recommend a refinement to the model to provide for a randomly moving point of central tendency that simulates the changing rate of inflation. This will result in a wider spectrum of interest rate tracks to cover plausible futures.

STEVEN P. MILLER:

My comments are directed toward Mr. Jetton's discussion of bias. He notes that, "Bias seems to be unavoidable with any probabilistic or stochastic method of interest rate generation." Accepting the fact that our models are in some way biased, our goal should be to know the implications of any bias and to avoid making bad decisions because of it. The purpose of this discussion is to examine some possible biases and their implications. We also offer a technique for removing a bias in Mr. Jetton's third stochastic model, at the cost of introducing another.

Option pricing models, using binomial lattices, are often characterized as "unbiased" if they eliminate arbitrage opportunities. In the course of creating a lattice that is arbitrage-free, certain "probabilities" are created. These numbers follow the mathematical definition of probabilities and thus have properties that are extremely useful in estimating the solution to the equations that are inherent in arbitrage pricing theory. There is no reason to believe that these probabilities can be used to forecast interest rates. As a matter of fact, these probabilities are based on the assumption that there is no expected benefit to investing in long bonds, despite historical evidence to the contrary.* This is not a flaw in option pricing theory, which seeks to find a portfolio of bonds that will match the cash flows under all interest rate paths, regardless of probability. Rather, it is a caution against making probabilistic statements by using numbers that are not intended to reflect the probability of economic events in the real world. We can see that the "bias" in this family of models comes from assuming that a model that works for one purpose will work for all purposes.

Mr. Jetton's third stochastic model has many desirable characteristics. For example, the yield curve is more likely to be inverted in high-interest-rate environments than in low; the mean-reverting property reflects the belief

*IBBOTSON ASSOCIATES, *Stock, Bonds, Bills and Inflation 1988 Yearbook.*

that authorities will "do something" if interest rates become unreasonable, and the calculation of other rates by weighting of long- and short-term rates produces yield curves that are consistent with history. However, Jetton makes no attempt to eliminate risk-free arbitrage opportunities.

Simply stated, arbitrage is the profit made by buying and selling exactly the same cash flows at different prices. It is believed that an unbiased model should not allow arbitrage without risk. The usual formulation for this is, "the expected present value of a cash flow N years from now should be equal to the value of a zero coupon bond that is purchased today to mature in N years." This is a sufficient but probably not necessary condition. A transaction may be expected to produce excess profits, but this may not be guaranteed.

Arbitrage in Mr. Jetton's third stochastic model may be demonstrated by the following example. Assume that an actuary is pricing a product in an environment in which one-year interest rates are 7.5 percent and twenty-year interest rates are 9.5 percent. All other rates are interpolated as suggested in Jetton's paper, and weighting factors not given are calculated by using linear interpolation. The liability exists for one year and is substantially equivalent in market value to a portfolio of bonds consisting of 54.4 percent 19-year bonds, 29.6 percent four-year bonds, and 16.0 percent one-year bonds. Target surplus is equal to 5 percent of liabilities, which is to be invested in one-year bonds, and the profit goal is 12 percent return on equity (ROE). In addition, the actuary would like to guarantee an ROE of at least 10 percent. Expenses are ignored.

The obvious investment strategy would be an exact match, with a profit margin of 60 basis points. This would guarantee a 12 percent ROE under all scenarios. Assume, however, that the actuary uses the third stochastic model to test the product under an investment strategy of 54.4 percent twenty-year bonds, 29.6 percent three-year bonds and 16.0 percent two-year bonds. By sampling 1000 scenarios, we obtain some surprising results. With no profit margin at all, the expected ROE is 12.3 percent and the minimum for any scenario is 10.1 percent. It appears that the actuary can exceed the profit goal with no profit margin at all.

Of course, this "profitable" investment strategy has been presented in a straightforward manner to emphasize that this model allows risk-free arbitrage. Unfortunately, reasoning similar to the above example may be one facet of a larger problem that is so complex that the presence of such illusory profits may not appear suspicious. In such a case, a company may test a strategy similar to the above example and thus project arbitrage profits that would be unavailable in the real world. For example, some asset liability management programs use linear programming techniques to find optimal investment strategies. Surely such a program would exploit any inconsistencies in the model but may not allow the fallacious reasoning to be easily noticed. The result would be underpriced products with the only evidence of the error buried deep in some intermediate calculation. Clearly, it would be useful to eliminate this bias.

