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OPERATIONS RESEARCH IN INSURANCE: A REVIEW

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ABSTRACT

Operations research methods have been applied to the modeling and the solution of numerous problems in insurance and actuarial science. This paper reviews the applications of these operations research methods in the insurance industry. The paper is organized according to the categorization of operations research methods. Specifically, various mathematical programming models and their applications are first introduced. Game theory and some new operations research approaches are discussed, along with their applications in insurance and actuarial science. The paper concludes with a general discussion of developments and trends in operations research and insurance.

For the student who has studied specific operations research techniques mandated by the SOA examination system, this paper provides a set of examples of techniques pertinent to actuaries and shows how the expanding field of general quantitative reasoning in risk management can have a positive impact on the insurance industry. For research actuaries, we finally present an updated bibliography of operations research applications in insurance cross-classified by authors, operations research methodologies, and insurance areas of application.

1. INTRODUCTION

Operations Research (OR) models have been formulated to solve a wide variety of problems in the insurance industry. In this paper, we review some insurance industry applications of quantitative reasoning techniques, often known as OR methods. Many of these techniques are studied by actuarial students in a non-insurance context (SOA Exam 130), and so a description of actual insurance applications can provide a useful addition and motivation for the educational process.

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Early overviews of OR in insurance were presented by Zubay [291], who discussed the feasibility of applying OR methods to the insurance industry. Wade et al. [279] presented an excellent annotated bibliography, which was published by the McCahan Foundation. Denenberg [98] provided a review of OR in insurance, and Jewell [151] provided another excellent survey. In addition, early on (in 1960s and 1970s) the Society of Actuaries published a series of insightful discussions on the potential usefulness of OR techniques in actuarial and insurance areas [102]. Since then, Jewell [155] and Shapiro [255] have provided updated surveys. More recently, Haehling von Lanzenauer and Wright [133] presented a very useful overview of the interface of OR and insurance in the broader context of risk management with a unique feature of explicitly dealing with the decision problems by insureds.

Most of the previous review papers were organized according to application in the insurance industry. Shapiro [255], however, presented the material according to specific OR methodologies. This paper is also categorized according to OR areas in an attempt to provide an updated overview of both new and classical OR methodologies and their applications in insurance. This paper is intended to be more technique-oriented, an approach that is consistent with how actuarial students in North America study OR. This survey is thus intended to supplement and motivate the material learned by actuarial students while providing convenient reference for the professional. In particular, actuarial students who have studied OR techniques from non-actuarial textbooks will find herein many applications of OR methods to insurance and finance. Accordingly, the relevance of the OR examination material to actuarial science research and practice is reinforced by the paper.

In addition, the insights gained by using general quantitative reasoning to address problems in risk management will become more apparent to the actuary, who is most responsible for implementing mathematical techniques in the insurance industry. Some mathematical formulations are illustrated, and the connections among various approaches are discussed. In addition, some new OR approaches are explored. Both modeling techniques and computational aspects are briefly considered. Our intent is to deliver a review of deterministic methods in insurance and actuarial science without formally discussing most probabilistic models and stochastic process models such as queuing processes of claim arrivals, and so on.[1]

[1]We recognize that uncertainty is the *raison d'etre* of risk and the business of insurance. We have chosen to limit our coverage of stochastic methods primarily to conserve space and to follow more directly the OR methods studied by actuarial students without immediately apparent applications in risk management. Several methodologies presented do, however, have stochastic components and

The paper is organized as follows. Various mathematical programming models and their applications in insurance are presented in the next section. In Section 3, some new OR methods (such as data envelopment analysis, expert systems and neural networks) together with game theory and their applications are illustrated. Next, conclusions and discussions are presented. Finally, a detailed bibliographic reference of OR applications to insurance is given. To make this bibliography useful to both researchers and practicing actuaries, it is cross-referenced in three ways: by author, by OR technique, and by insurance functional area of applications.

2. MATHEMATICAL PROGRAMMING

A major research direction and practical application approach within OR is mathematical programming. Accordingly, the major part of this review paper is dedicated to various mathematical programming models in a variety of risk management and insurance applications. In the following eight subsections, we introduce the developed or promising insurance applications of general linear programming, nonlinear programming, integer programming, and five other special mathematical programming approaches: network optimization, goal programming, dynamic programming, chance-constrained programming, and fuzzy programming.

A general mathematical programming problem can be formulated as[2]:

$$\begin{aligned} \text{Maximize} \quad & f(x, y) \\ \text{subject to:} \quad & g_i(x, y) = 0, \text{ for } i = l, \ldots, p; \qquad (1) \\ & g_j(x, y) \leq 0, \text{ for } j = p, \ldots, p + q, \\ & x \text{ is a non-negative real-valued } n\text{-vector} \\ & y \text{ is a non-negative integer-valued } m\text{-vector.} \end{aligned}$$

It is not difficult to show that the non-negativity restriction on the vectors x and y can be made without loss of any generality.[3] Notice also that the

statistical content. We also recognize that simulation methods are an important topic for actuaries using quantitative reasoning; however, we do not expand this subject here in the interests of space and because of the availability of other sources for the information.

[2]Throughout we use boldface lowercase letters to represent vectors and non-boldface subscripted letters to represent the components of the vector.

[3]To see this, consider the two cases, that is, negative variables and (upper and lower) bounded variables. If $x \leq 0$, then let $y = x$ and substitute $-y$ for x in the various functions to obtain a standard

general formulation above can encompass both minimization and maximization problems, since minimizing an objective function is equivalent to maximizing the negative of the objective function. It is also easy to transform a "greater than or equal" into "less than or equal to" inequality constraint by simply multiplying the inequality by -1. This confirms that Formulation (1) indeed encompasses the most general mathematical programming models. Accordingly, in this paper, we use either maximization or minimization interchangeably without explicit explanation.

In subsequent sections, we see how added restrictions on the objective function or on the constraints and variable domains generate the specific type of mathematical programming problems; this is made clear in the individual subsections. The order of these subsections is as follows. Because LP is the basis of many other mathematical programming approaches, linear programming (LP) is presented first. General nonlinear programming (NLP) is presented next. Finally, integer programming (IP), network optimization (NO), goal programming (GP), dynamic programming (DP), and chance-constrained programming (CCP) are introduced, and last, fuzzy programming (FP) is presented.

