

APPROXIMATE PROBABILITY STATEMENTS
ABOUT LIFE ANNUITY COSTS

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THE problem of determining proper contingency reserves against possible adverse mortality experience for collections of life annuity contracts has received considerable attention in modern American actuarial literature. Piper [8] in 1934 formulated the problem and suggested that, for suitably large groups of lives, the distribution of the total present value of life annuity costs can be adequately approximated by the normal distribution. Stone [10] in 1948 took another approach by using probability generating functions to make, for a specified distribution of the random variable time until death (1937 SA), exact probability statements concerning annuity costs. Then, in 1952, Taylor [11] suggested fitting Pearson Type III distributions to the total present value of life annuity costs. In 1956, Boermeester [2] tackled the same basic problem by using a Monte Carlo approach.

The problem has a great deal of appeal to actuaries interested in applying statistics to actuarial problems. Yet the problem is not devoid of some very practical aspects. Occasional large fluctuations in the amount of death claims arising from a group of insured lives during a fixed period of time are to be anticipated. However, because of the longer payout period, the impact of mortality variations in annuity systems is somewhat obscured. Nevertheless, deviations from expected results in such systems may be just as real and almost as financially painful, albeit the pain may be spread out, as in life insurance.

This problem is also related to the question of how to report the results of a pension system valuation. The traditional approach has been to report a cost figure determined from some sort of an equation of equivalence in which the present expected value of income is set equal to the present expected value of payments. Pension actuaries have long struggled with the problem of conveying to the managers of pension plans the idea that deviations from expected results as reported in the valuation are highly probable. Myers [7] has advocated supplementing a single present expected value pension cost figure with an associated interval of possible costs. This interval would be intended to span the costs that under reasonably favorable and adverse circumstances might be realized. How-

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ever, no probability statement would be attached to such intervals to help the user of this information measure the reliance that may be attached to the range of costs. The Society's study notes for Part 9E of its examinations contain a thought-provoking paragraph concerning the desirability of attaching probability statements to pension system liabilities. Then is added the sobering note that the state of the art has not progressed to this level.

This short note will not attempt to solve this fundamental problem. Its objective is simply to illustrate some of the alternative approaches to making approximate probability statements about the present value of a

TABLE 1
BOUNDS, WITH PROBABILITY OF EXCEEDING THESE BOUNDS APPROXIMATELY .05 OR .10, ON AVERAGE LIFE ANNUITY COSTS

I_{10}	MONTE CARLO		DISTRIBUTION ASSUMPTIONS				PROBABILITY INEQUALITIES					
			Normal		Pearson Type III		Tchebychef		Uspensky		Fourth Moment	
	.05	.10	.05	.10	.05	.10	.05	.10	.05	.10	.05	.10
1.....	14.07	13.61	14.42	13.76	14.38	13.76	19.42	17.10	19.22	16.81	17.68	16.04
2.....	14.39	13.49	14.53	13.85	14.49	13.84	19.71	17.30	19.50	17.01	17.86	16.18
5.....	15.01	14.56	15.35	14.48	15.25	14.45	21.93	18.88	21.67	18.50	19.22	17.24
10.....	16.03	15.61	16.58	15.44	16.39	15.38	25.27	21.23	24.92	20.73	21.10	18.72
25.....	17.63	16.67	18.34	16.81	18.06	16.71	30.05	24.62	29.58	23.95	23.58	20.72
50.....	18.38	17.28	19.35	17.58	19.02	17.48	32.77	26.54	32.23	25.77	24.92	21.81

collection of life annuity contracts assuming a specified distribution of the random variable—time until death—and a fixed interest rate. This is one aspect of pension costs where our knowledge of the distribution of the basic random variable, time until death, is fairly complete and where the mathematical tools available to the practicing actuary are sufficiently applicable to enable us to attempt at least a partial solution. We will leave for later development the question of determining intervals for life annuity costs with associated approximate probability statements where both time until death and the interest rate are random variables.

The burden of illustrating the various methods of approaching this problem is carried by Table 1. For this table the example used by Boermeester [2] to illustrate the Monte Carlo approach is expanded to show other alternative solutions. The illustration assumes the *a*1949 Male Mortality Table, $2\frac{1}{2}$ per cent interest, and ten lives all age 65. Each of the lives,

except life No. 10, receives one annual unit of life income, while life No. 10 receives the amount shown in the first column.

The entries in the table are of the number b , where

$$Pr \left[\sum_1^{10} I_j a_{\overline{T_j}|} \geq b \sum_1^{10} I_j \right] \leq .05 \text{ or } .10$$

and where T_j , $j = 1, 2, \dots, 10$ are the random variables time until death of life number j (these ten random variables are assumed to be mutually stochastically independent), and I_j is the annual income to life j . Thus the entries in the table provide upper bounds for average life annuity cost with the approximate probability of higher average cost being .05 or .10.

Let us now briefly review the alternative approaches to finding the distribution of total present value of life annuity costs that are illustrated in Table 1.

1. *Monte Carlo*.—This method is explained in detail by Boermeester [2]. It remains only to emphasize that, since the distribution of total annuity costs is estimated by a Monte Carlo (random) process, there is a random error connected with this estimation. The probability that this error will be large may be made small by planning to repeat the process a sufficient number of times. Yet this estimation error remains to complicate our analysis.

