INTERACTION BETWEEN ASSET LIABILITY MANAGEMENT AND RISK THEORY : AN UNSEGMENTED AND A MULTIDIMENSIONAL STUDY

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ABSTRACT

In this paper, we propose measures of the risk that the value of the liabilities becomes larger than the value of the assets, and this especially for insurance companies. We study probabilities of mismatching and a degree of mismatching in unsegmented and multidimensional models of the balance, in which we stress the influence of interest rates on the active and passive side of the balance.

The results are important as they are useful to determine ALM-strategies for insurance companies, but they are also interesting in the relation of an investor versus his broker.

Keywords: stochastic differential equation, Cox-Ingersoll-Ross process, Ornstein-Uhlenbeck process, probability of mismatching, ALM, option theory.

1. INTRODUCTION

Classical risk theory mainly concentrates on the liabilities of insurance companies and therefore on the study of claims, including their frequency and their amount. As explained in e.g. Parker (1997), the insurance companies do not only have to deal with insurance risk but also with investment risk like liquidity risk, risk of change and most importantly the interest rate risk. The insurance risk, due to the risk insured (e.g. mortality, fire, car accidents), can be decreased as the number of policies in the portfolio increases (however, a perfect and complete pooling will never occur). The investment risk does not decrease with an increasing number of policies since the rates of return are highly correlated. Interest rates both influence the active and the passive side of the balance and become very important when the insurance policies are long-term contracts.

It is therefore essential to devise asset liability models appropriate to insurance companies. In this paper, we concentrate on the mismatching of assets and liabilities, namely the times that the value of the assets $A(t)$ becomes lower than the value of the liabilities $B(t)$ and this according to some definitions which will be stated in section 2, namely perfect matching and final matching. In a model specified for insurance companies, we determine the probability that a mismatching happens and we propose a mismatching degree in case of final matching.

Concentrating on ALM for banks, Janssen (1992) modeled both the assets and the liabilities by geometric Brownian motions. We study an extension of the Janssen model in which the asset fund A takes into account fixed-income securities and this introduces asymmetry for A and B. This is particularly useful for insurance companies whose investments are more in bonds than in shares.

We start with a simple unsegmented model in order to obtain a first understanding of the different factors in ALM. We suppose that the asset portfolio can be modeled by a fund containing only pure-discount bonds which reflect the rates of return of the asset portfolio in the past and with maturity the time horizon of the period that we are interested in. We study the cases that the rates of return follow an Ornstein-Uhlenbeck process or a Cox-Ingersoll-Ross process.

We assume that the liability process is defined by a geometric Brownian motion with drift, which is correlated with the asset process in a constant way. In this generalized unsegmented Janssen model, we study the perfect matching and final matching of assets and liabilities by determining the probability of mismatching and the degree of mismatching. In case of perfect matching, we study the influence of the parameters of the asset and liability processes on the probability of no perfect matching in order to obtain general implications of the model to the company asset liability management and we look at the difference between the models with a Cox-Ingersoll-Ross process and an Ornstein-Uhlenbeck process. We further study a multidimensional model in which we assume that the assets and liabilities are segmented in respectively m and n pools, for example into different investments and into different insurance contracts. We first concentrate on the case that the assets contain only shares. Afterwards, we distinct shares and fixed-income securities which mainly differ since shares can be modeled by a geometric Brownian motion, but the interest-rate derivatives cannot. Since these financial instruments depend on the yield curve, we assume a stochastic interest rate process.

The results presented in this work, which are initially derived to measure the riskiness of an insurance company, are also useful in other settings. As an application, we look at the relation between an investor and his broker.

This paper is organized as follows. To begin with, we define perfect matching and final matching in section 2. The unsegmented Janssen model is presented in section 3.

Section 4 is devoted to the study of probabilities of mismatching in the unsegmented case. In section 5, we concentrate on a degree of mismatching between the assets and the liabilities. The influence of the parameters on the no-perfect-match probability in the unsegmented model is the subject of section 6. By a simple case study in section 7, we want to stress the usefulness of these indicators of risk to ALM. In section 8, we study the multidimensional model. Section 9 proposes an interesting application of our results in case of the relation between an investor versus his broker. Section 10 concludes the paper.

2. MISMATCHING AND ALM

Asset Liability Management may be seen as the simultaneous management of assets and liabilities. ALM is a way to quantify and control the various risks that a financial institution (like a bank or an insurance firm) encounters, and in particular the interest rate risk. The interest rates influence both the active side and the passive side of the balance, and not only the values of the present positions, i.e. assets and liabilities generated by past actions but also the cost and benefits of present and future actions will become different under changes in interest rates.

In theory, a financial institution would like to match its cash flows on the assets and liabilities sides. But in case of an insurance company, there often are considerable discrepancies between assets and liabilities when volumes and timing are concerned because an insurance company can have long-term contracts of premiums, which means that some of its assets may be short-term but its liabilities are long-term. So, in practice a perfect match is not always possible. Neither is it always desirable since the final objective of an insurance company is to make profit or better to maximize the net present value of the profit over a given time horizon. In order to achieve a higher return, they may accept a higher level of risk.

Until nowadays, ALM (especially in banks) is mainly concentrated on duration analysis and/or gap management. Another approach to ALM, is to use simulation models. Our goal is to measure the risk of the insurance company by using a stochastic model of both the asset and the liability side of the balance and we consider the possibilities of perfect matching and final matching. We say that the assets and liabilities have no perfect match in an observation period $[0,T]$, if at some date t in the observation period, the asset value $A(t)$ becomes lower than the liability value $B(t)$. In practice, perfect matching of insurance liabilities might be too demanding since low-risk investment strategies associated with the highest degree of matching possible usually produce lower expected returns. Therefore, we also observe final matching which means that we only check whether the assets cover the liabilities at the end of the period $[0,T]$: $A(T) > B(T)$.

We propose the probabilities of no perfect matching and no final matching as indicators of riskiness. We further concentrate on the difference between liabilities and assets at the time horizon. These measures are interesting for the management of the company who can check whether they stay within the risk limits and can approve their strategies with respect to investment, reinsurance, pricing and acceptation of policies. This kind of information would also be useful for the determination of a contingency reserve or the solvency of a portfolio of insurance policies.

We do not propose these indicators as an alternative of other ALM approaches but rather as a complement. Starting from a good database, we advise to use different ALM-tools like duration analysis, gap management, simulation and our mismatching probability and mismatching degree in order to obtain more useful information and a more complete idea of the situation of the company.

The proposed measures of risk are also useful from the point of view of regulating authorities. In fact the goals of an insurance company and regulatory bodies are the same to a certain degree. Insurance companies want to maximize their profit under an acceptable risk level. Regulators want insurance companies which are save for the clients. They are concerned with the social implications of a bankruptcy of an insurance company because not only the clients would be the victims, but also the third-party claimants. Therefore in most countries, the regulatory bodies introduced rules in connection with the contingency reserves and the solvency of the portfolio. They put constraints on the structure of the assets portfolio and they check the financial conditions of the insurance companies, as they also do in case of banks. E.g. in the U.S.A., the National Association of Insurance Commissioners (NAIC) subject insurance companies each year to computerized audits. The measures that we propose in this paper could be an example of an audit ratio test which could be used by the NAIC.

3. THE UNSEGMENTED JANSSEN MODEL

The most realistic model is to look at a portfolio of asset pools $A_1, A_2, \ldots, A_n$ with segments containing different investment instruments. Such a multidimensional model will be the subject of section 8.

