Let $V$ denote the value or price of a fixed-income security. If the cash flows of the
security are fixed and certain, the (continuous) yield to maturity or force of interest, $\delta$, is a
well-defined quantity and we may consider $V$ as a function of $\delta$,

$$ V = V(\delta). $$

Using Taylor's formula

$$ V(\delta + \varepsilon) = V(\delta) + V'(\delta)\varepsilon + V''(\delta)\varepsilon^2/2 + \ldots, $$

we can estimate the change in the price of the security as its yield changes. We note that

Jordan [Jo, p. 56, (2.49)] had given the approximation formula

$$ a_j^i = a_j^i - \frac{j - i}{1 + i} (a)_x. $$

In this note we consider the price, $V$, as a function of both yield $\delta$ and time $t$. Let

$V(t, \delta)$ denote the value of the security at time $t$ evaluated with the continuous yield rate $\delta$.

We also assume that $\delta$ is a differentiable function of time $t$, $\delta = \delta(t)$. We wish to examine
how the security price changes as time passes and as yield changes.

The change in the price of the security as time passes from $t$ to $t + dt$ is given by the
differential

$$ dV = V(t + dt, \delta(t + dt)) - V(t, \delta(t)). $$

Applying the chain rule in multivariate calculus, we have

$$ dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial \delta} d\delta. \quad (1) $$
[For example, the value at time t of a zero coupon bond that will pay 1 at time T, t < T, is

\[ V = e^{-(T-t)d}. \]

In this case formula (1) becomes \( dV = V \delta dt - V(T-t)\delta \).

Rewrite (1) as

\[
\frac{dV}{V} = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial \delta} d\delta. \tag{2}
\]

The partial derivative \( \partial V / \partial t \) gives the rate of change of the security value with respect to time, while everything else (i.e. interest rate) is held fixed. Thus the ratio

\[ \frac{\partial V}{\partial t} \]

is the yield rate \( \delta(t) \) of the security. The partial derivative \( \partial V / \partial \delta \) gives the rate of change of the security value with respect to yield, while everything else (i.e. time) is fixed. Defining

\[ D(t, \delta) = -\frac{\partial V}{\partial \delta}, \]

we can rewrite equation (2) as

\[
\frac{dV}{V} = \delta dt - D d\delta, \tag{3}
\]

which says that there are two components that make up the security's (instantaneous) total return — (i) yield (interest accrued) and (ii) capital gain or loss due to interest rate fluctuation. [For the zero-coupon bond, formula (3) is \( dV/V = \delta dt - (T-t)\delta \).] The quantity \( D \) is called duration. Formula (3) can be found in the book [Gr, p. 62, (4.3)]. Ayres and Barry [AB, (3)] also derive (3), but they use a different approach.

We now examine how the security price changes over a period of time, say, from time \( t = t_0 \) to time \( t = t_1 \). Integrating equation (3) from \( t = t_0 \) to \( t = t_1 \), yields

\[
\frac{V(t_1, \delta(t_1))}{V(t_0, \delta(t_0))} = \exp\left( \int_{t_0}^{t_1} \delta(t) dt - \int_{t_0}^{t_1} D(t, \delta(t)) d\delta(t) \right). \tag{4}
\]

(Formula (4) can be found in [Gr, p. 63, (4.7)].) It follows from (4) that the security price at
time $t_1$ is the product of three terms: (i) the security price at time $t_0$, (ii) the interest accumulation factor $e^{i dt}$ and (iii) the capital gain/loss factor $e^{-i dt}$.

Let us evaluate the integral $\int D d\delta$. Integrating by parts yields

$$
\int_{t_0}^{t_1} D d\delta(t) = \int_{t_0}^{t_1} D [\delta(t) - \delta(t_1)]
$$

$$
= [\delta(t_1) - \delta(t_0)]D(t_0, \delta(t_0)) + \int_{t_0}^{t_1} [\delta(t_1) - \delta(t)]dD.
$$

(5)

Since

$$
dD = \frac{\partial D}{\partial \delta} d\delta + \frac{\partial D}{\partial t} dt,
$$

we need to find the partial derivatives $\partial D/\partial \delta$ and $\partial D/\partial t$. The convexity of the security is defined by the formula

$$
C(t, \delta) = \frac{\partial^2 V}{\partial \delta^2}.
$$

Applying the quotient rule for differentiation gives

$$
\frac{\partial D}{\partial \delta} = -(C - D^2),
$$

(6)

which is a well-known formula and can be found in [KBT, pp. 148], [Gr, p. 40], [S1, p. 101] and [Re, (1.18)]. With the definition

$$
M(t, \delta) = C(t, \delta) - [D(t, \delta)]^2,
$$

we have

$$
\frac{\partial D}{\partial \delta} = -M.
$$

(7)