2.1 Linear Programming

For the general mathematical formulation (1), a linear programming problem is obtained when the objective function and the constraints are all linear in the unknown variables. Hence, a linear programming problem can be expressed as follows:

$$\text{Maximize } {}_{c}T_{x} \tag{2}$$

$$\text{subject to } \mathbf{Ax} \leq \mathbf{b}, \text{ and } \mathbf{x} \text{ is a real-valued vector.}$$

where $\mathbf{A}$ is an $m \times n$ real-valued matrix, $\mathbf{b}$ is an m-dimensional real-valued vector, and $\mathbf{c}$ is n-dimensional real-valued vector; that is, we are maximizing a linear function subject to linear inequality constraints.

We now present one illustration of LP in the insurance industry, a linear programming method for measuring the cost of whole life insurance (Schleef

formulation. If $x \in [l,u]$, then first, let $x' = x - l$, so that $x' \in [0, u-l]$. The second step is to introduce two non-negative and unbounded variables, y_1 and y_2, and let $x' = y_1 - y_2$. We then obtain two non-negative and unbounded variables and standard formulation by substituting $y_1 - y_2$ for x' and introducing the constraints $y_2 \leq y_1$ and $y_1 \leq u - 1 + y_2$.

[250]).[4] Compared to more traditional methods (the measure of interest adjusted surrender cost method and Linton's rate of return), the linear programming method requires fewer assumptions, because the only input required is the rate of return that is relevant to the policyholder. The method does not attempt to directly separate the protection and savings components of the whole life policy. It assumes that the insured individual requires a given level of protection and is not concerned with how the insurer breaks down the received premium into loading charges, reserves, and so forth. The method also has the additional flexibility of considering the time at which the insured requires protection. The flexibility of varying the year of required protection is the primary characteristic of the LP method that differentiates it from the more traditional methods.

In the LP formulation, the three types of decision variables are the amount w_t lent externally by the insured at the beginning of year t, the amount z_t borrowed externally by the insured at the beginning of year t, and u, the face value of insurance purchased at the time t=0. It is assumed that the rate of return, i, or borrowing and lending rate are the same (although this could be relaxed in the LP formulation), so only the net position $(w_t - z_t)$ appears in the formulation. The objective function is to maximize the discounted cash flows associated with a given policy, which is constrained by the amount that the insured is willing to budget for insurance, and the amounts of protection required in each year to the horizon. The linear programming formulation is shown below:

$$\text{Maximize} \quad (1 + i)^{-n} C_n u + \sum_{i=1}^{n} (1 + i)^{-(t-1)} (w_t - z_t)$$

$$\text{subject to} \quad P_t u + w_t - z_t \leq b_t, \text{ for } t = 1, \ldots, n;$$

$$u + \sum_{t=1}^{j} (1 + i)^{-(t-j)} (w_t - z_t) \geq I_j, \text{ for } j = 1, \ldots, n; \text{ and}$$

$$u, w_t, z_t \text{ are non-negative,}$$

where w_t, z_t, and u are decision variables; w_t is the amount lent externally by the insured at the beginning of year t; z_t is the amount borrowed externally by the insured at the beginning of year t; u is the face value of

[4]As Schleef [250] indicated, the LP model can also be applied to other types of life insurance such as term insurance and interest sensitivity products such as universal life.

insurance purchased at the time $t=0$; P_t is the net premium rate in year t; C_t is the cash-value rate at the end of year t; b_t is the amount budgeted by the insured at the beginning of year t; I_t is the insurance protection required at the beginning of year t; and n is the number of years in the planning period.

From this primal linear programming model, the dual linear programming model is obtained.[5] By using the "shadow price" interpretation of the dual parameters corresponding to the constraints in the primal problem, the dual LP model can be used to analyze the marginal discount factors for each year and the marginal discounted cost of increasing the death benefit requirement in each year.

Linear programming is a very general category programming problem. As shown later, many goal programs, integer programs, and network flow models can be formulated as linear programs. Hence, further applications of linear programming are discussed in separate subsections. There are many other interesting applications of LP to insurance. For example, Chan et al. [57], Schuette [253], and Hickman [141] provide theoretical discussion and formulation of LP approaches to graduation.

Financial management is another mature area in insurance and actuarial science in which the LP method has been widely used. Hofflander and Drandell [144], for example, use a linear programming model to discuss profitability, capacity and regulation problems in insurance management. Schleef [249] uses a linear programming model for decision-making in life insurance purchases.

Conwill [79] develops several linear programming models for maximizing policyholder value in problems of making combined decisions of life insurance product purchasing and asset investment. In his long paper, Conwill discusses the techniques used in building linear programming models for insurance problems, the computational issues involved in solving the linear programming problems, and the interpretation of the results produced from computation.

Haehling von Lanzenauer et al. [130] show how to formulate the problem of developing a manpower planning policy as a linear programming problem. Linear programming is also suggested by Jennergren [150] for use as an asset valuation method. Navarro and Nave [211] use linear programming for dynamic investment immunization problems. Indeed, there are many

[5]We recommend Hillier and Lieberman [143] for further reading about the definition of dual programming, how a primal linear programming transformed to its dual LP, what the relationship between the optimal solution to the primal and that to the dual is, and how one can interpret the dual (what the economic interpretation is).

applications of LP methods for problems in financial areas such as capital budgeting, portfolio management, duration matching, and immunization. These applications are also of substantial interest to actuaries and to insurance management.

2.2 Nonlinear Programming

In the general mathematical programming Formulation (1), nonlinear programming encompasses the least restrictive set of attributes imposed. In the general nonlinear programming model, the variables are free to be either real-valued or integer-valued, and the objective or the constraints can involve nonlinear functions. Hence, linear programming is actually a special form of nonlinear programming. Compared to linear optimization problems, however, nonlinear programming suffers more serious disadvantages. Nonlinear functions are typically more difficult to specify. In addition, although some special classes of nonlinear programming (such as convex programming) can be optimally solved, in nonlinear optimization there may be a multitude of local optima of the nonlinear objective function. Accordingly, a serious problem in nonlinear optimization is that commonly used solution methods such as Newton-Raphson techniques may find local rather than global optimal solutions. Occasionally, given the search technique used, the global optimum cannot be found within a reasonable time limitation.

