Pricing Practices for Joint Last Survivor Insurance

Heekyung Youn
Arkady Shemyakin
University of St. Thomas

Funded by the Society of Actuaries

Abstract

In pricing joint last survivor insurance policies, standard industry practice assumes independence of mortality rates among the insured and adds on loading factors to compensate for possible under-estimations of the joint last survivor mortality due to common life style, broken heart factor or common disaster.
Considering the fact that the separate mortality experiences for males and females are well documented, we study a dependence relation that describes the joint mortality of married couples in terms of marginal female and male mortality rates. We hope that this will help the insurance industry to utilize the existing single life mortality tables (male / female, smoker / non-smoker, and select / ultimate tables) in pricing joint life policies without relying on the independence assumption and yet reflecting more realistic joint mortality rates.
Using data from a large insurance company, we construct a parametric model for the joint survival function based on a Hougaard copula function. The data is from joint annuity contracts and represents mostly married couples. Age difference within a couple is a significant factor in our model. We investigate the association between the marginal survival functions and apply the results to price the joint last survivor insurance using single life tables.

Note. This is a preliminary version of a paper to be submitted for a publication. Because of possible modifications, this paper may not be cited without the permission of the authors.

1. Data Analysis

The data set is from 14,947 joint and last-survivor annuity contracts of a large Canadian insurer. The contracts were in pay out status over the observation period December 29, 1988 through December 31, 1993. We included 11,457 contracts in our study after eliminating duplicating contracts (3,432) and single sex contracts (58).
The fact that the data set reflects last- survivor annuities with guarantee periods raises a question of possible under-reporting.
Following the insurance industry practice of using integer ages at issue and since we compute discrete insurance premiums $A_{\overline{xy}}$, we round the female and male entry ages and termination ages to nearest integer values and denote them as E_F, E_M, T_F and T_M. We let D denote the age difference $E_M - E_F$.

The following table is a preliminary data analysis.

Table 1. Number of Policies and Observed Deaths by Age Difference *D*.

	All Data	D<0	D=0	D=1	D=2	D=3	D=4	D>4
Number of Policies (Total)	11457	2133	1037	1290	1246	1166	1025	3560
Female Deaths (FD)	449	103	43	70	47	44	39	103
Male Deaths (MD)	1245	201	102	149	144	131	119	399
Both Died (BD)	194	39	19	30	20	20	19	47
Ratio $(BD)/(FD \cdot MD/Total)$	3.98	4.02	4.49	3.71	3.68	4.05	4.20	4.07

(under-reporting of first deaths may have affected the Ratio $(BD)/(FD \cdot MD/Total)$)

If female and male lives are independent, we should have

$$\frac{\#of\ FD}{Total} * \frac{\#of\ MD}{Total} * Total \approx \#of\ BD$$

2. Univariate Model

We describe the procedures we used to obtain estimates of the univariate (single life) female and male survival functions. Computations were executed in Microsoft Excel and SPSS.

Two parameter Weibull survival functions for female and male can be expressed (following Frees et al.(1996)) as:

$$S_F(x) = \exp(-(x/m_F)^{m_F/s_F}),$$
$$S_M(y) = \exp(-(y/m_M)^{m_M/s_M}),$$

where m is the scale parameter and m/s is the shape parameter, with appropriate subscripts. Maximum likelihood estimation is carried out in Microsoft Excel with adjustments related to left truncation and right censoring of the data according to the scheme of Frees et al.(1996).

Two parameter Gompertz survival functions for female and male can be expressed as:

$$S_F(x) = \exp(e^{-m_F/s_F}(1-e^{x/s_F})),$$
$$S_M(y) = \exp(e^{-m_M/s_M}(1-e^{y/s_M})),$$

where parameters m and s, with appropriate subscripts, correspond to modes and scale measures of the Gompertz distribution. Maximum likelihood estimation with the likelihood function adjusted for left truncation and right censoring is carried out in Microsoft Excel according to Frees et al. (1996).