One method of correcting the problem involves the introduction of a number of correcting scenarios that eliminate simple arbitrage opportunities. Specifically, if $K-1$ scenarios are known and we desire no arbitrage, the Kth scenario that corrects the bias can be calculated by using the following iterative procedure:

$$B(k,m,\mathrm{n}) = [K\,P(m+n) - S]/V(k,n)$$

where $B(s,m,n)$ = price of an M-year zero coupon bond, in scenario s, at time n

$P(m+n)$ = price of an $m+n$ year zero coupon bond using today's yield curve

$V(k,n)$ = product of $B(k,1,t)$ for $t=0, \ldots, n-1$

S = sum of $V(s,n)\,B(s,m,n)$ for $s=1, \ldots, k-1$

$V(k,1)$ = $P(1)$.

The yield curve for par bonds can be created from the zero coupon bond prices by using:

$$Y\,(k,m,n) = \frac{1 - B(k,m,n)}{\sum_{t=1}^{m} B(k,t,n)}$$

when $Y(k,m,n)$ is the yield of an m-year par bond in scenario k at time n.

For example, if $k=2$, this procedure will produce one correcting scenario for every scenario generated by the original model. If it is desired, boundary conditions can be placed on the scenarios so that the correcting scenarios are "reasonable." Should boundary conditions constrain the correcting scenario, another scenario would be generated to eliminate the rest of the arbitrage.

An actuary could make statements regarding the probability of ruin using the original model, but use the correcting scenarios to estimate the amount of profit due to arbitrage. In the above example, the correcting scenarios give an expected value equal to -1 times the value given in the original model. Thus, all profits in our example are due to arbitrage.

The obvious question is, "But what is the probability that any of these correcting scenarios actually happen?" This leads us back to the question of bias, and so it goes. The moral of the story is that one must design an interest rate model around the question at hand and never assume that this model is valid for any other question.

ERIC S. SEAH:

The author points out quite correctly that there are "more efficient methods for generating Z that are normally distributed with $\mu = 0$ and $\sigma = 1$". These methods include the Box-Muller method (also known as the Polar method), the odd-even method, the rectangle-wedge-tail method, and the ratio method. A very good description of these methods can be found in Knuth [1, pp. 117–127].

The author describes a method for producing binomial distributions with parameters n and p $(=0.5)$. The method calls for generating n random numbers and enumerating those which are odd (or even). It works well for small n. For large n, more efficient methods are available, see [1, p. 131].

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ELIAS S.W. SHIU:

The paper points out that "interest rate scenarios can be used to answer 'what if' questions about both sides of the asset-liability equation and to develop an easily comprehended distribution of profit (or loss) results under a wide range of possibilities." An earlier attempt to address this issue is the Society of Actuaries monograph [9], which deals with provisions for the risks of adverse deviation in a valuation carried out under Generally Accepted Accounting Principles (GAAP) as required by the Audit Guide for Stock Life Insurance Companies. Section 2.13 of [9] discusses the generation of interest rate scenarios.

The hypothesis of no arbitrage is "one of the most basic unifying principles of the study of financial markets" [8, p. 56]. In a well-developed financial market with rational, profit-seeking individuals, arbitrage opportunities should be very rare. As soon as such opportunities arise, profit-maximizing agents will attempt to exploit them. In an interest-rate movement model, if the yield curves are always flat, then there are arbitrage opportunities; this has been pointed out by actuarial authors such as Boyle [2] and

Milgrom [6]. The binomial lattice method described in this paper is not arbitrage-free. In their discussion of [3], Tilley, Noris, Buff and Lord have pointed out that the *put-call parity* relationships for European options do not hold in the binomial model presented in [3]. Indeed, Bookstaber, Jacob and Langsam [1, p. 17] write: "Despite the pernicious nature of such lattice inconsistencies, it appears to have not been widely treated in the academic or professional literature. The potential for arbitrage-inconsistent lattices extends beyond the option pricing models to interest rate simulation methodology. Interest rate simulations are applied broadly for applications in which the complexity of the option feature of financial instruments makes the usual option pricing models unworkable. . . . Adjustable rate mortgages, CMOs (collateralized mortgage obligations) and a number of financial products, such as the universal life programs and single-premium deferred annuities, are typical candidates for simulation analysis. A simulation model that does not explicitly consider the full span of rates for the relevant portion of the yield curve and that is not founded on an arbitrage-free construction may not give dependable results for either pricing or exposure management."