First, we concentrate on a less realistic but more treatable model in order to obtain an increased understanding of different influences. Instead of dividing the assets up in different classes, we suppose that we can model the assets as one group of interest rate sensitive securities, reflecting the rates of return of the asset portfolio in the past. Since insurance companies invest particularly in bonds, we model the asset portfolio by assuming that it contains N zero-coupon bonds which are modeled by the rates of return which have been obtained by the portfolio over the last years.

The maturity $\tilde{T}$ of the bonds representing the asset portfolio certainly should be larger than or equal to the time horizon T of the observation period $[0,T]$ we are interested in. In order to simplify the notations, we choose $\tilde{T}=T$. The results about the mismatching probabilities can easily be generalized to longer maturities. In case of the proposed risk measure of final mismatching for stochastic rates of return, however, it makes a difference whether $\tilde{T} = T$ or whether $\tilde{T} > T$.

In Deelstra and Janssen (1998a), we studied the case that the rates of return follow an Ornstein-Uhlenbeck process of the form

$$dr_t = \kappa(\theta - r_t)dt + \eta dZ_t,$$

where $(Z_t)_{t\geq 1}$ is a Brownian motion and where $\kappa, \theta, \eta \in \Re^+$. In this paper, we also concentrate on the case of a Cox, Ingersoll and Ross process of the form

$$dr_t = \kappa(\theta - r_t)dt + \eta\sqrt{r_t}\,dZ_t$$

and compare it with the Ornstein-Uhlenbeck case. Both processes have the realistic property of being mean reverting towards the long term value θ where the speed of adjustment is determined by the parameter κ. The rates in the Vasicek (1977) model can be negative but in our opinion, negative rates of return are possible since assets can be invested in many different financial instruments. If the rates of return remain positive with certainty like for a portfolio of only bonds, the Cox-Ingersoll-Ross model is recommended.

We assume that financial markets are complete and frictionless and that trading takes place continuously. In this setting, Harrison and Kreps (1979) have shown that there exists a unique risk-neutral probability.

Under these assumptions, the assets A_t, modeled by the investment in N pure-discount bonds with maturity T, are modeled in case of the Ornstein-Uhlenbeck process by (see e.g. Vasicek (1977)):

$$dA_t = A_t\left(r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right)dt - A_t\frac{\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)dZ_t$$

with $A_T = N_1$ and in case of the Cox-Ingersoll-Ross process by (see e.g. Cox, Ingersoll and Ross (1985))

$$dA_t = A_t r_t\left(1 - \lambda K(t,T)\right)dt - A_t K(t,T)\eta\sqrt{r_t}\,dZ_t$$

with

$$K(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda + \kappa)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma},$$

$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\eta^2}$, λ the market risk parameter and with $A_T = N_2$. Remark that in fact, Briys and de Varenne (1997) also model the portfolio of assets of a life insurance company by looking at the asset dynamics in the case of the Ornstein-Uhlenbeck process, although they add another diffusion term in order to capture asset risk other than the interest rate risk.

We model the liability process B by a lognormal process with positive constants μ_B, σ_B which is correlated with $(Z_t)_{t\geq 1}$. Cummins and Ney (1980) argue that the lognormal distribution is a reasonable model for insurer liabilities if there is a good reinsurance program to hedge catastrophic jumps in the liabilities.

4. PROBABILITIES OF MISMATCHING IN THE UNSEGMENTED MODEL

4.1 The mismatching process

Using the unsegmented Janssen model presented in the previous section, we study the relations between the assets process A and the liabilities process B in order to point out some management principles. In order to determine the probabilities of mismatching, we consider the mismatching process $a = (a_t, t \geq 0)$, which has been defined in Janssen (1992), namely $a_t = \ln\left(\frac{A_t}{B_t}\right)$ and $a_0 = \ln\left(\frac{A_0}{B_0}\right)$. This process has the same meaning as the surplus process in risk theory. We recall from Deelstra and Janssen (1998a) the dynamics of the mismatching process $a = (a_t, t \geq 0)$ in case of an Ornstein-Uhlenbeck process:

Theorem 1
The stochastic process $a = (a_t, t \geq 0)$ is a solution of the stochastic differential equation

$$da_t = \mu(r_t, t, T)dt + \tilde{\sigma}(t,T)d\overline{W}_t$$

where

$$\mu(r_t, t, T) = r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \mu_B - \frac{\eta^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 + \frac{\sigma_B^2}{2}$$

$$\tilde{\sigma}^2(t,T) = \frac{\eta^2}{\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 + \sigma_B^2 + \frac{2\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\varphi\sigma_B$$

and where $\overline{W} = (\overline{W}_t, t \geq 0)$ denotes a standard Brownian motion and where φ is the correlation parameter between assets and liabilities.

In case of a Cox-Ingersoll-Ross process, Ito's lemma leads to an equivalent theorem:

Theorem 2
The stochastic process $a = (a_t, t \geq 0)$ is a solution of the stochastic differential equation

$$da_t = \mu(r_t, t, T)dt + \tilde{\sigma}(r_t, t, T)d\overline{W}_t$$

where

$$\mu(r_t,t,T)=r_t(1-K(t,T)\lambda)-\mu_B-\left(K(t,T)^2\,\eta^2 r_t-\sigma_B^2\right)/\,2$$
$$\tilde{\sigma}^2(r_t,t,T)=K(t,T)^2\,\eta^2 r_t+2K(t,T)\,\eta\sqrt{r_t}\,\varphi\sigma_B+\sigma_B^2$$
$$K(t,T)=\frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\lambda+\kappa)\left(e^{\gamma(T-t)}-1\right)+2\gamma}$$

and where $\overline{W}=\left(\overline{W}_t,t\geq 0\right)$ *denotes a Brownian motion,* $\gamma=\sqrt{(\kappa+\lambda)^2+2\,\eta^2}$ *and where* φ *is the correlation parameter between assets and liabilities.*

Remark that in case of the Ornstein-Uhlenbeck process, the mismatching process $a=(a_t,t\geq 0)$ is a Gaussian process. In the Cox-Ingersoll-Ross case, the process $a=(a_t,t\geq 0)$ is not a Gaussian process since r_t, the rate of return at time t, does not have a Gaussian distribution but is distributed as a non-central χ^2 - random variable.

4.2 Perfect matching

As the assets and liabilities have no perfect match if for some $t\geq 0$ the asset value $A(t)$ becomes lower than the liability value $B(t)$, a no-perfect-match occurs if $a(t)$ becomes negative (see Janssen (1992) and Ars and Janssen (1994,1995)).

Therefore, we define the first mismatching time in the period $[0,T]$ as

$$\tau=\inf\{t:0\leq t\leq T,a(t)\leq 0\}.$$

In order to determine the probability of no perfect match, we have to concentrate on the crossing probabilities $P[\tau<T]$, which cannot be obtained explicitly in the general model. To obtain more insight, we first treat deterministic rates of return.

4.2.1 Special case: Non-stochastic rates of return

First, let us assume that the volatility coefficient η equals zero so that the rates of return are deterministic and equal

$$r_t=e^{-\kappa t}(r_0-\theta)+\theta.$$

a/ Constant rates of return

In order to be able to use the nice and well-known results in case of a Brownian motion, we first concentrate on the special case of constant rates of return *r*. Clearly, if $r_0=\theta$, then $a=(a_t,t\geq 0)$ is a Brownian motion with drift and denoting $\mu=\theta-\mu_B+\dfrac{\sigma_B^2}{2}$ and $\sigma=-\sigma_B$.