Table 2. MLE of Parameters of Univariate Survival Functions

Gender	Parameters	Weibull	St.Error	Gompertz	St.Error
Female	m_F	92.62	0.79	91.81	0.67
	s_F	9.28	0.54	8.14	0.43
Male	m_M	86.32	0.32	85.86	0.30
	s_M	10.87	0.42	10.6	0.42

3. Age Difference as a Factor

Since most of the pairs in our data set, and probably most of the customers of last survivor insurance policies, represent couples who share wealth and lifestyle, we can list several possible sources of association in female-male mortality rates such as

- common lifestyle,
- common disaster,
- broken-heart factor.

The first of these three factors has an effect throughout a couple's lifetime and will affect the correlation between X and Y – their ages at death. However, the other two factors correspond to events happening simultaneously (common disaster) or close in time (broken-heart factor). Therefore, one could expect an increased number of cases when couple's deaths occur closely one after the other in real (chronological) time. For a couple with age difference $D \neq 0$, the proximity of their deaths does not imply a proximity of their ages at deaths X and Y. We will call this phenomena as *chronological-time source of association*.

One way to deal with this issue would be to shift the time scale introducing a new variable $Z = Y\text{-}D$, "***lifelength of the male partner since DOB of the female partner***" or "***wife's age at the time of husband's death***". The effect of chronological-time source of association translates naturally into correlation between X and Z. Therefore, closeness of the times of deaths of a couple will indicate closeness of the values X and Z, regardless of couple's age difference. Although obtaining a meaningful marginal distribution for Z might be difficult, we study the correlation between X and Z to confirm our intuition that the age difference D is indeed an important factor to consider.

Table 3. Nonparametric Correlations

(*Z=Y-D: Wife's age at the time of husband's death*)

Variables	***X,Y***	***X,Z***	***X,Y/ D=0***	***X,Y/ D=1***	***X,Y/ D=2***	***X,Y/ D=3***	***X,Y/ D=4***
Kendall's $\hat{t}$	0.639	0.906	0.838	0.904	0.892	0.802	0.893

Ratio to $\hat{t}$ (X,Y)	1	1.42	1.31	1.41	1.40	1.26	1.40
Spearman's $\hat{r}$	0.795	0.975	0.934	0.975	0.961	0.895	0.967
Ratio to $\hat{r}$ (X,Y)	1	1.23	1.17	1.23	1.21	1.13	1.22

4. Bivariate Model

In some recent studies (see Hougaard et al. (1992), Frees et al. (1996)), the method of copula functions was suggested for construction of joint survival functions. According to this method, the joint survival function of X and Y is represented as

$$S(x,y) = C(S_F(x), S_M(y)), \text{ where}$$

$$S_F(x) = P(X \geq x) \quad \text{and} \quad S_M(y) = P(Y \geq y)$$

are female and male marginal survival functions, and *C(u,v)* is a **copula** – a function with special properties, mixing up the marginals with a certain association parameter. Frees et al. (1996) suggest using Gompertz or Weibull marginal survival functions and Frank's copula function (Frank (1979)):

$$C(u,v) = u + v - 1 + \frac{1}{a}\ln\left(1 + \frac{(e^{a(1-u)} - 1)(e^{a(1-v)} - 1)}{e^{a} - 1}\right)$$

with the association parameter $a < 0$ for the construction of a bivariate spousal survival function. We followed the same approach, considering also another class of copula functions with association parameter $a > 1$, introduced by Hougaard (1988):

$$C\ (u,v) = \exp\left\{-\left[(-\ln u)^{a} + (-\ln v)^{a}\right]^{1/a}\right\}.$$

For these models, the likelihood function was built for the available data sample (11,457 joint annuity policies observed for the period of 5 years), and then the maximum likelihood estimate was constructed for the 5-dimensional vector parameter (m_F, s_F, m_M, s_M, a), where the first 4 components correspond to Gompertz or Weibull marginals, and the last one is the parameter of association. The technique introduced by Frees et al. (1996) and replicated by the authors allows for left truncation and right censoring of the data. The maximum likelihood estimates presented in **Table 4** correspond to parameters of Gompertz (three first columns) and Weibull (three last columns) distributions evaluated for the univariate (compare **Table 2**) and bivariate (shaded) cases. Parameters *m* and *s* with appropriate subscripts correspond to modes and scale measures of the Gompertz distribution, and to the scale and scale/shape parameters of the Weibull distribution, so that we can expect these values to be close to each other.