In addition to the arbitrage problem, the yield curve jumping method described in the paper contains the following bias. The transition matrix $P = (P_{ij})$ determines a *finite-state Markov chain* [5, p. 502]. Properties of transition matrices and Markov chains are well-known in the literature. For an irreducible ergodic Markov chain, it can be shown that

$$\lim_{n \to \infty} P^n$$

exists and that each of its rows is the vector $(\pi_1, \pi_2, \pi_3, \ldots)$, where the numbers $\{\pi_j\}$ uniquely satisfy the steady-state relations [5, p. 510]:

$$\pi_j > 0,$$

$$\pi_j = \sum_i \pi_i P_{ij}, \quad j = 1, 2, 3, \ldots,$$

$$\sum_j \pi_j = 1.$$

The number π_j is the probability that, after a large number of transitions, the interest rate curve is the j-th one. The steady-state probabilities $\{\pi_j\}$ are independent of the initial curve! When modelers prescribe the transition probabilities $\{P_{ij}\}$, they may think that they are just putting down conditions for random generation of interest rate curves and may not realize that at the same time they are specifying the long-run trend of interest rates.

I now sketch another approach for modeling interest rate scenarios. Assume that the interest rate process $I(t)$ satisfies the stochastic differential equation:

$$\frac{dI(t)}{I(t)} = \mu(t, I)\, dt + \sigma(t,I)\, dZ(t), \tag{1}$$

where $Z(t)$ is a Gauss-Wiener process with incremental variance dt. Consider the function $y(t,I) = \log_e(I)$. Applying Itô's lemma, we have

$$\begin{aligned} dy &= \frac{\partial y}{\partial t}\, dt + \frac{\partial y}{\partial I}\, dI + \frac{1}{2}\frac{\partial^2 y}{\partial I^2}\,(dI)^2 \\ &= 0 + \frac{dI}{I} + \frac{1}{2}(-I^{-2})\,(\sigma^2 I^2\, dt) \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ. \end{aligned}$$

If μ and σ are constant functions, then

$$y(t) - y(s) = (\mu - \sigma^2/2)(t - s) + \sigma[Z(t) - Z(s)],$$

or

$$I(t) = I(s) \exp\{(\mu - \sigma^2/2)(t - s) + \sigma[Z(t) - Z(s)]\}. \tag{2}$$

The formula

$$I_{t+n} = I_t \exp[VF(Z_1 + Z_2 + \ldots + Z_n)]$$

in the paper may be viewed as a special case of (2).

A problem with the log-normal process (2) is that the variance grows linearly with time. As the variance becomes large, the probability for very high or very low interest rates becomes substantial. "There is reason to think interest rates are mean reverting, since abnormally high rates will lead to a shift in monetary policy to reduce rates while unusually low rates will lead to a less restrained policy which will lead rates to increase" [1, p. 21].

For a mean-reverting interest rate process, one can put $\mu(t, I) = \kappa(\theta - I)$ in (1), where κ and θ are positive constants. This corresponds to a continuous time first-order autoregressive process where the randomly moving interest rate $I(t)$ is elastically pulled toward a long-run mean θ. The parameter κ determines the speed of adjustment. In a path-breaking paper [4], Cox, Ingersoll and Ross postulate that the spot interest rate (local interest rate) process $r(t)$ satisfies the equation:

$$dr = \kappa(\theta - r)\,dt + \theta\sqrt{r}\,dZ.$$

In the context of an intertemporal general equilibrium asset pricing model, they derive a closed-form formula [4, (23)] that determines the values of noncallable and default-free bonds. In their model, the yield curve is rising for low spot rates and falling for high spot rates. There is a range of spot rates that produces a humped yield curve. The volatility of "long" rates decreases with maturity. Interest rates are never negative. The variance of the interest rate is an increasing function of the interest rate. As these are reasonable properties of interest rates, one may consider using their model to generate interest rate scenarios. More complex models involving more than one state variable are also available in the literature.