2. *Distribution assumptions*.—

a) *Normal distribution*. The use of the normal distribution in problems such as this has been suggested many times in recent actuarial literature ([6], [8], [9]). The justification of this approach rests on generalizations of the well-known Central Limit Theorem. Menge [6] specifically mentions an extension by Liapounoff. A precise statement of this extension and its proof may be found in Cramér [3, p. 215]. As might be expected, the Liapounoff Theorem depends upon an additional condition beyond the existence of the mean and variance that are required by the Central Limit Theorem. Cramér points out that “the object of such additional condition is, generally speaking, to reduce the probability that [an individual component random variable] will yield a relatively large contribution to the [sum].” The concentration of income on one life in some small pension systems, which could possibly cause a relatively large contribution to the total costs by one of the component lives and hence slow the approach to the normal distribution of total life annuity costs, is one of the reasons for caution in adopting the normal distribution in studying small systems.

b) *Pearson Type III*. The use of this distribution is not supported by any specific limiting distribution theorem. Rather it was felt by Taylor [11] that introducing a measure of skewness, as well as of location (mean) and dispersion (variance), would result in a closer approximation to the distribution of total annuity costs than could be obtained with the two-parameter normal distribution. This conviction was supported by the observation that concentrating income on one life in a small system tends to increase the skewness of the distribution of total costs. Note from Table 1 that the inequalities found using this distribution are slightly tighter than those found by using the normal distribution. This results from the fact that for the *a1949 Male Mortality Table* the random variable, time until death, for a life age 65 has a third central moment that is slightly negative.

3. *Probability inequalities*.—It seemed of interest to see what bounds could be obtained using general probability inequalities. The advantage of such bounds would be their independence of any specific assumption as to the structure of the distribution of total life annuity costs and possibly their ease of computation.

a) *Tchebychef inequality*. This well-known and very general inequality depends only on the mean and the variance of the random variable. Therefore, it is not surprising that it produces rather high bounds.

b) *Uspensky inequality* [12, p. 198]. This inequality has been attributed to various authors. It is a one-sided inequality which, like the Tchebychef, depends only on the mean and the variance of the random variable and which produces a slight improvement over the Tchebychef. In searching for general inequalities that might produce lower bounds than the Tchebychef, we also tried the Bernstein inequality and its various improvements made by Bennett [1] and discussed by Hoeffding [5]. The Bernstein inequality depends on the mean, variance, and bound on the absolute value of the component random variables ($I_{10}a_{44}$ in our case). Unfortunately, because the bound in our example is large relative to the variance, the Bernstein inequality produced higher bounds for the probabilities in which we were interested than did the Uspensky inequality.

c) *Fourth-moment inequality*. An obvious approach to further shrinking the bound in our probability statement would be to incorporate moments higher than the second in our inequality. Using a standard fourth-moment inequality [3, p. 256], some of the desired reduction was achieved. However, because of the complexity of computing the fourth moment for a sum of independent random variables, it did not seem attractive to go ahead and use inequalities of the same family depending on yet higher moments.

APPENDIX

PROOFS OF PROBABILITY INEQUALITIES

We will present here short proofs, applicable to our illustration, of the inequalities that are used in Table 1 and which may not already be familiar to actuarial readers. The proofs will be of the following type. We let $g(X)$ be a nonnegative function of the random variable X , which has a probability density function $f(x)$, and we let the set $S = [x:g(x) \geq c]$. Then

$$E [g (X)] \geq c \cdot \sum_S f (x) = cPr [g (X) \geq c] .$$

An elementary application of this inequality in actuarial science is immediate if we let $g(X) = a_X$, and $f(x) = {}_x|q_{x_0}$, $x = 0, 1, 2, \dots$, and zero elsewhere. Then

$$a_{x_0} \geq a_{\bar{y}} Pr (a_{\bar{x}} \geq a_{\bar{y}}) = a_{\bar{y}} Pr (X \geq y) = a_{\bar{y}} p_{\bar{y} : x_0} , \quad y \geq 0 .$$

Uspensky. We let X be a positive random variable,

$$E (X) = m , \quad E (X^2) = v , \quad g (X) = \left[1 + \frac{m (k - 1) (X - k m)}{v + k^2 m^2 - 2 m^2 k} \right]^2 ,$$

$$k > 1 , \quad c = 1 ,$$

and $S = [x:g(X) \geq 1]$. Then, following the pattern outlined above, we have

$$Pr (X \geq k m) \leq Pr [g (X) \geq 1] \leq E [g (X)] = \frac{v - m^2}{v + k^2 m^2 - 2 m^2 k} .$$

Fourth moment. We let X be a positive random variable,

$$E (X^2) = v , \quad E (X^4) = t , \quad k > 1 , \quad c = 1 ,$$

$$g (X) = \left[1 + \frac{v (k - 1) (X^2 - k v)}{t + k^2 v^2 - 2 v^2 k} \right]^2$$

and $S = [x:g(X) \geq 1]$. Then, once more following the pattern previously outlined, we have

$$Pr (X \geq \sqrt{k v}) \leq Pr [g (X) \geq 1] \leq E [g (X)] = \frac{t - v^2}{t + k^2 v^2 - 2 v^2 k} .$$

In our application, the role of X is played by

$$\sum_{j=1}^{10} I_j a_{\overline{T_j}} / \sum_1^{10} I_j .$$

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