Table 4. MLE of the Parameters of Joint Survival Function $S(x, y)$

Model		Univariate	Bivariate		Univariate	Bivariate	
Marginals		Gompertz			Weibull		
Copula			Frank	Hougaard		Frank	Hougaard
Female	m_F	91.81	88.55	88.98	92.62	89.09	89.51
	s_F	8.14	8.4	8.29	9.28	8.92	8.99
Male	m_M	85.86	85.2	85.45	86.32	85.74	85.98
	s_M	10.6	10.92	11.19	10.87	10.99	11.24
Association	a			1.615			1.638
			-3.8			-3.75	
Kendall's $\hat{t}$			.37	.38		.37	.39
Spearman's $\hat{r}$			.54			.53	

Let us observe though that the univariate estimates of the marginals (columns 3 and 6 of **Table 4**, see also **Table 2**) sometimes differ substantially from the corresponding components of the vector parameter (m_F, s_F, m_M, s_M, a) estimated simultaneously. For m_F, for instance, this difference is higher than four standard errors.
The estimates obtained for the association parameter a indicate a strong statistical dependence between female and male mortalities in a married couple (see the last two rows of the table). There exist direct relationships between values of a and such nonparametric correlation measures as Kendall's (for both classes of copulas) and Spearman's (Frank's copulas). For example, for Hougaard's copula $t = 1 - a^{-1}$.

However, we believe that the unconditional survival function $S(x, y)$ does not capture all the association between X and Y. Some additional dependence between female and male mortalities could be captured if an additional observable d - "the age difference between husband and wife", is introduced into the picture, and conditional survival functions are studied.

A rationale for this is given by the results of maximum likelihood estimation for alternative models presented in **Table 5** below**.** Here we assume Weibull marginals and Hougaard's copula function. ***Model A*** is obtained by changing time scale to calendar time (estimating the joint survival function of X and Z instead of X and Y); ***Model AD1*** (d = 0,1,2,3,4) estimates $S_d(x, y)$ by separate subsamples $D = d$. All these estimation results show **higher values of association** than those presented in **Table 4.** The log-likelihood value for ***Model A*** is –7,473 while for the best of parametric models in **Table 4** we obtain –7,851, and for the Weibull-Hougaard model we have –7,875. The main problem with this model is the fact that obtaining a reasonable marginal distribution for variable Z is not as easy as for X and Y.

Table 5. Study of Association

Parameters		*Model A (X,Z)*	*Model AD1 d=0*	*d=1*	*d=2*	*d=3*	*d=4*
Female	m_F	86.59	91.07	86.17	87.97	87.43	85.82
	s_F	6.6	10.41	6.99	8.36	9.61	7.26
Male	m_M	82.66	86.94	83.85	84.75	84.78	85.14
	s_M	7.61	10.69	8.26	12.92	10.32	9.46
Association	a	1.81	2.44	1.68	1.93	2.33	1.88
Kendall's $\hat{t}$		.45	.59	.40	.48	.57	.47

We will use Weibull-Hougaard copula models similar to those above with one distinction: the association parameter a is allowed to depend on d. The choice of Hougaard's copula versus Frank's is explained by simpler overall expression while used with Weibull marginals, which was essential in our search of a convenient functional form of dependence $a = a(d)$.

The joint survival function of X and Y given d is represented as

$$S_d(x, y) = C_d(S_F(x), S_M(y)).$$

Here

$$S_F(x) = \exp(-(x/m_F)^{m_F/s_F}),$$
$$S_M(y) = \exp(-(y/m_M)^{m_M/s_M}),$$

are Weibull survival functions, and $C_d(u,v)$ is a Hougaard's copula with association parameter $a(d) > 1$. Then the survival function given d is

$$S_d(x, x+d) = C_d(S_F(x), S_M(x+d)) = \exp\left\{ -\left[\left(\frac{x}{m_F}\right)^{a(d)\frac{m_F}{s_F}} + \left(\frac{x+d}{m_M}\right)^{a(d)\frac{m_M}{s_M}} \right]^{1/a(d)} \right\}$$

Marginals do not change with d, so the only parameter depending on age difference is association. We choose a parametric model for $a = a(d)$ to be a "Cauchy-type" function $a(d; b, g) = 1 + \frac{b}{1 + gd^2}$ with hyperparameters (b, g). This model is referred as *Model AD2* in the following Table 6. *Model ONE* is the model beased on one survival function for all age difference groups, thus referring to the model ignoring the age difference factor.