The probability of no perfect match in the period $[0,T]$ turns out to be (see for an overview e.g. Deelstra (1994)):

$$P[\tau < T] = P\left[\sup_{0\le t\le T}\left(W_t - \frac{\mu}{\sigma}t\right) \ge \frac{a_0}{\sigma}\right].$$

$$=1 \qquad \text{if } \frac{a_0}{\sigma} \le 0$$

$$=1-\Phi\left(\frac{a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right) + e^{-2a_0\mu/\sigma^2}\Phi\left(\frac{-a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right) \qquad \text{if } \frac{a_0}{\sigma} > 0,$$

where $\Phi(\ )$ denotes the cumulative Normal distribution function.

An interesting measure of the mismatching risk is to calculate the probability of mismatching in the period $[0,\infty)$. For T tending to infinity, we find that:

$$P[\tau < \infty] = e^{-2\mu a_0/\sigma^2} \qquad \mu, a_0 > 0$$

$$= 1 \qquad \mu \le 0 \ or\ a_0 \le 0.$$

Therefore, if μ is negative or a_0 is negative, then there will be no perfect match with probability 1. Otherwise, the probability of having at least once a mismatch equals $e^{-2\mu a_0/\sigma^2}$. This probability decreases if $a_0 = \ln\left(\frac{A_0}{B_0}\right)$ or $\mu = r - \mu_B + \frac{\sigma_B^2}{2}$ increases and/or $\sigma = \sigma_B$ decreases. So, the initial assets should be as large as possible in comparison with the liabilities. The instantaneous drift and the volatility of the liabilities should be as low as possible. As motivated before, this information number can be interesting for the managers of the company, the regulators as well as the clients and everyone who has to deal with the insurance company because it is a measure of the risk position of the company.

b/ Time-dependent rates of return

If $r_0 \ne \theta$, the determination of the crossing probability $P[\tau < T]$ is not so easy since the drift term of $a = (a_t, t \ge 0)$ is time-dependent and therefore, we cannot rely on results about Brownian motions crossing (piecewise) linear boundaries. In this case, we can rewrite the first mismatching time τ as

$$\tau = \inf\left\{t : 0 \le t \le T,\ W_t \ge \frac{1}{\sigma_B}\left(a_0 + \frac{r_0-\theta}{\kappa} - \left(\mu_B - \theta - \frac{\sigma_B^2}{2}\right)t + \frac{e^{-\kappa t}}{\kappa}(\theta - r_0)\right)\right\},$$

which is the crossing time of a standard Brownian motion to a boundary $l(t)$ which is wholly convex for $r_0 < \theta$, and wholly concave for $r_0 > \theta$.

In such a situation, Durbin (1992) gives successive approximations for the first-passage densities and some error bounds. Also Sacerdote and Tomassetti (1996) propose approximations for the first passage probabilities and indicate error bounds by using a series expansion for the solution of an integral equation for the first-passage time probability density function.

4.2.2 General case of Ornstein-Uhlenbeck and of Cox-Ingersoll-Ross process

Let us now concentrate on the first mismatching process in case of stochastic rates of return modeled by an Ornstein-Uhlenbeck process or a Cox-Ingersoll-Ross process with $\eta \neq 0$. In Deelstra and Janssen (1998a), we treated the case of an Ornstein-Uhlenbeck process. We used approximations by Durbin (1985) for the crossing probabilities and the first-passage density of a continuous Gaussian process $y(t)$ at a boundary $l(t)$ at $u = t$. But the Cox-Ingersoll-Ross process is no Gaussian process and therefore the approach used in case of an Ornstein-Uhlenbeck process cannot be applied.

In case of the Cox-Ingersoll-Ross process, we looked at an alternative way to approximate first mismatching times. We first approximated r_S by a constant upper bound U and by a constant L, which are determined in such a way that for both bounds the probability that the process crosses the bound in $[0,T]$ is for example *5%.* These boundaries follow from results of Giorno et al. (1989) and of Sacerdote and Tomassetti (1996).

Afterwards, we could compare the time of first mismatching τ of the process $a = (a_t, t \geq 0)$, by the times of first mismatching $\tau_1, \tau_2, \tau_3, \tau_4$ with L or U substituted in stead of r_S. The crossing probabilities $P[\tau_i < T]$, *i=1, 2, 3, 4,* could be determined by the method of Durbin (1985) since by the substitution of L and U, one obtains a continuous Gaussian process $(y(t))_{t \geq 0}$. Finally, we could use the bounds

$$\min_i P[\tau_i < T] \leq P[\tau < T] \leq \max_i P[\tau_i < T] .$$

Unfortunately, this way of approximating the crossing probabilities is very complicated. Therefore, one can better use simulations to obtain the mismatching probabilities in case of the Cox-Ingersoll-Ross model. Another reasonable approximation seems to use the no-perfect-match probabilities in case of the Ornstein-Uhlenbeck model. We did many simulations (see section 6) which usually showed only small differences between the mismatching probabilities on a short period when the parameters were estimated by the same rates of return.

4.3 Final mismatching

In practice, perfect matching of insurance liabilities might be too demanding since low-risk investment strategies associated with the highest degree of matching possible usually produce lower expected returns. Therefore, we also observe final matching which means that we only check whether the assets cover the liabilities at the end of the period $[0,T]$: *A(T)>B(T).* Therefore, the probability of no final matching is the probability

$$P[A_T < B_T] = P[a_T < 0]$$

where $(a_t)_{t\geq 0}$ is the process of mismatching defined above and this probability follows from the distribution of a_T.

In the non-stochastic case and in the Gaussian case, the mismatching process $(a_t)_{t\geq 0}$ is a Gaussian process and therefore a_T has a Normal distribution with mean *m(T)* and standard deviation $\sigma(T)$. The probability of no final matching simply equals

$$P[A_T < B_T] = P[a_T < 0] = 1 - \Phi\left(\frac{m(T)}{\sigma(T)}\right),$$

where $\Phi(z)$ denotes the cumulative standard normal distribution function in *z*.

In order to calculate the probability of no final match in the general Cox-Ingersoll-Ross case, we have to know the distribution of a_T since

$$P[A_T < B_T] = P[a_T < 0] = \int_{-\infty}^{0} f_{a_T}(u)\,du.$$

Unfortunately, we do not know the distribution of a_T since the process of mismatching satisfies a stochastic differential equation where the rate of return r_t shows up in both the drift term and the volatility term. Therefore, one has to approximate the no-final-match probability by using simulations or by using the results in case of the Ornstein-Uhlenbeck process. In any case, the probability of no final matching is always lower than the probability of no perfect matching since final matching puts only a restriction on the portfolio at time *T*.

5. MISMATCHING DEGREE

In case of no final matching, we want to have a risk measure of final matching which gives an idea of the difference between liabilities and assets at the time horizon *T*. We use the approach of Cummins (1988) in his calculation of risk-based premiums, and of Kusakabe (1995) in his discrete ALM model, and we propose as a measure of risk at time *t*:

$$M_t(B_T - A_T) = E\left[(B_T - A_T)^+ e^{-\int_t^T i_u du} \mid F_t\right]$$

with $(i_t)_{t\geq 0}$ modeling the short-term interest rates and with F_t the sigma-field of information until time *t*. This implies that the risk measure equals zero if the assets *A(T)* are higher than the liabilities *B(T)*.

At time *T* itself, we know that the measure M_T equals

$$(B_T - A_T)^+ = \max(B_T - A_T, 0).$$

The value at time *t* can be obtained by using techniques from option theory and in particular from the formulae of Black and Scholes (1973) and Merton (1973).