In this section, rather than pursuing further discussion on general nonlinear programming, we describe two applications of nonlinear programming in the insurance industry: a quadratic programming model for insurance portfolio analysis (Markle and Hofflander [197]) and an information theoretic approach to mortality table graduation (Brockett [38]).

Among the early efforts at combining underwriting and investment into an integrated portfolio analysis is the work by Markle and Hofflander [197]. As an extension of the Markowitz portfolio model, their combined portfolio analysis indicates an efficient portfolio that may be relevant for insurers' financial decision-making. A similar philosophy is used in Crum and Nye [85], in which generalized network flow models are proposed to obtain optimal insurance and investment portfolios. Other extensions include that of Brockett, Charnes, and Li [40], who consider simultaneously the optimal selection of investment vehicles and insurance lines of business decisions for a casualty insurance company.

Markle and Hofflander's model is a quadratic programming model; that is, the objective function used is quadratic in the unknown decision variables of interest and the constraints involve only linear functions. The objective function involves maximizing the expected portfolio return, and this maximization is done subject to two types of constraints: institutional and regulatory solvency constraints, and accounting types of constraints. The decision variables they use are selected balance sheet variables representing the allocation of assets (including various bonds and stocks) and liabilities (including the premiums written in multiple insurance lines). After all the variance and covariance matrices within and between the asset and liability variables have been estimated, the objective function is written as a quadratic function in the decision variables. All the constraints are linear. With such a formulation, not only can the unique optimal portfolio be found (because the constraint set is convex), but also a sensitivity analysis can be conducted of the effect of changes in the regulatory solvency constraints on the optimal expected portfolio return. Such a quadratic programming formulation for portfolio analysis is widely used in finance and investment literature. In addition, the algorithms for solving quadratic programming problems are also efficient and commercially available for easy use.

We next introduce another application of nonlinear programming of interest to life actuaries, namely, mortality table construction and graduation. The information theoretic methodology (Brockett and Zhang [46], Brockett [38], Brockett et al. [43], and Brockett et al. [45]) can be used for selecting statistical models for analysis when the true underlying distributions are unknown, which is typical of mortality table construction. The graduation of the mortality table using empirical data is a particular problem of interest. A complete discussion of the information theoretic methodology illustrated here can be found in Brockett [38].

Let the vector $\mathbf{u}$ denote the observed series of values $\mathbf{u} = (u_1, u_2, \ldots)$ that are to be graduated into a mortality table and the vector δ denote the resultant smooth or graduated series of values $\delta = (\delta_1, \delta_2, \ldots)$. The information distance between the observed series and the graduated series is defined as

$$I(\delta/\mathbf{u}) = \sum_i \delta_i \ln\left[\frac{\delta_i}{u_i}\right].$$

It represents a measure of closeness of the observed and graduated series with $I(\delta/\mathbf{u})=0$ if and only if $\delta=\mathbf{u}$.

The objective of the graduation process is to find a graduated series that is as close as possible to the observed series but satisfies certain constraints. Thus, the objective function is

$$\text{Min}_\delta I(\delta/\mathbf{u}) = \sum_i \delta_i \ln\left[\frac{\delta_i}{u_i}\right].$$

The first constraint is the non-negativity on the graduated series. Other constraints on δ occur because the true underlying pattern of mortality rates is (a) smooth, (b) increasing with age, that is, $\Delta\delta_x = \delta_{x+1} - \delta_x \geq 0$, and (c) more steeply increasing at the higher ends of the range, that is, $\Delta^2\delta_x \geq 0$. Additional constraints in the graduation process are that (d) the graduated number of deaths using δ equals the observed number of deaths using $\mathbf{u}$ and (e) the total of the graduated ages at death equals the total of the observed ages at death. The measure of smoothness is given by $\Sigma(\Delta^3\delta_x)^2 \leq M$, which can be formulated as a quadratic constraint $\delta^T \mathbf{A} \delta \leq M$, where the matrix $\mathbf{A}$ is:

$$\left|\begin{matrix} -1 & 3 & -3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -3 & 1 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right|.$$

The constant M determines the degree of smoothness obtained for the graduated series.

The resulting convex programming model can be solved by using general nonlinear programming codes such as generalized reduced gradient algorithm (GRG2) (Lasdon [173] and Lasdon et al. [174], and [175]).

As mentioned in Brockett [38], other objective functions could be used for graduation, and the quadratic one associated with Whittaker-Henderson graduation provides a case in point. Illustrative of using a mathematical programming approach to the Whittaker-Henderson graduation is the paper of Lowrie [190]. Chan et al. [57] also show that the problem of minimizing the Whittaker-Henderson objective function $F_p(\mathbf{u}) + \lambda S_q(\mathbf{u})$ over $\mathbf{u} \geq 0$, using the fit measure

$$F_p(\mathbf{u}) \equiv \sum w_x |u_x - u_x''|^p$$

and the smoothness measure $S_q(\mathbf{u}) \equiv \Sigma / \Delta^z u_x|^q$, $p \neq q$, can be formulated as a linear programming problem when $p=1$ and $q=\infty$, as a quadratic programming problem when either p or q is 2. These general graduation methods are all amenable to solution by using nonlinear programming methods.

2.3 Integer Programming

An integer program (IP) is obtained from the general mathematical programming Formulation (1), when the decision variables are restricted to being integers. Integer programming problem can involve nonlinear functions in its objective or constraints. For illustrative convenience, we introduce only integer LP.

Integer LP, as a special case of LP, arises in resource allocation or assignment problems, facility location, traveling salesman or vehicle routing problems, and many other combinatorial problems. Graph theory and integer programming are often interrelated. Many graph theory problems are formulated and solved as IP problems, whereas many IP developments are directly related to graph theory study. Because of the integer restriction, the solution obtained by simply rounding the corresponding real-valued LP solutions to integer values is often suboptimal. Accordingly, some other effective algorithms are required to solve integer LP problems.

There are many IP problems known for the NP-complete or NP-hard problem; that is, it is not very likely that polynomial bounded algorithms (in sense of the time required to find the optimal solution) will be found for these classes of problems. In spite of the prevalence of NP-completeness/ NP-hardness in IP, many efficient algorithms have recently been developed. Among them are the branch-and-bound search, the Lagrangian relaxation method, the subgradient technique, the decomposition method, and the constraint aggregation method. These algorithms have proven to be computationally successful for complex IP problems, although some of them achieve computation time efficiency by finding the satisfactory, but not necessarily optimal, solution. Several excellent books or papers specializing in integer programming are listed in the bibliography. In particular, for those readers who are interested in the complexity of algorithms in general, we suggest Garey and Johnson [116], while for those who are particularly interested in integer programming modeling, we recommend Nemhauser and Wolsey [212].