Table 6. MLE of the Parameters of Conditional Survival Function $S_d(x, y)$

Parameters		***Model AD2***	***Model 0NE***
Female	m_F	89.02	89.51
	s_F	8.89	8.99
Male	m_M	85.82	85.98
	s_M	11.13	11.24
Association	$a(0)$	2.02	1.64
	$a(5)$	1.67	1.64
	$a(10)$	1.33	1.64
	b	1.018	
	g	.021	

5. Premium Computation: Last Survivor Insurance Premium $A_{\overline{xy}}$

We calculate the joint last survivor insurance premiums for each model developed above and compare the results by the ratios of premiums based on different models. We used *i*=5%.

Independence Assumption

Age Difference	-10	-5	0	5	10
Female Age					
50	.112	.126	.139	.150	.158
55	.142	.160	.176	.189	.199
60	.180	.203	.222	.238	.250
65	.227	.254	.278	.297	.311
70	.284	.316	.343	.365	.381
75	.349	.386	.416	.440	.457
80	.422	.462	.495	.519	.537

Model AD2

Age Difference	-10	-5	0	5	10
Female Age					
50	.135	.163	.182	.183	.183
55	.170	.203	.226	.226	.226
60	.211	.251	.276	.276	.278
65	.260	.305	.333	.333	.338
70	.316	.365	.397	.397	.405
75	.378	.430	.467	.467	.479
80	.444	.497	.539	.539	.556

Model ONE

Age Difference	-10	-5	0	5	10
Female Age					
50	.141	.160	.173	.179	.180
55	.176	.200	.215	.222	.221
60	.218	.246	.264	.271	.269
65	.267	.300	.320	.327	.323
70	.321	.359	.383	.390	.385
75	.380	.424	.450	.459	.454
80	.442	.491	.521	.531	.526

We notice that for female ages 55 and above, the premiums are higher when the age difference is 5 than when the age difference is 10.

Ratio of Premium Values: Model AD2/Independence

Age Difference	-10	-5	0	5	10
Female Age					
50	1.207	1.293	1.311	1.225	1.157
55	1.191	1.269	1.280	1.194	1.133
60	1.170	1.237	1.242	1.159	1.109
65	1.143	1.199	1.20	1.123	1.085
70	1.113	1.157	1.156	1.090	1.064
75	1.082	1.115	1.114	1.061	1.047
80	1.052	1.076	1.078	1.039	1.035

Ratio of Premium Values: Model AD2/Model ONE

Age Difference	-10	-5	0	5	10
Female Age					
50	.958	1.018	1.051	1.021	1.015
55	.961	1.018	1.049	1.021	1.024
60	.966	1.018	1.046	1.020	1.034
65	.974	1.017	1041	1.019	1.044
70	.983	1.016	1.036	1.018	1.051
75	.994	1.015	1.030	1.017	1.056
80	1.005	1.013	1.024	1.016	1.056

6. Application

We applied the association factors found in our study to 15-year select female and male non-smoker mortality table provided to us by a local insurance company. This mortality table reflects the company's insurance experience and is used to price its joint last survivor insurance policies. Thus this model is a bivariate copula model with non-parametric marginal survival functions.

Model: BivSelect AD:* (*Model AD2 with select $S_F(x)$ and $S_M(y)$ as marginal survival functions)

$$S_d(x,y) = S(x,y \mid D=d) = C_d(S_F(x), S_M(y)),$$

where $S_F(x)$ and $S_M(y)$ are constructed using a 15-year select table from a local company.
We calculate the premiums based on two different formulas: One calculation is done based on what we call a full bivariate model and the other based on what we call a partial bivariate model. The following is the rationale for the two methods.