Indeed, it is well-known that the value of a call option at time *t* which gives the right (but not the obligation) "to buy" at time *T* the liabilities B_T, modeled by the geometric Brownian motion

$$dB_t = \mu_B B_t dt + \sigma_B B_t dW_t,$$

at the exercise value $K = A_T$ of the assets at time T, equals

$$E\left[\left(B_T - A_T\right)^+ e^{-\int_t^T i_u du} \mid F_t\right]$$

where the conditional expectation is taken with respect to the risk-neutral probability.

Under the assumption that interest rates are constant and $\tilde{T} = T$, we can use the well-known Black and Scholes (1973) formula:

$$M_t(B_T - A_T) = e^{-i(T-t)} E\left[\left(B_T - A_T\right)^+ \mid F_t\right] = B_t \Phi(z) - e^{-i(T-t)} K \Phi(z - \sigma_B \sqrt{T-t})$$

with

$$z = \frac{\log\left(B_t / K\right) + \left(i + \sigma_B^2 / 2\right)(T-t)}{\sigma_B \sqrt{T-t}}$$

and where $\Phi(z)$ denotes the cumulative standard normal distribution function in z. Substituting t=0, delivers us the risk measure at time t=0 of the expected deficit at time T, i.e. the expected discounted value of the difference between the liabilities and the assets when there is no final match.

Under the assumption of stochastic interest rates, the value of the risk measure also follows from results obtained in finance (see e.g. Merton (1973), Rabinovitch (1989)).

If $\tilde{T} > T$, the results remain the same in case of deterministic rates of return with

$$\mathrm{K} = A_T = A_0 \exp\left(\theta T + \frac{r-\theta}{\kappa}\left(1 - e^{-\kappa T}\right)\right).$$

In the general case, however, the risk measure of no final match has to be determined numerically since now not only the liabilities B_T but also the assets A_T at time T are random.

6. SIMULATIONS IN THE UNSEGMENTED MODEL

In order to determine the no-perfect-match probabilities in the time-interval $[0, T]$, we have used an Euler scheme to simulate the processes ln *A(t)* and ln *B(t)* and this with rates of return first modeled by an Ornstein-Uhlenbeck process and afterwards by a Cox-Ingersoll-Ross process. Therefore, we used 1,000 steps and took the average over 1,000 paths. So, for the calculation of each probability of no perfect matching, we generated 2,000,000 Normal random numbers.

For the simulations presented in this paper, we have chosen the parameters of the rates-of-return process as in Chan et al. (1992), namely for the Ornstein-Uhlenbeck process: θ=0.0866, κ=0.1779, η=0.02 and for the Cox-Ingersoll-Ross process: θ=0.0808, κ=0.2339, η=0.0854; with in both cases r_0=0.01. The drift and the volatility of the liability process and the initial value of the assets are taken from Ars and Janssen (1994), who estimated the parameters from data of a real Belgian company: μ_B=0.1008, σ_B=0.0261 and A_0=24,406 million Belgian francs. The value of the initial liabilities, namely B_0=23,389 million Belgian francs, was chosen by us to have a probability of no perfect match of around one half so that both increases and decreases should be clear. In the model, we suppose at a first stage that φ=0 and λ=0.

We have checked the influence of the different parameters of the asset process in order to find out which parameters have the largest influence.

A first conclusion of our simulations is that if the maturity is small, e.g. T=3 years, and the parameters of the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process are estimated to fit the same rates of return, then the probabilities of no perfect matching are almost the same. Since in the Cox-Ingersoll-Ross model, the rates of return remain always positive, they are expected to be higher than the rates of return in case of the Ornstein-Uhlenbeck process and consequently the probability of no perfect matching to be lower than in case of the Ornstein-Uhlenbeck process.

We notice that in general the influence of the initial values r_0, A_0 and B_0 are very high since no good match in the beginning can lead to an immediate mismatching. The probability of no perfect matching increases as the initial value of the liabilities increases or the initial value of the assets decreases. The probability of no perfect matching also decreases with the initial rate of return. The initial values, however, become less important if the maturity increases.

The influence of θ is the same as the influence of r_0, namely a decreasing θ leads to higher probabilities of no perfect matching, but the effect is not as large as that of r_0 since θ is a long-term value.

The influence of the volatility parameter η is also quite important: increasing values for η lead to increasing values of the probability of no perfect matching. The influence in case of the Ornstein-Uhlenbeck model however is much higher since the volatility term in the Ornstein-Uhlenbeck model is ηdZ_t and in the Cox-Ingersoll-Ross model $\eta\sqrt{r_t}dZ_t$. So in case of the Cox-Ingersoll-Ross model, the increase in the volatility parameter η is tempered by the multiplication of the square root of the rates of return.

The influence of the speed-of-adjustment parameter κ depends on the difference of $r_0 - \vartheta$. If r_0 almost equals θ, then there is almost no influence. If the difference is positive and quite large, then the probabilities of no perfect match increases if κ decreases. For a negative difference, the opposite statement is true.

The influence of the market-risk parameter λ differs for both models since they appear in a different way in each model. In both cases, however, the market-risk parameter λ has a high influence and this might cause a problem as the market-risk parameter λ is difficult to estimate.

Increases in the correlation parameter φ leads to both increases and decreases of the probability of no perfect matching. The dependence of the no-perfect-match probability on the correlation parameter φ is not clear.

7. CASE STUDY

In this section, we give some examples of how the measures of no perfect matching and of no final matching can be applied. To obtain an increased understanding, we want to be able to calculate the probabilities by using a calculator and we restrict ourselves to the case where the portfolio can be modeled by zero-coupon bonds with maturity date the final date of the observation period and with a fixed constant rate of return *r*. This constant rate can be thought of as the average of the rates of return of the portfolio in the last years. As explained above, for more difficult situations we have to recur to a computer program but the underlying ideas of using the indicators of risk are the same.

For the first example, we use the data which is presented in Ars and Janssen (1994), namely of a Belgian firm during the period 1980-1991 with A_0=24,406 million Belgian francs and B_0=22,631 million Belgian francs. Supposing that the liabilities follow a geometric Brownian motion with drift, Ars and Janssen (1994) estimated the parameters: $\hat{\mu}_B = 0.1008$ and $\hat{\sigma}_B = 0.0261$. As in the period 1980-1991 the interest rates and the returns were high, we suppose that the constant rate of return equals r = 0.1 and that the market interest rate equals i = 0.09.

In this situation, the probability of having once a mismatch (in an infinite period) equals 1 since

$$\mu = r - \mu_B + \frac{\sigma_B^2}{2} = \text{-0.00045939} < 0.$$

The probability of no perfect matching in a finite period *T*, however, does not equal 1 but equals

$$P[\tau < T] = 1 - \Phi\left(\frac{a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right) + e^{-2a_0\mu/\sigma^2}\Phi\left(\frac{-a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right)$$

with $\Phi(\ )$ the cumulative Normal distribution function and where we recall that τ denotes the first mismatching time. For example, if we are interested in an observation period of 3 years, then the probability of no perfect matching equals

$$P[\tau < T = 3years] = 1 - \Phi(1.64) + 1.10735\,\Phi(-1.70) \approx 10\%.$$

The probability of no final matching in $[0, T = 3\,years]$ is always smaller than the probability of no perfect matching and equals

$$P\left[A_T < B_T\right] = 1 - \Phi\left(\frac{a_0}{\sigma\sqrt{T}} + \frac{\mu\sqrt{T}}{\sigma}\right) = 1 - \Phi(1.64) \approx 0.0505 .$$

To have an idea of the mismatch at the final date, we calculate the risk measure given by the Black and Scholes formula:

$$M_0(B_T - A_T) = B_0\Phi(z) - e^{-iT}K\,\Phi(z - \sigma_B\sqrt{T})$$

with

$$z = \frac{\log\left(B_0/K\right) + \left(i + \sigma_B^2/2\right)T}{\sigma_B\sqrt{T}}$$

and $K = A_T = A_0\exp(rT)$. Substituting the different parameters, this measure of a deficit at $T = 3$ years turns out to be

$$M_0(B_T - A_T) = 22{,}631\Phi(-2.31) - 25{,}150\Phi(-2.36) \approx 6.5$$

millions of Belgian francs.