(We note that Fong and Vasicek [FV1; FV2] use $M^2$ to denote our $M$, and Bierwag [Bi] calls the quantity $M$ inertia.) The partial derivative of $D$ with respect to $t$, $\partial D/\partial t$, turns out to be quite simple:

$$
\frac{\partial D}{\partial t} = -\frac{\partial^2}{\partial t \partial \delta} \log_e V = -\frac{\partial^2}{\partial \delta \partial t} \log_e V = -\frac{\partial}{\partial \delta} \delta = -1.
$$

(8)
It follows from (7) and (8) that
\[
\int_{t_0}^{t_1} [\delta(t_1) - \delta(t)] \, dD = -\int_{t_0}^{t_1} [\delta(t_1) - \delta(t)] \, M \, d\delta(t) - \int_{t_0}^{t_1} [\delta(t_1) - \delta(t)] \, dt.
\] (9)

It also follows from (7) and (8) that \( \partial M/\partial t = 0 \). Hence the function \( M \) is a function of \( \delta \) only, i.e.,
\[
M(t, \delta) = M(\delta).
\]

Thus, by the mean value theorem for integrals, there exists a point \( \xi \) between \( t_0 \) and \( t_1 \) such that \( \delta(\xi) \) is between \( \delta(t_0) \) and \( \delta(t_1) \), and
\[
-\int_{t_0}^{t_1} [\delta(t_1) - \delta(t)] \, M \, d\delta(t) = -\frac{1}{2} \int_{t_0}^{t_1} M(\delta(t)) \, d[\delta(t_1) - \delta(t)]^2
\]
\[
= -\frac{1}{2} M(\delta(\xi)) \, [\delta(t_1) - \delta(t_0)]^2.
\] (10)

Applying formulas (4), (5), (9) and (10), we obtain the formula
\[
V(t_1, \delta(t_1)) = V(t_0, \delta(t_0))\exp\{(t_1 - t_0)\delta(t_1) - D(t_0, \delta(t_0))\delta(t_1) - \delta(t_0)) + M(\delta(\xi))(\delta(t_1) - \delta(t_0))^2/2\}.
\]

This is a valuation equation at time \( t_1 \). We can rewrite it as one at time \( t_0 \):
\[
V(t_1, \delta(t_1))\exp[-(t_1 - t_0)\delta(t_1)]
\]
\[
= V(t_0, \delta(t_0))\exp[-D(t_0, \delta(t_0))(\delta(t_1) - \delta(t_0)) + M(\delta(\xi))(\delta(t_1) - \delta(t_0))^2/2].
\] (11)

Note that, for the formulas
\[
\partial \log V/\partial t = \delta(t), \ t \in [t_0, t_1],
\]
and
\[
\partial D/\partial t = -1, \ t \in [t_0, t_1],
\]
to hold, there should be no payment or cash flow occurring in the time interval \([t_0, t_1]\).
Remarks

There is another approach to obtain (11). Note that

$$V(t_1, \delta(t_1)) \exp[-(t_1 - t_0)\delta(t_1)] = V(t_0, \delta(t_1)).$$

Apply the formula

$$\frac{V(t, \delta)}{V(t_0, \delta_0)} = \exp\left\{ \int_{\delta = \delta_0}^{\delta = \delta_1} \frac{\partial}{\partial \delta} \log V(t, \delta) \, d\delta \right\} = \exp\left\{ - \int_{\delta = \delta_0}^{\delta = \delta_1} D(t, \delta) \, d\delta \right\},$$

and expand $D(t, \delta)$ as a Taylor series in $\delta$. The derivatives of $D$ with respect to $\delta$ can be obtained in terms of cumulants; see pages 100 and 101 of [S1]. Also see [Re].

If we assume $\delta(t)$ to follow a diffusion process, we need to apply Itô's lemma to calculate the differential $dV$. Then there is an extra term

$$\frac{1}{2} \frac{\partial^2 V}{\partial \delta^2} (d\delta)^2$$

in formula (1). See [Bo], [Al], [Le] and [Ma].

This note can be viewed as an attempt to have a better understanding of Redington's theory of immunization. For an approach to extend Redington's theory, see [FV1], [FV2], [MP], [S2] and [S3].

One may want to generalize the above to the case in which the cash flows are not fixed and certain. However, what is the yield rate for a stream of stochastic cash flows?
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References