We notice that

$$A_{\overline{xy}} = \sum v^{k+1}({}_k p_{\overline{xy}} - {}_{k+1}p_{\overline{xy}}),$$

where

$${}_k p_{\overline{xy}} = \frac{S(x+k,y) + S(x,y+k) - S(x+k,y+k)}{S(x,y)} .$$

Let us define conditional (superscripted) survival probabilities and net-single premiums where the superscripts denote the spousal stastus.

$${}_k p_x{}^y = \frac{S(x+k,y)}{S(x,y)}, \quad A_x{}^y = \sum v^{k+1}({}_k p_x{}^y - {}_{k+1}p_x{}^y),$$

$${}_k p_y{}^x = \frac{S(x,y+k)}{S(x,y)}, \quad A_y{}^x = \sum v^{k+1}({}_k p_y{}^x - {}_{k+1}p_y{}^x),$$

$${}_k p_{xy} = \frac{S(x+k,y+k)}{S(x,y)}, \quad A_{xy} = \sum v^{k+1}({}_k p_{xy} - {}_{k+1}p_{xy}).$$

Then,

$$A_{\overline{xy}} = A_x{}^y + A_y{}^x - A_{xy}$$

and when the joint last survivor premium is computed using this formula, we call this model ***Full Bivariate Model***.

On the other hand, when we approximate ${}_k p_x{}^y$ by ${}_k p_x = \frac{S(x+k)}{S(x)} = \frac{S(x+k,0)}{S(x,0)}$, and likewise ${}_k p_y{}^x$ by ${}_k p_y = \frac{S(y+k)}{S(y)} = \frac{S(0,y+k)}{S(0,y)}$, then we have $A_{\overline{xy}} \cong A_x + A_y - A_{xy}$

and when the joint last survivor premium is computed using this formula, we call this model ***Partial Bivariate Model***.

The following premiums are calculated based on full and partial bivariate models.

Full BivSelect AD $A_{\overline{xy}} = A_x{}^y + A_y{}^x - A_{xy}$

X\D	-10	-5	0	5	10
50	0.1087	0.1314	0.1496	0.1511	0.1511
55	0.1347	0.1611	0.1816	0.1823	0.1842
60	0.1656	0.1953	0.2170	0.2192	0.2240
65	0.2009	0.2326	0.2573	0.2624	0.2673
70	0.2434	0.2793	0.3086	0.3124	0.3242
75	0.2893	0.3298	0.3586	0.3698	*
80	0.3460	0.3866	0.4230	*	*

Independent (Select)

X\D	-10	-5	0	5	10
50	0.1009	0.1151	0.1285	0.1403	0.1500
55	0.1271	0.1445	0.1607	0.1745	0.1862
60	0.1590	0.1799	0.1987	0.2152	0.2292
65	0.1967	0.2206	0.2429	0.2626	0.2771
70	0.2415	0.2704	0.2979	0.3196	0.3385
75	0.2916	0.3264	0.3552	0.3818	*
80	0.3519	0.3888	0.4255	*	*

Ratio: *Full BivSelect AD/Indepentent(Select)*

X\D	-10	-5	0	5	10
50	1.0773	1.1414	1.1642	1.0770	1.0068
55	1.0601	1.1146	1.1303	1.0447	0.9892
60	1.0414	1.0856	1.0922	1.0182	0.9770
65	1.0211	1.0540	1.0594	0.9995	0.9645
70	1.0081	1.0329	1.0359	0.9773	0.9576
75	0.9922	1.0105	1.0097	0.9684	*
80	0.9832	0.9944	0.9941	*	*

Model: Partial BivSelect AD $A_{\overline{xy}} \cong A_x + A_y - A_{xy}$

X\D	-10	-5	0	5	10
50	0.1144	0.1396	0.1601	0.1642	0.1636
55	0.1424	0.1721	0.1960	0.2007	0.2007
60	0.1760	0.2104	0.2371	0.2432	0.2443
65	0.2151	0.2529	0.2832	0.2911	0.2922
70	0.2609	0.3052	0.3420	0.3509	0.3544
75	0.3116	0.3622	0.4002	0.4126	*
80	0.3719	0.4245	0.4700	*	*

Ratio: ***PartialBivSelect AD / Full BivSelect AD***

X\D	-10	-5	0	5	10
50	1.0526	1.0627	1.0703	1.0862	1.0833
55	1.0569	1.0687	1.0789	1.1004	1.0900
60	1.0628	1.0771	1.0925	1.1096	1.0909
65	1.0706	1.0875	1.1007	1.1093	1.0931
70	1.0718	1.0928	1.1082	1.1233	1.0932
75	1.0770	1.0982	1.1160	1.1158	*
80	1.0748	1.0981	1.1110	*	*