The Nelson and Siegel paper [7] introduces a parametrically parsimonious model for yield curves that has the ability to represent the shapes generally associated with yield curves: monotonic, humped and S-shaped. "The ability of the fitted curves to predict the price of the long-term Treasury bond with a correlation of 0.96 suggests that the model captures important attributes of the yield/maturity relation" [7, p. 473].

Finally, I would suggest that the binomial method in Appendix A for generating normal random variates is an unusual application of the central limit theorem. Many very efficient methods for generating normal random variates are available in the literature. The Box-Muller method is one of them.

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JOSEPH H. TAN:

I congratulate the author for writing a timely and informative paper describing the various methods of creating interest rate scenarios. The methods presented are a useful tool for analyzing interest rate risk and facilitating the communication of the analysis of such risk.

My discussion deals with extensions of the author's paper. It is hoped that my discussion can contribute to the value of the methods presented by the author.

Advantages of the Random Generation Method

The author has listed various advantages of the random interest rate generation method over the rudimentary method (that is, specifying the rates one by one for as many scenarios as the creator wants). I believe another advantage of the random interest rate generation method is that it simplifies the task of forecasting interest rates over an extended period of time (the projection period). That is, under the rudimentary method, the creator has to specify the interest rate for the entire projection period; while under the random generation method, the creator only needs to determine the rate for the next period (perhaps with the use of a probability distribution) given the rate for the current period and maybe the prior periods. The task under the random generation method is simpler because it is easier and more reliable to predict the rate for one period (for example, next year) than for the next 20 or 40 successive periods.

Problems and Misconceptions Related to the Random Generation Method

Although the creation of interest rate scenarios is used for various purposes (for example, pricing, "what if" testing, New York Regulation 126 filing, testing the adequacy of reserve), my discussion focuses on the test for reserve adequacy. To the best of my knowledge, the random generation method is currently not required for reserve adequacy by any regulatory authority but is internally used in the company to satisfy the actuary and management that the reserve is adequate.

Reserve adequacy testing is often done by using a cash flow test and is a very complicated process, with the creation of the interest rate scenarios being only a part of the process. Other tasks include the setting of the various assumptions required by the actuary's model (for example, lapse rate, mortality rate, asset call, expense), the construction of the model, and the generation of the projection. My discussion focuses only on the task of random generation of interest rate scenarios and its implications. To this extent, my discussion assumes that the other tasks involved in the reserve adequacy testing are done appropriately.

Because both reserve adequacy and inadequacy scenarios often are present in a set of interest rate scenarios, it is often difficult for the actuary to conclude whether the reserve is adequate. And in the case of the random generation method, the issue is complicated by the fact that any set of randomly generated scenarios (for example, 50) is a very small sample of the set of all possible scenarios (often called the population) that can be generated by the model. This is because the size of the population is extremely large. Consider a simple example.

Assume that given this year's interest rate, there are only three possible rates for next year. Over a forty-year period, the total number of possible scenarios is 3^{40} (or 1.21×10^{19}). Even though some of these scenarios can be ruled out by specifying a maximum rate ceiling and a minimum rate floor, there will still be millions and billions of possible scenarios in this simple model. Given that the creator of the interest rate scenarios (for example, valuation actuary) only generates 50 (or perhaps 100) scenarios:

- How can he or she be 75 percent, 90 percent (or even 5 percent) confident that a "safe" conclusion can be drawn from the results of the model (for example, that the reserve is adequate)?
- How many generated scenarios are enough? Are 50 (or perhaps 1,000) scenarios sufficient?
- Are the chosen scenarios appropriate and representative of the population (that is, the set of all possible scenarios)? Could the next 50 randomly generated scenarios from the same model show significantly different results from the current 50 scenarios? If so, how should the actuary interpret the results from the model?