It turns out that in this example, the probability of mismatching is quite high (a mismatch with probability 1 over an infinite period!) and the expected discounted difference between the assets at the final date is quite important. This firm should try to increase its expected rates of return on the assets portfolio, should decrease the drift of the liabilities and / or should adapt the initial proportion of assets to liabilities.

Let us now consider a second example with rates of return and interest rates more adapted to the present situation. We suppose that the constant average of the rates of return equals $r = 0.05$, that the constant market spot interest rate equals $i = 0.042$, that the estimated drift and volatility of the liability process equal respectively $\hat{\mu}_B = 0.045$ and $\hat{\sigma}_B = 0.009$, and we choose the initial values of the assets and liabilities quite arbitrarily equal to respectively $A_0 = 23{,}000$ and $B_0 = 22{,}631$ millions of Belgian francs.

In this situation, the probability to have a mismatch (in an infinite period) equals

$$P[\tau < \infty] = e^{-2\mu a_0/\sigma^2} \approx 13.4\%$$

The probability of obtaining a mismatch within 3 years equals

$$P[\tau < T = 3years] = 1 - \Phi(2.008) + 0.1336\Phi(-0.0675) \approx 0.085 .$$

And the probability of no final match in [0, T=3 years] turns out to be

$$P\left[A_T < B_T\right] = 1 - \Phi(2.008) \approx 0.0222 .$$

The order of mismatch at the final date of T = 3 years follows from substituting the different parameters in the risk measure of no final matching:

$$M_0(B_T - A_T) = 22{,}631\Phi(-2.57) - 23{,}559\Phi(-2.585) \approx 1.16$$

millions of Belgian francs.

These two examples show that it is important to calculate different measures of risk. For instance, when calculating the probability of having a mismatch within 3 years, the second company has a result which is only slightly better than the first company. However, when concentrating on an infinite period, one can conclude that the situation in the second example is much better since the company in the first example has a mismatching in an infinite period with probability 1 and the second one of only 13,4 %. Also the measures of final matching show that the situation of the second firm is more stable.

8. THE MULTIDIMENSIONAL MODEL

Following Janssen (1992), Ars and Janssen (1994), Bergendahl and Janssen (1994) and Janssen (1995), we concentrate now on the following multidimensional model. Let us denote the total value of the assets at time *t* by *A(t)* and the total value of the liabilities by *B(t)*. Both the asset side *A(t)* and the liability side *B(t)* can have a segmentation in respectively *m* and *n* pools:

$$A_1(t),\ldots,A_m(t) \text{ such that } \sum_{i=1}^{m} A_i(t) = A(t)$$

and

$$B_1(t),\ldots,B_n(t) \text{ such that } \sum_{i=1}^{n} B_i(t) = B(t).$$

The assets can be segmented into different investments like bonds with different maturities and coupons, shares, options, currency et cetera. The segmentation of the liabilities represents different types of contracts like fire insurance, automobile insurance, civil responsibility insurance or in case of a life-insurance company, temporary life-insurance or endowment contracts for e.g. smokers, non-smokers, females, males. Remark that the unsegmented situation corresponds with $m=n=1$.

Janssen (1995) studied the matching of this multidimensional ALM model in the particular case that all segments are a constant fraction of respectively the total assets *A(t)* and the total liabilities *B(t)* which are both modeled by geometric Brownian motions.

In this paper, we want to treat the general model but therefore, we introduce a weaker form of perfect matching. For perfect matching, one usually assumes that for all *t* in a period $[0,T]$ both the asset side *A(t)* and the liability side *B(t)* have a segmentation in *n* pools:

$$A_1(t),\ldots,A_n(t)$$
$$B_1(t),\ldots,B_n(t)$$

and that each segment $A_i(t)$ covers the liability part $B_i(t), 1 \le i \le n$, or equivalently

$$\forall i \quad \ln\big(A_i(t)\big) \ge \ln\big(B_i(t)\big) \qquad \forall t \in [0,T].$$

This condition implies that the arithmetic mean of the logarithms of the assets should be larger than the one of the logarithms of the liabilities:

$$\frac{1}{n}\sum_{i=1}^{n}\ln\big(A_i(t)\big) \geq \frac{1}{n}\sum_{i=1}^{n}\ln\big(B_i(t)\big) \qquad \forall t \in [0,T].$$

This inequality is equivalent with the inequality between the geometric means of respectively assets and liabilities

$$\sqrt[n]{\prod_{i=1}^{n} A_i(t)} \geq \sqrt[n]{\prod_{i=1}^{n} B_i(t)} \qquad \forall t \in [0,T].$$

This inequality is not equivalent with perfect matching but is implied by it. In the following, we will look at the first time that the geometric mean of the assets becomes lower than the geometric mean of the liabilities since at that time there is certainly no perfect match anymore. We admit that it would be more intuitive to work with arithmetic means but as we will show below, it is mathematically easier to check matching in geometric mean. Because of difficulties with the arithmetic mean of lognormal distributions, other authors have already suggested in other settings to use the geometric mean, e.g. Vorst (1992) or Nielsen and Sandmann (1998).

We generalize this definition of matching in geometric mean to the general basic multidimensional model above with *m* asset pools and *n* liability segments by the condition:

$$\sqrt[m]{\prod_{i=1}^{m} A_i(t)} \geq \sqrt[n]{\prod_{j=1}^{n} B_j(t)} \qquad \forall t \in [0,T].$$

We remark that if the number *m* of asset pools equals the number *n* of liability segments, then no matching in geometric mean implies no perfect matching. If the number *m* of asset pools differs from the number *n* of liability segments, then the definition of perfect matching cannot be applied but the generalized definition of matching in geometric mean can be used.

Following the ideas in case of the unsegmented model, we introduce the mismatching process $(a_t)_{t\geq 0}$ defined by

$$a_t = \frac{1}{m}\sum_{i=1}^{m}\ln\big(A_i(t)\big) - \frac{1}{n}\sum_{j=1}^{n}\ln\big(B_j(t)\big)$$

As before, we look at the first mismatching time τ:

$$\tau = \inf\{t : 0 \leq t \leq T, a(t) \leq 0\}$$

and at the probability that the first mismatching time τ is smaller than the final date of the period we are observing:

$$P[\tau < T],$$

which is the probability of no matching in geometric mean. This probability is in the segmented (*m*=*n*) model always smaller than the probability of no perfect match. Thus, if the probability of no matching in geometric mean is higher than a certain risk limit, then we know that the probability of no perfect matching is even higher and that the insurance company has to conclude that they have to adapt the strategies with respect of pricing, reinsurance, investment et cetera.

As there exist only explicit results about crossing probabilities if the process $(a_t)_{t\geq 0}$ is a Brownian motion with drift, we first look at a basic model in which the segments follow a geometric Brownian motion. In section 8.2, we will turn to the more realistic model with a mixed asset portfolio, containing fixed-income securities like bonds.