A_x

X\D	-10	-5	0	5	10
50	0.1779	0.1779	0.1779	0.1779	0.1779
55	0.2162	0.2162	0.2162	0.2162	0.2162
60	0.2603	0.2603	0.2603	0.2603	0.2603
65	0.3081	0.3081	0.3081	0.3081	0.3081
70	0.3723	0.3723	0.3723	0.3723	0.3723
75	0.4336	0.4336	0.4336	0.4336	*
80	0.5056	0.5056	0.5056	*	*

$A_x^{\ y}$

X\D	-10	-5	0	5	10
50	0.1756	0.1737	0.1703	0.1671	0.1669
55	0.2135	0.2109	0.2059	0.2007	0.2017
60	0.2570	0.2534	0.2455	0.2400	0.2427
65	0.3042	0.2990	0.2900	0.2851	0.2867
70	0.3663	0.3589	0.3479	0.3395	0.3460
75	0.4264	0.4191	0.4040	0.3992	*
80	0.4972	0.4864	0.4727	*	*

A_y

X\D	-10	-5	0	5	10
50	0.1372	0.1686	0.2054	0.2478	0.2928
55	0.1686	0.2054	0.2478	0.2928	0.3439
60	0.2054	0.2478	0.2928	0.3439	0.4031
65	0.2478	0.2928	0.3439	0.4031	0.4611
70	0.2928	0.3439	0.4031	0.4611	0.5296
75	0.3439	0.4031	0.4611	0.5296	*
80	0.4031	0.4611	0.5296	*	*

$A_y^{\ x}$

X\D	-10	-5	0	5	10
50	0.1338	0.1646	0.2026	0.2455	0.2912
55	0.1636	0.1997	0.2438	0.2900	0.3419
60	0.1983	0.2396	0.2875	0.3402	0.4004
65	0.2374	0.2814	0.3360	0.3974	0.4576
70	0.2813	0.3314	0.3941	0.4554	0.5257
75	0.3288	0.3852	0.4491	0.5212	*
80	0.3856	0.4424	0.5156	*	*

REFERENCES

AGRESTI, A., FINLAY, B. (**1997)** *Statistical Methods for the Social Sciences*, Prentice Hall, Upper Saddle River, NJ,

ANDERSON, J.E. , LOUIS, T.A., HOLM, N.V. and HARVALD B. (**1992**) "Time Dependent Association Measures for Bivariate Survival Functions", *Journal of the American Statistical Association*, Vol. 87, 419

BOWERS, N., GERBER, H., HICKMAN,J., JONES, D. and NESBITT, C. (**1997)** *Actuarial Mathematics*, Schaumburg, Ill.: Society of Actuaries,

DABROWSKA, D. (**1988**) "Kaplan-Meier Estimate on the Plane", *Annals of Statistics*, Vol. 16, 4, 1475-1489

FRANK, M.J. (**1979**) "On the Simultaneous Associativity of $F(x,y)$ and $x+y-F(x,y)$", *Aequationes Mathematicae*, Vol. 19, 194-226

FREES, E., CARRIERE, J. and VALDEZ, E. (**1996**) "Annuity Valuation with Dependent Mortality", *Journal of Risk and Insurance*, Vol. 63, 229-261

FREES, E. and VALDEZ, E. (**1998**) "Understanding Relationships Using Copulas", *Actuarial Research Clearing House, Proceedings, 32nd Actuarial Research Conference, August 6-8, 1997*, p.5

HOUGAARD, P. (**1986**) "A Class of Multivariate Failure Time Distributions", *Biometrika*, Vol. 73, 671-678

HOUGAARD, P., HARVALD, B., and HOLM, N.V. (**1992**) "Measuring the Similarities Between the Lifetimes of Adult Twins Born 1881-1930", *Journal of the American Statistical Association*, Vol. 87, 17-24

LONDON, D. (**1988**) *Survival Models and Their Estimation*, ACTEX, Winsted, CT

MILLER, M.D. (**1946**) *Elements of Graduation*, The Actuarial Society of America, American Institute of Actuaries

PRUITT, R.C. (**1993)** "Small Sample Comparison of Six Bivariate Survival Curve Estimators", J. Statist. Comput. Simul., Vol. 45, 147-167