In the remaining discussion, I address these three questions. These questions arise because the valuation actuary does not know the underlying probability distribution of the population and hence finds it difficult to make probability statements regarding the population based on the results of a small set of randomly generated scenarios. In other words, to make probability statements regarding reserve adequacy for the set of all possible scenarios in the model, the valuation actuary needs to know the probability distribution of the ending surplus (for example, 40th-year surplus). However, such a probability distribution can take various forms, depending on the particular company's situations and assumptions. That is, based on the company's investment, reinvestment, prepayment, lapse, premium, and so on, experience and assumptions, the underlying probability distribution of the ending surplus could be normal, log-normal, or any other form of known or unknown probability distributions.

Some people believe that by taking a large enough sample size (that is, large number of generated scenarios), the underlying probability distribution would be approximately normal. Their belief, which is mainly due to their mistaken understanding of the central limit theorem, is clearly incorrect. The central limit theorem states that the *sample average* of a random sample of size n ($\overline{X}_n$) will approach a normal distribution as n approaches infinity. In applying this theorem to our random generation method, we can only claim that the *average* ending surplus of the n (say 50) randomly generated scenarios is approximately normally distributed. And the probability distribution of the *individual* outcome is still unknown.

Unless future research can clearly identify the probability distribution of the individual (versus sample average) ending surplus, probability statements regarding the ending surplus of the population (that is, the set of all possible rate scenarios) cannot be made. However, based on known statistical results, probability statements regarding the *proportion* of adequate scenarios can still be made. I believe that such probability statements are also relevant in addressing the reserve adequacy issue.

Probability Statements Regarding the Proportion of Adequate Scenarios

Based on the proportion of adequate scenarios in the set of randomly generated scenarios, the valuation actuary can make probability statements regarding the minimum proportion of adequate scenarios in the *population*, that is, the set of all possible scenarios of the model.

Let us use the following notation:

N = the size of the set of all possible scenarios of the model, usually called the population size

P = the proportion of adequate scenarios in the population

n = the randomly generated sample size, for example, 50
X = the number of adequate scenarios in the sample
p = the small p denotes the proportion of adequate scenarios in the random sample and is equal to X/n
Pr = probability
Z = the standard Normal variate, that is, Z has a Normal distribution with mean 0 and variance 1
r = the minimum acceptable confidence level. This could be 90 percent, 95 percent, or whatever level the valuation actuary desires.

If the random sample is taken with replacement (that is, repetition of the same scenario is allowed in the sample), then the number of adequate scenarios in the sample has a Binominal distribution

$$f(x) = {}_nC_X P^X (1 - P)^{n-X}$$

where ${}_nC_X$ denotes combination n taken X.

If the random sample is taken without replacement, that is, no scenario is allowed to be repeated, then the distribution would be a Hypergeometric distribution. But since the population of all possible scenarios is extremely large, it is highly unlikely that a relatively small random sample will contain any repeated scenario, and hence the binomial distribution also suffices for sampling without replacement.

A well-known statistical result is that the Normal distribution is a good approximation to the Binomial distribution. That is,

> "If n is large and P is not too close to 0 or 1, the probability distribution of X (and hence p) can be approximated by a Normal distribution. Experience indicates that the approximation is fairly accurate as long as $nP > 5$ when $P \leq ½$ and $n(1-P) > 5$ when $P > ½$."

(This statement can be found in a number of statistical references. For instance, it can be found on page 153 of Mansfield's book *Statistics for Business and Economics*, New York: W. W. Norton & Company, 1980.) The following table shows the required sample size n for some selected P values to make the Normal approximation fairly accurate.

P	Required Sample Size n
0.75	21
0.8	26
0.85	34
0.9	51
0.95	101

in the population. Note that:

1. For a given acceptable confidence level r, the lower bound of P (that is, the quantity to the right of the inequality) gets larger as p or n increases. Hence by increasing the sample size, the actuary can obtain a higher lower bound for P. And by transforming the above equation, we can easily derive the minimum sample size n needed to ensure (with some confidence level) that the difference between the sample p and the population P be less than some quantity (say 2 percent).
2. Some statisticians have recommended the use of a continuity correction in the above equation. This is a correction that is used when a discrete probability distribution (for example, binomial) is approximated by a continuous one (for example, Normal). The correction is relatively minor and is normally done by subtracting 0.5 from X (or $1/2n$ from p).