8.1 The basic model

In this section, the dynamic evolution of the (*m*+*n*)-dimensional stochastic process $(A_1(t),\dots,A_m(t),B_1(t),\dots,B_n(t))_{t\geq 0}$ are assumed to be governed by the following system of differential equations

$$\begin{aligned} dA_i(t) &= \mu_i A_i(t)dt + \sigma_i A_i(t)dZ_i(t) \qquad i = 1,\dots,m \\ dB_j(t) &= \overline{\mu}_j B_j(t)dt + \overline{\sigma}_j B_j(t)d\overline{Z}_j(t) \quad j = 1,\dots,n \end{aligned}$$

with $\mu_i, \sigma_i,\ i = 1,\dots,m;\ \overline{\mu}_j, \overline{\sigma}_j,\ j = 1,\dots,n$ positive constants representing the instantaneous return and volatility of either assets or liabilities. The (*m*+*n*)-dimensional process $(Z_1(t),\dots,Z_m(t),\overline{Z}_1(t),\dots,\overline{Z}_n(t))_{t\geq 0}$ is an (*m*+*n*)-dimensional Brownian motion with mean 0 and covariance matrix

$$\Lambda_Z = \begin{pmatrix} \Lambda & \overline{\overline{\Lambda}} \\ {}^t\overline{\overline{\Lambda}} & \overline{\Lambda} \end{pmatrix}$$

with three different types of elements, namely for $1 \leq i, j \leq m$

$$E\left[dZ_i(t)\cdot dZ_j(t)\right] = \lambda_{ij}dt,$$

for $1 \leq i, j \leq n$

$$E\left[d\overline{Z}_i(t)\cdot d\overline{Z}_j(t)\right] = \overline{\lambda}_{ij}dt$$

and for $1 \leq i \leq m, 1 \leq j \leq n$

$$E\left[dZ_i(t)\cdot d\overline{Z}_j(t)\right] = \overline{\overline{\lambda}}_{ij}dt = \overline{\overline{\lambda}}_{ji}dt.$$

We include this general covariance matrix in order to take into account the possibility of multidimensional correlation between the different asset and liability sections.

In order to study the probability of mismatching, we concentrate on the process of mismatching $(a_t)_{t\geq 0}$ whose stochastic differential equation is easily deduced by Ito's lemma and which turns out to be easy : the drift term of the mismatching process $a = (a_t, t \geq 0)$ is constant. The only difficulty in directly solving the stochastic differential equation of the mismatching process $a = (a_t, t \geq 0)$ defined by looking at the geometric mean, is that

$$\overline{Z} = (Z_1(t),\dots,Z_m(t),\overline{Z}_1(t),\dots,\overline{Z}_n(t))_{t\geq 0}$$

is a correlated Brownian motion. Therefore, we express the components of this (*m*+*n*)-dimensional Brownian motion in terms of an (*m*+*n*)-dimensional standard Brownian motion

$$\left(W_1(t),\dots,W_m(t),W_{m+1}(t),\dots,W_{m+n}(t)\right)_{t\geq 0}.$$

Indeed, it is possible to define a matrix of constants with *m*+*n* rows and *m*+*n* columns

$$C=\left(c_{ij}\right)_{1\leq i\leq m,1\leq j\leq n}$$

such that

$$\overline{Z}_i=\sum_{k=1}^{m+n}c_{ik}W_k, \quad i=1,\dots,m+n.$$

The determination of the constant matrix *C* is equivalent to the matrix factorization of the covariance matrix $\overline{Z}=\left(Z_1(t),\dots,Z_m(t),\overline{Z}_1(t),\dots,\overline{Z}_n(t)\right)_{t\geq 0}$, namely

$$\Lambda_Z=C^tC.$$

This problem is called the Cholesky factorization and it is known from linear algebra that there exists a unique solution *C* which is lower triangular since a covariance matrix is always symmetric. For a procedure to determine the constant matrix $C=\left(c_{ij}\right)_{1\leq i\leq m,1\leq j\leq n}$, we refer to Hahad, Erhel and Priol (1993).

Having determined the matrix *C*, the stochastic differential equation of the mismatching process can be formulated as follows:

Theorem 3

The stochastic process $a=\left(a_t,t\geq 0\right)$ is a solution of the stochastic differential equation

$$da_t=\mu\,dt+\sigma d\overline{W}_t$$

with

$$\mu=\frac{1}{m}\sum_{i=1}^{m}\left(\mu_i-\frac{\sigma_i^2}{2}\right)-\frac{1}{n}\sum_{j=1}^{n}\left(\overline{\mu}_j-\frac{\overline{\sigma}_j^2}{2}\right)$$

$$\sigma^2=\sum_{k=1}^{m+n}\left(\frac{1}{m}\sum_{i=1}^{m}\sigma_ic_{ik}-\frac{1}{n}\sum_{j=1}^{n}\overline{\sigma}_jc_{m+j,k}\right)^2$$

and where $\overline{W}$ is a Brownian motion.

Remark that the mismatching process $a=\left(a_t,t\geq 0\right)$ is a Brownian motion with drift and therefore, modulo parameters, the results are the same as those in section 4.2. For example, let us recall the simple formula for the probability of no matching in geometric mean in the infinite period $[0,\infty[$:

$$P[\tau<\infty]=e^{-2\mu a_0/\sigma^2} \qquad \mu,a_0>0$$
$$=1 \qquad \mu\leq 0\ or\ a_0\leq 0.$$

Therefore, if μ is negative or a_0 is negative, then there will be no matching in geometric mean with probability 1 and consequently, there also will be no perfect match with probability one since the probability of no perfect match in a segmented (m=n) model is always higher than the probability of no matching in geometric mean.

Otherwise, the probability of having at least once a mismatch in geometric mean equals $e^{-2\mu a_0 / \sigma^2}$. This probability decreases if a_0 or μ increases and/or σ^2 decreases.

As motivated before, this probability of no matching in geometric mean can be interesting for the managers of the company, the regulators as well as the clients and everyone who has to deal with the insurance company because it is a measure of the risk position of the company. Moreover, it is a lower bound of the no-perfect-match probability in the segmented (m=n) model.

8.2 A general multidimensional model

As mentioned before, the modeling of the liabilities by a geometric Brownian motion is an acceptable assumption if there are no catastrophes or if the insurance company has an excellent reinsurance program (see Cummins and Ney (1980)). A geometric Brownian motion for the assets seems only appropriate if the assets portfolio contains only shares. However, it is more realistic to assume a segmentation of the assets containing not only shares but also bonds, futures, options et cetera.

In the unsegmented model, we have worked with the rates of return in order to reflect the evolution of the asset portfolio in the past without explicitly modeling different pools. Since we take in a multidimensional model the most important segments of the assets into account, we do not model a process of the global rates-of-return but we need a yield curve to price the interest rate derivative securities.

Let us from now on use the notation $(r_t)_{t\geq 0}$ to denote the short-term interest rates. The segments of the shares remain modeled by a geometric Brownian motion but the sections of interest rate derivative securities usually are not. In case of most fixed-income securities with maturity T_i, the stochastic differential equation of the segment has a stochastic differential equation of the form

$$dA_i(t) = A_i(t)\mu_i(r_t, t, T_i)dt + A_i(t)\sigma_i(r_t, t, T_i)dZ_i(t),$$

where the instantaneous drift and/or volatility can depend on the short-term interest rates.