Applications of IP models abound in the insurance industry. As discussed in a later section, network flow models can be used in many situations such as financial planning, cash management, and so on. These network flow problems constitute a special case of IP problems under certain reasonable assumptions. In this subsection, we illustrate the application of IP to the problem of reorganizing the sale regions for a life and annuity insurance company (Gelb and Khumawala [118]). The first example in Section 2.5, Goal Programming, provides another integer programming illustration in insurance.

In 1982, a Houston-based company, Variable Annuity Life Insurance Company (VALIC), was interested in reorganization of its sales force of 336 individuals. At that time, there were 16 sales regions, each with a manager and regional office. The regions had evolved as combinations of 57 geographical segments involving states or portions of states. The reorganization plan investigation was specifically undertaken to determine a least-cost solution to determining the number of regions and the geographical configuration of the regions. It was also desirable to compute the cost savings that would be obtained if the suggested sales region configurations were adopted. Three constraints were imposed by the company: (1) the number of regions should not decrease; (2) the number of regions should at most double; and (3) disproportion in market potential should not be exacerbated. The primary task was to improve profitability by either increasing market potential or decreasing the costs incurred. This problem of reorganization of the insurance sales force was formulated by Gelb and Khumawala [118] as follows:

$$\text{Maximize} \quad \sum_{ij} C_{ij}x_{ij} + \sum_{i} F_i y_i$$

$$\text{subject to:} \quad \sum_{i} x_{ij} \geq D_j, \text{ for all } j;$$

$$\sum_{j} x_{ij} \leq S_i y_i, \text{ for all } i; \text{ and}$$

$$\text{for all } i,\ y_i = 1 \text{ if } x_{ij} \geq 0;\ y_i = 0 \text{ if } x_{ij} < 0,$$

where C_{ij} is the sum of variable costs (operating expenses and cost of lost sales) relating to the i-th regional office and j-th geographical segment; F_i the fixed costs of the i-th potential office; D_j the market potential of the j-th geographic segment; S_i the capacity of the i-th potential office; y_i the integer decision variable indicating the utilization (y_i=1) or nonutilization (y_i=0) of the i-th regional office; and x_{ij} the integer decision variable denoting the amount of market potential of the j-th segment to be served from the i-th regional office.

This formulation is a typical facility location problem. For an insurance company this model has utility not only for reorganizing existing regional sales force structure geography but also for designing a regional sales office configuration for an insurance company intending to expand its business geographically. Gelb and Khumawala [118] used a branch-and-bound implicit search procedure to solve the problem. The solution showed that if the total number of regional offices was allowed to increase from 16 to 25, the total cost could be reduced from \$18,826,000 to \$9,993,000, a saving of \$8,833,000.

The problem size, however, will expand with the number of potential offices. The branch-and-bound approach may not be able to handle the case of a very large number of potential offices because branch-and-bound techniques basically use an enumerated search approach. Although the search method is wisely designed to potentially reduce the search time substantially, in the worst case, the search time is an exponential function of the problem size (see Garey and Johnson [116] for the precise definition of problem size). For large IP problems, other algorithms such as the Lagrangian relaxation may be required.

Integer programming approaches to problem-solving have also been successfully applied in finance and other business areas. Many portfolio management problems (Faaland [109], and Nauss [209], [210]) and capital budgeting problems (Gonzalez et al. [126], Laughhaunn [176], and Pettway [224]) have been modeled as integer programming problems when assets are indivisible or projects must be adopted or rejected in their entirety. Insurance, in its role as a financial intermediary, has many other potential applications of integer programming. Two such applications of IP in insurance and finance are given in the next section.

2.4 Network Optimization

A network model is denoted by $G(N, A)$, where N is a set of nodes and A is a set of arcs, while G relates the network optimization to graph theory. Each arc $(i, j) \in A$ represents an ordered relationship between two nodes, $i, j \in N$. Thus a network is a directed graph. There are three main types of network flow problems: the shortest path problem (SPP), the maximum flow problem (MFP), and the minimum cost flow problem (MCFP). In fact, both the SPP and MFP are special cases of MCFP; hence, below we give only the general network formulation for a minimum cost flow model. We refer readers to an excellent book by Ahuja, Magnanti, and Orlin [5] for thorough discussion on network flows.

A general network formulation is as follows:

$$\text{Maximize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{\{i:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} p_{ji}x_{ji} = b(i), \text{ for } i \in N; \text{ and}$$

$$0 \le l_{ij} \le x_{ij} \le u_{ij}.$$

where $\{j:(j,i)\in A\}$ denotes the set of nodes j that have an arc leading to node i, while $\{j:(i,j)\in A\}$ denotes the set of nodes j to which an arc originating from node i. For one explanation of the formulation given above, x_{ij} represents the amount of flow from node i to node j; p_{ij} is the transmitting efficient rate of the arc (i, j), which is usually less than or equal to 1; $b(i)$ is the extra demand or excess supply of node i; and c_{ij} represents the benefit of unit flow on arc (i, j) because of the maximization of the objective function.

Networks are pervasive. They arise in numerous application settings and in many forms. Physical networks are perhaps the most common and the most readily identifiable networks. Network flow problems, however, also arise in surprising ways for problems that, on the surface, might not appear to involve networks at all. Sometimes the nodes and arcs have a temporal dimension that model activities that take place over time. Many scheduling applications have this flavor. In any event, network models arise in a wide variety of problems in project management; equipment and crew scheduling (say, claims adjusters or auditors); location layout theory; warehousing and distribution; production planning and control; and social, medical, and financial contexts.

Network flow models have also been used in the insurance industry. In this paper, we present a class of network models applicable to insurance and investment portfolio management.

Our first illustrative application of network flow models is project management. As early as 30 years ago, Zubay [291] suggested potential applications of network models in the insurance industry. In this section, we show three basic models of project management: determination of minimum project duration, just-in-time scheduling, and the time-cost trade-off project scheduling problem.