To assess whether the current reserve is adequate given the results of the randomly generated sample, the valuation actuary can use a procedure as follows:

1. Determine a minimum acceptable confidence level, for example, 90 percent.
2. The reserve will be considered adequate if the lower bound of P, the proportion of adequate scenarios in the population, is at least some amount P^* (for example, 0.80).

Let us illustrate the application of the procedure with a simple example. Assume that in a set of 60 randomly generated scenarios, there are 54 adequate scenarios. Then we can say with a 90 percent confidence that

$$P \geq 0.9 - 1.29 \sqrt{\frac{0.9(1 - 0.9)}{60}} = 0.85$$

where 1.29 is the Z value with 90 percent probability to the left of it. That is, there is a 90 percent probability that the proportion of adequate scenarios in the set of all possible scenarios of the model will be at least 0.85. If a lower bound of 0.8 is specified for P, then the reserve would be judged to be adequate.

If the actuary wants to increase the confidence level from 90 percent to 95 percent, the lower bound of P will be somewhat smaller,

$$P \geq 0.9 - 1.645 \sqrt{\frac{0.9(1 - 0.9)}{60}} = 0.84$$

And, for a 99 percent confidence level, the lower bound of P is 0.81.

Summary

Unless future research clearly identifies the underlying probability distribution of the ending surplus, it will be difficult for the valuation actuary to extend the randomly generated results to the population. However, the actuary can use our suggested procedure to make probability statements regarding the proportion of adequate scenarios in the set of all possible scenarios that can be generated from the creator's (actuary's) model. Such a procedure is based on well-known statistical results and is simple to apply. It is hoped that future research can uncover more elaborate and precise methods than the simple method illustrated here.

It should be reiterated that any conclusion about reserve adequacy based on the result of a model is highly dependent on the appropriateness of the model and its assumptions. The latter, plus the determination of what is adequate (80 percent, 95 percent, and so on), involves subjective judgment. To this extent, it is better to use the suggested procedure as a management analytical tool and not as a regulatory requirement.

(AUTHOR'S REVIEW OF DISCUSSION)

MERLIN F. JETTON:

I thank the discussants for their fine comments, which enhance the value of this paper.

Mr. Strommen offers some interesting variations on the third stochastic model. Regarding using a range of values for $T_1(\infty)$, I would advise also increasing the coefficients of $f(t)$ to achieve a more suitable behavior. Without such a change, the mean-reversion property would be greatly weakened.

His method of calculating the twenty-year Treasury rate is creative, and it has some nice properties. It provides a built-in solution to the case for which the starting yield curve has an atypical slope. The fact that it uses parameters that can be based on historical interest rates is, I believe, a definite plus.

Mr. Gurski provides some strong statistical support for the hypothesis that changes in interest rates are normally distributed. It is a welcome addition to my paper, because the assumption is an important one and I did not furnish such support.

He describes an alternative model that has much in common with my third stochastic model. The major difference between his model and mine is the method of calculating another point on the yield curve, given only one point. His method relies more heavily on statistics, which a user may find desirable.

A point worth noting, however, is that heavier reliance on conventional statistical methods called for an additional constraint, because such methods fall short of mirroring the real world. Regarding my model, he states that one should use the standard deviation of the error or residual random variable e from the regression equation $Y=aX+b+e$ rather than the standard deviation of the spreads. Strictly speaking, he is correct. Practically speaking, they are nearly the same.

Mr. Gurski claims that historical spreads $(T_{20}-T_1)$ are not normally distributed and that they are serially correlated. The question "Are historical spreads normally distributed?" has no clear answer. It depends on the data you look at and the statistical test (type and confidence level) used. In my opinion there is more than adequate justification for assuming that spreads are normally distributed. The second claim is correct, given that the time interval between measurements is short enough. I would agree that this characteristic of spreads should be recognized in a model in which interest rates are determined more frequently than semiannually or annually.

Mr. Gurski states that he has used a linear mean-reverting formula. I tested a linear formula in the course of developing my cubic formula and found it inadequate to handle the full range of differences between $T_1(t)$ and $T_1(\infty)$. If the coefficient was suitable for large (small) differences, then it was too large (small) for small (large) differences.

I commend Mr. Gurski for having tested output from his model against historical interest rates.