Therefore, a general multidimensional model of both the asset and the liability side of the balance $(A_1(t),\ldots,A_m(t),B_1(t),\ldots,B_n(t))_{t\geq 0}$ can be determined by the following system of differential equations

$$
\begin{aligned}
dA_i(t) &= \mu_i A_i(t)dt + \sigma_i A_i(t)dZ_i(t) && i = 1,\ldots,l \\
dA_i(t) &= \mu_i\left(r_t,t,T_i\right)A_i(t)dt + \sigma_i\left(r_t,t,T_i\right)A_i(t)dZ_i(t) && i = l+1,\ldots,m \\
dB_j(t) &= \overline{\mu}_j B_j(t)dt + \overline{\sigma}_j B_j(t)d\overline{Z}_j(t) && j = 1,\ldots,n
\end{aligned}
$$

with $\mu_i, \sigma_i,\ i = 1,\ldots,l;\ \overline{\mu}_j, \overline{\sigma}_j,\ j = 1,\ldots,n$ positive constants and where $\mu_i, \sigma_i,\ i = l+1,\ldots,m$ can be depending on the time, the maturities and the short-term interest rates. The $(m+n)$-dimensional process $Z=\left(Z_1(t),\ldots,Z_m(t),\overline{Z}_1(t),\ldots,\overline{Z}_n(t)\right)_{t\geq 0}$ is as before an $(m+n)$-dimensional Brownian motion with mean 0 and covariance matrix

$$
\Lambda_Z = \begin{pmatrix} \Lambda & \overline{\overline{\Lambda}} \\ {}^t\overline{\overline{\Lambda}} & \overline{\Lambda} \end{pmatrix}.
$$

For this model, the process of mismatching $\left(a_t\right)_{t\geq 0}$ is a solution to the stochastic differential equation

$$
da_t = \mu\left(r_t,t,T_i(l+1 \leq i \leq m)\right)dt + \sigma\left(r_t,t,T_i(l+1 \leq i \leq m)\right)d\overline{W}_t
$$

where $\overline{W}$ is a Brownian motion, but where the drift and volatility term can be quite complicated.

In order to determine the probability of no matching in geometric mean, we have to study the crossing probability $P[\tau < T]$ where τ is the first mismatching time

$$
\tau = \inf\{t : 0 \leq t \leq T,\ a(t) \leq 0\}
$$

and where T is the final date of the period that we are interested in.

It is possible to model the interest rates by a constant rate, by deterministic rates, by an Ornstein-Uhlenbeck process or by a Cox-Ingersoll-Ross process. The dynamics of the different interest rate derivatives in the asset portfolio follow from the chosen short-term interest rate model. Using Ito's lemma, the stochastic differential equation of the first mismatching process is then derived and it is possible to obtain explicit or approximating results of the probability of no matching in geometric mean by applying the methods proposed in section 4, e.g. by Durbin (1992, 1985) and by Sacerdote and Tomassetti (1996).

As an example, we study in the following subsection a (2+1)-dimensional model where the assets contain both shares and bonds, as Bacinello and Ortu (1993) did for pricing guaranteed securities-linked life insurance.

8.3 A particular multidimensional model

In this section, we assume that the liability side of the balance is unsegmented and modeled by a geometric Brownian motion. We further concentrate on the asset portfolio which we model as consisting from a segment A_1 of shares and a section A_2 of bonds with maturity

T. The short-term interest rates are supposed to follow an Ornstein-Uhlenbeck process and this assumption leads to the stochastic differential equation of the section A_2 of bonds.

Summarizing, the particular multidimensional model in this section is governed by the following stochastic differential equations

$$
\begin{aligned}
&dr_t = \kappa(\theta - r_t)dt + \eta dZ_2(t)\\
&dA_1(t) = A_1(t)\mu_1 dt - A_1(t)\sigma_1 dZ_1(t)\\
&dA_2(t) = A_2(t)\left(r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right)dt - A_2(t)\frac{\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)dZ_2(t)\\
&dB_t = \mu_B B_t dt + \sigma_B B_t d\overline{Z}_1(t)
\end{aligned}
$$

with $A_2(T)=N$, the number of bonds in the asset portfolio, and with $\left(Z_1(t), Z_2(t), \overline{Z}_1(t)\right)_{t\geq 0}$ a three-dimensional Brownian motion with mean 0 and covariance matrix $\Lambda_Z = \left(\varphi_{ij}\right)_{1\leq i,j\leq 3}$.

In this case, the mismatching process is a solution of the stochastic differential equation

$$
da_t = \mu\left(r_t, t, T\right)dt + \sigma(t)d\overline{W}_t
$$

with

$$
\mu\left(r_t, t, T\right) = \frac{\mu_1}{2} - \frac{\sigma_1^2}{4} + \frac{1}{2}\left(r_t + \frac{\eta\lambda}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right) - \frac{1}{2}\frac{\eta^2}{\kappa^2}\left(1 - e^{-\kappa(T-t)}\right) - \mu_B + \frac{\sigma_B^2}{2}
$$

and

$$
\sigma^2(t) = \frac{\sigma_1^2}{4} + \frac{1}{4}\left(\sigma_1\varphi_{12} - \frac{\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\sqrt{1-\varphi_{12}^2}\right)^2 + \frac{1}{4}\left(\sigma_1\varphi_{13} - \frac{\eta}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\alpha - \sigma_B\beta\right)^2
$$

with

$$
\alpha = \frac{\varphi_{23} - \varphi_{12}\varphi_{13}}{\sqrt{1-\varphi_{12}^2}}
$$

$$
\beta = 1 - \varphi_{13}^2 - \frac{\left(\varphi_{23} - \varphi_{12}\varphi_{13}\right)^2}{1-\varphi_{12}^2}
$$

and where $\overline{W}$ is a Brownian motion.

Notice that the mismatching process $\left(a_t\right)_{t\geq 0}$ is a Gaussian process and straightforward calculations show that the probability of no matching in geometric mean is in fact the crossing probability of the continuous Gaussian process $y = \left(y(t); 0 \leq t \leq T\right)$ defined by

$$
y(t) = \frac{-1}{2}\int_0^t e^{-\kappa s}\int_0^s \eta e^{\kappa u} dZ_u ds - \int_0^t \sigma_s d\overline{W}_s
$$

to the boundary $l(t) = a_0 + \int_0^t \left(\mu\left(r_s, s, T\right) - r_s\right)ds + \int_0^t E\left[r_s\right]ds$. Therefore, the approximations of Durbin (1985) can be used (see also Deelstra and Janssen (1998b)). Indeed, under mild restrictions on $l(t)$ and on the covariance function $\rho(u,t) = \mathrm{cov}\left(y(u), y(v)\right)$, Durbin (1985) derives approximations for the crossing probabilities and the first-passage density of a continuous Gaussian process $y(t)$ at a boundary $l(t)$ at $u = t$.

9. OTHER APPLICATION: AN INVESTOR VERSUS HIS BROKER

In this section, we want to explain another interesting application of the probabilities of matching, which has been suggested by an anonymous referee.

Suppose that an investor sells short through his broker a non-dividend paying stock. As a collateral, he deposits with the broker a zero-coupon bond which matures at time *T*. Let us assume that the broker marks to market continuously. Then, the investor is interested in calculating the probability that he will be asked by the broker to put additional funds into his account to cover a deficit before the bond maturity date *T*.

As it is common use in finance (see e.g. Merton (1971)), the non-dividend paying stock is modeled by a geometric Brownian motion. For pricing the pure-discount bond, we have to model the yield curve. For example, if we choose the short-term interest rates to follow a Cox-Ingersoll-Ross process, then we are in fact observing the model:

$$dr_t = \kappa(\theta - r_t)dt + \eta\sqrt{r_t}dZ_t$$
$$dA_t = A_t r_t\left(1 - \lambda K(t,T)\right)dt - A_t K(t,T)\eta\sqrt{r_t}dZ_t$$
$$dB_t = \mu_B B_t dt + \sigma_B B_t dW_t$$

with $A_T = N$, with $\mathrm{E}\left[dZ_t . dW_t\right] = \varphi dt$ and with

$$K(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \lambda + \kappa)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma},$$

$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\eta^2}$ and λ the market risk parameter.