For an application of network methodology to project scheduling, suppose we are given a set of jobs required to complete a project (for example, a new rate filing or policy filing case). We are also given the order in which the jobs are to be done, as certain jobs must proceed others, while other tasks can be accomplished simultaneously. These constraints on the order in which the jobs can be done are known as the precedence relationships. The objective is to determine the minimum project duration, that is, the least possible amount of time needed to complete the entire project. This problem is a typical shortest path problem. Let $u(i)$ and $u(j)$ denote the earliest possible start times for job i and j. Then the problem has the following formulation:

$$\begin{aligned} &\text{Minimize} && u(t) - u(s) \\ &\text{subject to} && u(j) - u(i) \geq c_{ij}, \text{ for } (i, j) \in A; \text{ and} \\ &&& u(j) \text{ unrestricted in sign for all } j \in N, \end{aligned}$$

where nodes s and t represent the starting point and the finishing point, respectively; A is the set of precedence relationships, and $c_{ij}=u_{ij}$ represents the time duration of job i.

In the previous formulation, there were no restrictions on the variables except for the precedence constraints. In some cases, however, certain jobs in the project might have an absolute time restriction; that is, a job must be started within a specified time limit from the start of some precedent jobs, for example, constraints on the time available after notice of a claim to make payment or the deadlines for rate filing. The objective in this case is still to minimize the entire project duration. The so-called "just-in-time scheduling" problem is an extension of the previous formulation with the additional class of constraints: $u(j) \leq u(i) - \alpha_{ij}$, for all $(i, j) \in A$, where α_{ij} means that job j must start within α_{ij} units of time from the start of job i. The "just-in-time scheduling" problem can also be formulated as a minimum cost flow model.

In some circumstances, the durations of jobs can be reduced by allocating extra resources (manpower, equipment, or money) to them; that is, there are time-resource trade-off curves on certain jobs. If the curves are linear, then the dual program of the primal linear program is the minimum cost flow problem.

As mentioned previously, network flow models are well suited to the situations in which there is a set of entities and flows of some sort between entities. The transportation problem, with minimum cost as the objective, and the traffic light control problem, with maximum flow per unit time as the objective, illustrate classical network flow problems. The best known flow-type problem in insurance involves the flow of cash or funds between suborganizations of an insurance firm and between the insurance firm and other sources or uses of funds. For example, Crum and Nye [85] designed general network flow models for three operations of a multiple-line property-casualty insurance firm: insurance portfolio operations, investment portfolio operations, and the capital acquisition operations. We introduce the first network model and refer readers to Crum and Nye [85] for the other two case studies.

In insurance network flow models, there are four types of nodes: the nodes representing the cash balance, the nodes designating the insurance lines of business, the nodes representing existing claims, and the nodes representing new claims. A network model of the insurance portfolio of an insurance company with two lines of business and spanning three time periods is shown in Figure 1 (see Crum and Nye [85] for the original work).

Corresponding to four types of nodes, there are four cash flow equations balancing dollars in to dollars out of the nodes (these are similar to the equations of balance in asset share calculations and theory of interest). These equations[6] are:

$$\sum_{i=1}^{m} INS_{ij}(1 - UE_{ij}) - \sum_{i=1}^{m} \sum_{k=1}^{j} CP_{ijk} + \sum_{i=1}^{m} PPC_{ij} + CB_{j-1,j} - CB_{i,j+1} = ICB_j,$$

for $j = 1, \ldots, n + 1$; $k = 1, \ldots, n$;

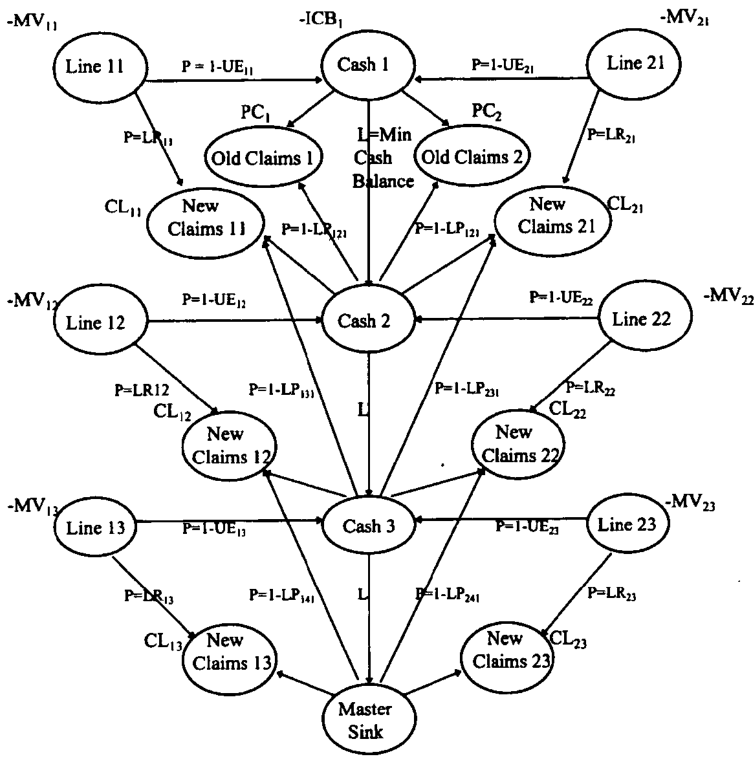
$$-INS_{ij} - CR_{ij} = MV_{ij}, \text{ for } i = 1, \ldots, m, j = 1, \ldots, n;$$

$$\sum_{j=1}^{n} (1 - LP_{ij})PPC_{ij} = PC_i, \text{ for } i = 1, \ldots, m; \text{ and}$$

[6]In the original paper, the second term of the first equation is: $\Sigma_{i=1}^{m} \Sigma_{k=1}^{n} CP_{ijk}$, we think the range for k should be $[1, j]$ instead of $[1, n]$.

FIGURE 1

INSURANCE PORTFOLIO OPERATIONS (TWO LINES AND THREE PERIODS)*



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$$\sum_{k=1}^{n} (1 - LP_{ijk})CP_{ijk} + LR_{ij}CR_{ij} = CL_{ij},$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

where i is the index of business lines; j the index of time periods; k the index of (time) steps; INS_{ij} the insurance actually sold; UE_{ij} the underwriting expense ratio; CP_{ijk} the dollar amount paid for claims; $CB_{j,j+1}$ the cash balance carried over between periods; MV_{ij} the maximum volume of insurance; CR_{ij} $(=MV_{ij}-INS_{ij})$ the additional insurance that could have been sold; ICB_j the initial cash balance; LP_{ij} the penalty cost incurred from payment delay; LR_{ij} the loss ratio; PC_{ij} the amount of claims; PPC_{ij} the payments in a period to satisfy claims PC_i; and CL_{ij} the maximum levels of claims, $CL_{ij}=LR_{ij}*MV_{ij}$.