Messrs. Buff and Lassow should be thanked for their list of references, many of which were published after my paper was submitted for publication. They raise several good questions about creating interest rate scenarios. I will respond to a couple of their questions about comparing different models.

First, they asked how scenarios based on the third stochastic model compare to those based on the second. It would be easiest to answer with a graph. Suppose a number of scenarios were generated from both models and frequency curves were drawn from the one-year Treasury rates generated by each model. Then the curves would be similar to those in Figure 4.

Second, they asked how important is the choice of the long-term interest rate trend assumption. Again a graph seems to be the best means of response. Suppose two sets of scenarios are generated, one with $T_1(\infty)=T_1(0)$ and the second with $T_1(\infty)=T_1(0)+1\%$, and frequency curves are drawn from the one-year Treasury rates. Then the curves would be similar to those in Figure 5.

FIGURE 4

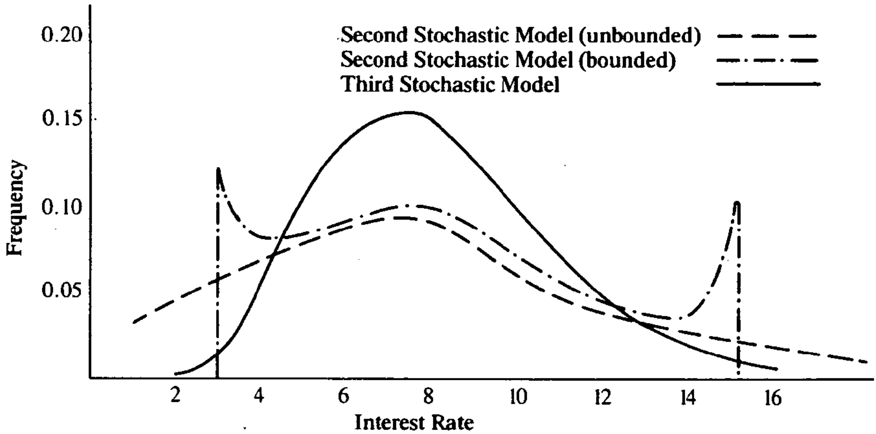
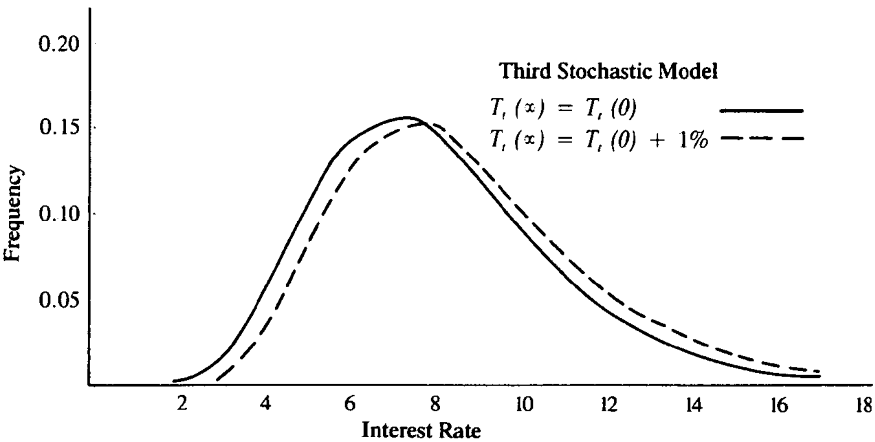


FIGURE 5



Mr. Lord also deserves thanks for his references to other literature on modeling interest rates.

He points out the relative simplicity of the models presented in the paper. My goal was to provide skeletons and discuss major characteristics, which could be readily understood. I have found that much of the other literature is very technical, is not made for easy reading, and/or contains models described in words without the mathematics.

Mr. Lord seems to believe that the mean reversion formula should produce a more rapid convergence toward $T_1(\infty)$. The formula I gave does not have much effect when the difference between $T_1(t)$ and $T_1(\infty)$ is small. I do not believe there is an objective answer here. It would be very difficult to base an answer on historical interest rates, for it would be hard to distinguish between interest rate movements that are "random" and those that are due to "mean reversion." Anyone who agrees with Mr. Lord can, of course, use a different formula more suitable to his/her tastes.