Since the broker marks to market continuously, the probability that the investor will be asked by the broker to put additional funds, is in fact the probability of no perfect matching in case of a Cox-Ingersoll-Ross process, which has been discussed in section 4.

Notice that in the situation of this section, $(r_t)_{t\geq 0}$ denotes the process of the short-term interest rates and not the process of the rates of return as in sections 4 to 7. This terminology, however, does not change the results in these sections.

We stress that the example above of an investor versus a broker could be generalized to situations in which the investor sells short several non-dividend-paying stocks and deposits as a collateral a number of different securities like shares and pure-discount bonds. If the broker marks to market continuously, then this situation can be modeled by a multidimensional model as in section 8 and the results of that section can be used.

It is also possible, however, to suppose that the broker does not mark to market continuously but only at some fixed date *T*. Then, the investor is interested in the probability of final matching and then, the investor is usually not only interested in the probability that he has to pay additional funds but also in the expected amount of money that he has to pay in case of a deficit. This means that he is interested in the risk measures of final matching that we have proposed.

We conclude that the results in this paper are not only useful for the asset liability management of an insurance company, but also in the relation of an investor versus his broker.

10. CONCLUSION

We have successfully extended the Janssen model in such a way that the asset fund A takes into account fixed-income securities. This is important for insurance companies whose investments are more in bonds than in shares, especially for life-insurance companies.

We have first considered a model in which we assume that the assets can be represented by only zero-coupon bonds which reflect the historical rates of return. We concentrated on the cases that the rates of return of the portfolio in the past are presented by an Ornstein-Uhlenbeck process or by a Cox-Ingersoll-Ross process. In this unsegmented Janssen model, we have studied the probability of mismatching of the assets and liabilities of the company in a period $[0,T]$ by introducing the first mismatching time $\tau = \inf\{t : 0 \le t \le T, a(t) \le 0\}$ where $a = (a_t, t \ge 0)$ is defined by $a_t = \ln\left(\frac{A_t}{B_t}\right)$ and where T can be assumed to be infinity. Further, we have proposed a risk measure of no final matching which indicates the difference between the assets and the liabilities at time T. The influence of the parameters and the difference between the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process have been studied by simulations. The usefulness of the different measures of risk proposed in this paper, has been stressed by a simple case study.

Afterwards, we have studied a more realistic multidimensional model by introducing mismatching in geometric mean.

Finally, we have shown that the results in this paper are not only useful for the asset liability management of an insurance company, but also in the relation of an investor versus his broker.

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REFERENCES

1 Ars P. and J. Janssen, 1994, "Operationality of a Model for the Asset Liability Management", *Proceedings of the AFIR 4th Session*, 877-905, Orlando.

2 Ars P. and J. Janssen, 1995, "Stochastic Model with Possibility of Ruin and Dividend Repartition for Insurance and Bank", *Proceedings of the AFIR 5th Session*, 281-310, Brussels.

3 Bacinello A.R. and F. Ortu, 1993, "Pricing Guaranteed Securities-Linked Life Insurance Under Interest-Rate Risk", *Proceedings of the AFIR 3th Session*, 35-55, Roma.

4 Bergendahl G. and J. Janssen, 1994, "Principles for the Control of Asset Liability Management Strategies in Banks and Insurance Companies", *submitted.*

5 Black F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81, 637-654.

6 Buonocore A., A.G. Nobile, L.M. Ricciardi, "A New Integral Equation for the Evaluation of the First-Passage-Time Probability Densities", *Advances in Applied Probability*, 19, 784-800.

7 Chan K.C., G.A. Karolyi, F.A. Longstaff and A.B. Sanders, 1992, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate", *Journal of Finance*, 47, 1209-1277.

8 Briys E. and F. de Varenne, 1997, "On the Risk of Insurance Liabilities: Debunking Some Common Pitfalls", *The Journal of Risk and Insurance*, Vol. 64(4), 673-694.

9 Cox J.C., J.E. Ingersoll and S.A. Ross, 1985b, "A Theory of the Term Structure of Interest Rates", *Econometrica*, 53, 385-407.

10 Cummins J.D., 1988, "Risk-Based Premiums for Insurance Guaranty Funds", *Journal of Finance*, 43, 823-839.

11 Cummins J.D. and D.J. Ney (1980), "The Stochastic Characteristics of Property-Liability Insurance Profits", *Journal of Risk and Insurance*, 47.

12 Deelstra G., 1994, "Remarks on 'Boundary Crossing Result for Brownian Motion'", *Blaetter*, 21 (4), 449-456.

13 Deelstra G. and J. Janssen (1998a), "Interaction between Asset Liability Management and Risk Theory", *Appl. Stochastic Models Data Anal.,* 14, 295-307.

14 Deelstra G. and J. Janssen (1998b), "Interaction between Asset Liability Management and Risk Theory: A Multidimensional Study", *Proceedings of the second International Congress "Insurance: Mathematics and Economics" at Lausanne.*

15 Durbin J., 1985, "The First-Passage Density of a Continuous Gaussian Process to a General Boundary.", *J. Appl. Prob.*, 22, 99-122.

16 Durbin J., 1992, "The First-Passage Density of the Brownian Motion to a Curved Boundary.", *J. Appl. Prob.*, 29, 291-304.

17 Giorno V., A.G. Nobile, L.M. Ricciardi and S. Sato, 1989, "On the evaluation of the first-passage-time probability densities via nonsingular integral equations", *Adv. Appl. Prob.*, 21, 20-36.

18 Hahad M., J. Erhel and T. Priol, 1993, "A new approach to parallel Sparse Cholesky factorization on distributed memory parallel computers", *rapport de recherche INRIA*.

19 Harrison J.M. and D.M. Kreps, 1979, "Martingales in Multiperiod Securities Markets", *J. Economic Theory*, 20, 381-408.

20 Janssen J., 1992, "Modèles stochastiques de gestion d'actif-passif pour les banques et les assurances", *Transactions of the 24th International Congress of Actuaries*, ICA-ACI, Montréal, 131-139.

21 Janssen J., 1993, "Asset liability management for banking and insurance", *Proceedings of the ISI 49th Session*, Firenze, 253-269.

22 Janssen J., 1995, "A dynamic stochastic ALM model for insurance companies", in: *Advances in Stochastic Modelling and Data Analysis*, Kluwer Academic Publishers, Dordrecht.

23 Kusakabe T., 1995, "Asset Allocation Model for Japanese Corporate Pension Fund from Liability Aspects", *25th International Congress of Actuaries*, 499-526, Brussels.

24 Merton R.C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", *Journal of Economic Theory,* 3, 373-413.

25 Merton R.C., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, 4, 141-183.

26 Nielsen J.A. and K. Sandmann, 1998, "Asian Exchange Rate Options under Stochastic Interest Rates: Pricing as a Sum of Delayed Payment Options", *working paper.*

27 Parker G., 1997, "Stochastic Analysis of the Interaction Between Investment and Insurance Risks", *North American Actuarial Journal*, 1 (2), 1-26.

28 Rabinovitch R., 1989, "Pricing Stock and Bond Options when the Default-Free Rate is Stochastic", *Journal of Financial and Quantitative Analysis*, 24 (4).

29 Sacerdote L. and F. Tomassetti, 1996, "On Evaluation and Asymptotic Approximations of First-Passage-Time Probabilities", *Adv. Appl. Prob.*, 28, 270-284.

30 Vasicek O., 1977, "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, 5, 177-188.

31 Vorst T, 1992, "Prices and hedge ratios of average exchange rate options", *International Review of Financial Analysis*, 1(3), 179-193.