In these four equations, summations preceded by a negative sign represent cash outflows from the node, and positive coefficient terms represent cash inflows to the node. Crum and Nye [85] gave an illustrative example of an insurance portfolio network that contains two lines of business over three time periods. The objective function is expressed to maximize the value of the firm after all incremental capital acquired by the model has been repaid. This can be shown to be equivalent to maximizing the value of the existing equity—the appropriate objective for a public corporation.

The investment portfolio problem can be similarly formulated as a minimum cost flow problem. Combined with the investment portfolio network, the whole network model represents flow of funds over time for the multiple-line insurance company. Given the maximum period premium volume, the initial asset amount, the investment choices, and some other parameters, the network flow model is able to find the optimal portfolios for the company. However, the decision of a major insurance company depends upon how the company acquires external capital such the capital structure (for example, debt-equity composition) and how well the company manages its assets and liabilities. Hence, Crum and Nye used network models to formulate the capital acquisition problem and investment portfolio problem in addition to the above-cited model for the insurance portfolio problem. These three models combined with the objective function form the complete model for a major insurance company. Such an insurer can either be a multiline property and casualty company or a life company including life insurance products and annuities.

To further illustrate the usefulness of network flow models in finance and insurance, we briefly discuss how network optimization might be used to find arbitrage opportunities in currency exchanges for a multinational insurance company. Suppose an American company is the target company and dollar is the target currency; that is, the company has an amount of excess cash that can be used for currency exchange. There exists a set of foreign currencies available for trading. A network model for this problem, as usual, consists of a set of nodes and a set of arcs. The nodes represent currencies, one node for each currency. The arcs represent the possible exchange between the two currencies with an exchange rate attached to the arc. The problem is simply to increment the amount of dollars by finding optimal exchanging paths and amounts. To implement the model, a source node and a sink node are artificially added. The source node has an initial excess of capital (the amount to be invested). An arc connects the source node to the dollar node. Another arc connects the dollar currency node to the sink node. The problem is then transformed into a maximum flow problem (that is, the company is trying to maximize its current dollar holdings by circulating currencies within the market). Clearly, if the exchange rates are spot rates and are fed into the model in real time, the model can be integrated into the company's whole financial management system. Under different specifications, the model can be extended to currency swap or interest swap problems. We refer interested readers to Kornbluth and Salkin [169] for more details.

2.5 Goal Programming

Charnes and Cooper [60] first provided the foundations of goal programming and developed a strategy that is capable of handling multiple, incompatible and/or incommensurable goals. In a typical goal programming model, each goal is formulated as a constraint. There are two variables associated with each goal (each constraint): overachievement deviation and underachievement deviation. The value of these two deviational variables measure how well the corresponding goal is accomplished. For example, if both deviational variables in the final solution to the goal programming model are close or equal to zero, the corresponding goal is well achieved. However, the two deviational variables of each goal cannot be zero at the same time in a feasible solution since it is unusual to have a goal overachieved and underachieved simultaneously. After two deviational variables have been assigned for each goal, the next essential step in setting up a goal programming model is to build an objective function. An objective function in a goal programming formulation is usually a linear function in deviational variables. Specifically, the objective function takes the weighted summation of the deviational variables. The weights assigned to a deviational variable indicate the importance of the corresponding goal in the decision-making process. The objective is thus to minimize the weighted sum of deviations from goal achievement, that is, to accomplish the best overall achievement.

The goal programming method has been extensively applied to problems in management and finance. In the insurance industry, Klock and Lee [164] suggested a goal programming model for an insurance company with profit, current asset returns, and legal bounded goals, while Drandell [103] demonstrated that the goal programming model developed is equivalent to the original linear programming model of optimum allocation of assets. O'Leary and O'Leary [214] also used goal programming to address a problem faced by the financial and personnel departments in many firms: choosing an investment manager.

To further demonstrate the utility of goal programming in the insurance industry, we introduce two applications: capital budgeting in an insurance company (Lawrence and Reeves [179]) and insurance agency management (Gleason and Lilly [123]).

Given a set of projects, such as investment projects for an insurance company and given a set of multiple objectives for these projects (multiple strategies), the capital budgeting problem is to determine which particular projects should be selected in each given time. Specifically, Lawrence and Reeves [179] formulated the above problem utilizing seven objectives or goals that are desired to be met to the extent possible:

(1) Achieve at least a certain minimum level of project rate of return
(2) Do not exceed a certain maximum level of anticipated penalty cost associated with project lateness
(3) Achieve at least a certain minimum level of additional premiums
(4) Achieve at least a certain minimum level of additional agents' earnings
(5) Do not exceed budget
(6) Achieve at least a certain minimum level of social responsibility and
(7) Do not use more than maximum level of resources.

The decision variables, x_{ij}, associated with each potential project are an indicator variable of whether each individual project i is selected in time period j. Accordingly, these decision variables are zero-one (integer) variables. For a three-time-period horizon capital budgeting problem, 21 pairs of deviational variables (d^-, d^+) arise in this seven-goal, three-period setting. The seven types of constraints corresponding to the seven goals are duplicated in order below, followed by certain non-goal (system-characterizing) constraints:

Project rate of return: $$\sum_{i=1}^{7} \sum_{j=1}^{3} R_{ij}x_{ij} + d_1^- - d_1^+ = TR;$$

Project lateness penalty: $$\sum_{i=1}^{7} \sum_{j=1}^{3} PL_{ij}x_{ij} + d_2^- - d_2^+ = TPL;$$

Additional premiums: $$\sum_{i=1}^{7} AP_{ij}x_{ij} + d_{2-j}^- - d_{2+j}^+ = TAP_j;$$

for $j = 1,2,3$;

Additional agents' earnings: $$\sum_{i=1}^{7} AE_{ij}x_{ij} + d_{5-j}^- - d_{5+j}^+ = TAE_j;$$

for $j = 1,2,3$;

Budget: $$\sum_{i=1}^{7} AE_{ij}x_{ij} + d_{8-j}^- - d_{8+j}^+ = TB_j;$$

for $j = 1,2,3$;

Social responsibility: $$\sum_{i=1}^{7} \sum_{j=1}^{3} B_{ij}SR_{ij}x_{ij} + d_{12}^- - d_{12}^+ = TSR;$$

Resources: $$\sum_{i=1}^{7} h_{ijk}x_{ij} + d_{9-3j-k}^- - d_{9+3j+k}^- = H_{jk};$$

for $j = 1,2,3$, for all k,

Non-goal constraints: $$\sum_{i=1}^{3} (x_{1j} + x_{3j}) \geq 1; \text{ and}$$

$$x_{ij+1} - x_{kj} \leq 0.$$

where R is the forecast rate of return of project in a certain period; PL the expected penalty cost of project lateness; AP the forecast level of additional premiums that will be written in a period; AE the expected amount of additional agents' earnings associated with the selection of the project in a certain period; B the projected cost of the project in a period; and SR the percentage of the project directly associated with matters of social responsibility and public service (for example, special inserts on household safety, auto care, and so on). Resources include manpower, computer system running time, and so on, and h is the number of work days of a type of resource required to complete the project in a period. Accordingly, the right-hand side of each equation is the target, that is, the goal. For example, TR for the first equation is the target (total) rate of return; TAP_j is the total (target) additional premium in year j; and so on. The subscripts, $9+3j+k$, for some of the deviational variables can be read as follows: when $j=1$ and $k=1$, $9+3j-k=13$, which is the subscript for the human resource goal in the first period; when $j=3$ and $k=3$, $9+3j+k=21$, which is the subscript for the very last (human resource) goal. It is not surprising to see conceptually that because of the indivisibility of projects, the model is an integer programming formulation.

The first illustrative non-goal constraint represents the dependency between two projects, for example, at least one of project 1 and 3 should be selected over the total time period. The second non-goal constraint specifies that project i cannot begin before project j is complete and is thus able to be used to model multiyear projects. Many other constraints can be added depending upon desired system requirements. The objective function is given below:

$$\text{Maximize} \quad Z = P_1 d_1^- + P_2 d_2^+ + P_3 \sum_{j=3}^{5} d_j^- + P_4 \sum_{j=6}^{8} d_j^- + P_5 \sum_{j=9}^{11} d_j^+ + P_6 d_j^- + P_7 \sum_{j=13}^{21} d_j^- ,$$

where $P_k \geq 0$, $k=1, \ldots, 7$, are the preemptive priorities associated with the objectives; that is, those priorities are determined a priori, with higher value meaning greater importance or contribution of a specific goal to the overall decision problem. Obtaining appropriate assignment of weights to the goals is based on the preference of the decision-maker as well as on common sense. For example, since higher rather than lower rate of return is usually favored by financial decision-makers, the reasonable objective is to minimize underachievement in rate of return. That is why the first term appears in the objective (minimization) and the overachievement deviational variable disappears. We leave readers to determine the meaning of other terms in the objective function.

With such a model, the optimal solution is obtainable for each given set of parameters, such as the projected rates for return and expected penalty costs. Also, by adjusting the priorities associated with goals, a sensitivity analysis on the trade-offs between different goals is obtainable.

The above-mentioned goals are neither all-encompassing nor constant among insurance agencies. One goal of an agency might be to increase the amount of premiums written. Income maximization is a concern to an agency in the long run, while growth may be more important in the short run. An agency may have the incentive to represent as many insurers as possible. On the other hand, overdependency on a single insurer might translate into a higher business risk, whereas receiving more services from the insurer reduces business costs. An agency also has the option to specialize in different insurance business lines. For example, independent agencies tend to concentrate in commercial lines of insurance and deemphasize personal lines. In summary, an agency may have multiple business goals, and these goals can be conflicting and compatible. For this reason, Gleason and Lilly [123] modeled the agency operation by using a goal programming problem.

Gleason and Lilly considered six goals grouped into four levels of priorities. The priority 1 goals are: "expand premiums written" and "expand the number of insurers the agency represents." The priority 2 goal is "do not become too dependent upon any single insurer." Two goals are categorized as priority 3 goals: "obtain cost reduction services from the insurers" and "maximize gross income." The lowest priority goal is "to shift from personal to commercial lines."

Each of the six goals is formulated as a goal constraint. Some additional constraints arise from practical agency operation limitations: for example, the annual growth rate is limited to no more than 20%; the business in certain lines might be restricted; and so on. With minimizing the total overachievements and underachievements of the goals as the objective, the goal programming problem is well formulated. The decision variables in this problem are the amount of premiums for each insurance class to be written by the agency using each insurer. In this application then, the decision variable is not required to be integral. Thus, this goal programming can be solved as a linear programming problem.

Goal programming can also be used to model working capital management problems (Satoris and Spreill [247]). Many other risk management problems, such as those involving environmental pollution management (Charnes et al. [63], [65], and [77]) and those involving senior-level decision-making, such as company mergers and acquisitions, can also be modeled and solved as goal programming problems.

In summary, a goal program can be favorably used when there are multiple competing goals involved in the decision-making. Although it is not difficult to set a target for a specific goal and then transform the goal formulation to a constraint, the spirit of goal programming lies in utilizing the deviations from the goals in the objective function formulation, assigning subjective weights to each such deviation, and then minimizing the total weighted deviation. These deviations are essentially treated as decision variables when the resultant linear or nonlinear programming problem is solved computationally. Controversies can arise from the goal weighting strategy, that is, the way that weights are determined and the rationality of the concept that different goals are equalized by assigning quantitatively different weights.

2.6 Dynamic Programming

Dynamic programming is another general approach to problem-solving. In general, the problems that dynamic programming are capable of handling have several basic features. The problem contains a series of stages either physically or conceptually, for example, over time periods or over conceptual stages. In each stage, a prespecified set of states represents the potential outcomes at this stage. A policy decision can cause changes in states from stage to stage. The likelihood of being in any specific state in the next stage is completely determined by the current stage and the policy decisions made within it, but is independent of the states that might have occurred in previous stages. This is the Markov property, which is often assumed to simplify the modeling process. The effects of transitions from state to state over stages are quantified in utility or cost form, and the objective of the problem is to determine a series of (possibly state- or stage-dependent) policy decisions that maximize the total or final utility or that minimize the total or final cost.

Usually, a recursive relationship is developed to solve the dynamic programming problem. The recursive relationship is a formula describing how a state in the subsequent stage is determined from the states in the current stage. If the state in the next stage is determined with certainty by the state in the current stage (together with the adopted policy decision), then the dynamic programming is called deterministic. If the states in the subsequent stage are determined according to some probabilistic distribution (which will generally depend upon the value in the current stage and the adopted policy), then the dynamic programming is called probabilistic. The solution procedure in either case involves first determining the optimal decision strategy in the final stage without recourse to previous stages. From this optimal decision strategy in the final period, the derived recursive relationship is used to derive an optimal solution in the next-to-the-last stage. This backward calculation of optimal decision strategy is continued until the optimal strategy for the current stage is derived.

In this paper, we illustrate this technique with a dynamic model of insurance company management [115]. The model deals only with determining the optimal policy of dividends for a stock insurance company over time. It is assumed that the objective of the company is to maximize the expected utility of the dividend payments, which is calculated according to the distribution of claims. The utility function of dividend payments and the distribution of claim arrivals is not explicitly specified in Frisque, so this dynamic model is a general theoretical construction.

Let $U(d_1, d_2, ..., d_j, ...)$ represent the shareholder's current utility or order of preferences corresponding to the systems of dividend payments $(d_1, d_2, ..., d_j, ...)$, where d_j denotes the payment of dividend made by the company during year j. Also, to account for the time value of consumption in different periods, it is assumed that U is of the following form:

$$U(d_1, d_2, ..., d_j, ...) = u(d_j) - vU(d_2, d_3, ..., d_j, ...),$$

where $u(d_1)$ denotes the shareholders' one-period utility function and v is a factor expressing the shareholders' preferences for an early dividend, $0<v<1$. Thus,

$$U(d_1, d_2, ..., d_j, ...) = \Sigma_j \, v^{j-1} \, E[u(d_j)],$$

assuming that utility function is time additive. Let S_j denote the expected reserve level of the company at the beginning of year j, and assume the reserve dynamics follows the equations of balance,

$$S_j + 1 = S_j - d_j + k_j \left(P - \int_0^\infty x dF(x)\right),$$

where $F(x)$ is the distribution of claims; k_j is the part of the insurance portfolio retained by the company in a quota reinsurance system; and P is the amount of premiums received during year j. By introducing a function $U_j(S_j)$ as the discounted average utility of the dividends d_j, d_{j+1}, ..., evaluated at decision points $j, j+1, \ldots$ when initial reserve is S_j, an optimal policy is followed with respect to payment of dividends in all subsequent periods. Specifically,

$$U_j(S_j) = \underset{0 \le d_j \le S_j, 0 \le k_j \le 1}{\text{Maximize}} \left[u(d_j) \div v \int_0^\infty U_{j+1}(S_j - d_j + k_j(P - x))dF(x) \right].$$

$$\text{subject to} \int_0^\infty (S_j - d_j + k_j (P - x))dF(x) \ge 0$$

Thus we obtain a dynamic programming model in which the decision variables are the series of dividend payments, d_j, and also the series of reinsurance fractions, k_j. A more practical dynamic model, however, might further examine the effect of introducing additional decision variables (such as insolvency constraints), the soundness of the utility functions used in the calculation, and the sensitivity of the optimal strategy to such parameters of the model as the distribution functions and the discount factor v.

In the previous model, dynamic programming was illustrated with the stages in the decision process being time periods. Bouzaher et al. [33] provided an example of a dynamic programming formulation in which the "stages" are defined differently.

In fact, a deterministic multiple-stage dynamic programming can be equivalently formulated as a one-stage mathematical programming problem (cf., Bellman and Dreyfus [22] and Denardo [97]). We leave readers to verify this claim. This equivalent transformation, however, may not be favored for two main reasons. First, the recursive (multiple-stage) formulation appears more intuitive and straightforward and reveals how the process proceeds from one stage to the next stage. If the underlying process is decision-making, then a decision-maker has certain rules that should be followed to proceed smoothly from stage to stage. Another critical reason may be that the algorithm can be more easily implemented computationally when a recursive equations system is provided. Also, with less variables involved in the recursive formulation, less computer resources (such as memory) are required to solve the problem. We refer readers to Bellman and Dreyfus [22] and Denardo [97] for general discussions about dynamic programming.

2.7 Chance-Constrained Programming

Chance-constrained programming (CCP) (Charnes et al. [68]) is a mathematical programming method dealing with optimization when some of the variables are stochastic. In some circumstances the variables involved in the calculations are only known with uncertainty (for example, are random variables), and it may be impossible to write a constraint that holds deterministically. The main idea of CCP is to allow certain factors, such as risk, to be realization of random variables. The decision variables are then selected such that these random variables are constrained to lie within an acceptable range of values with a pre-specified high probability; that is, the constraint is of the form $Pr(x \geq L) \geq \alpha$, rather than the deterministic equality or inequality of mathematical programming. The objective function may be either maximization or minimization. Among the applications of CCP in insurance are Agnew et al. [4], Pyle and Turnovsky [226], Thompson et al. [270], Kahane [157], McCabe and Witt [203], and Brockett et al. [42]. We use McCabe and Witt [203] for illustration of chance-constrained programming.

According to McCabe and Witt [203], for an insurance firm, the overriding objective is to maximize the profit from underwriting earnings and investment income; however, this objective is constrained by insolvency regulation requirements. A simplified financial model is analyzed for the insurer's behavior under uncertainty for both underwriting and investment income. Since the underwriting earnings and investment income are stochastic, the profits (as a function of underwriting earnings and investment income) are also stochastic. The objective is expressed as the maximization of the expected cash flow, that is, $E(\pi) = E(PQ) + E(I) - E(L) - E_1$, where P is the price per standard exposure unit (*SEU*), Q is the number of *SEU*'s written, I is the investment income, L is the total losses and loss adjustment expenses, and E_1 is the non-loss or underwriting expenses [see McCabe and Witt [203] for the composition of each category and the definition of *SEU*]. Along with the profit-maximization objective, the firm should be primarily concerned with the risk of technical insolvency. This insolvency risk is quantified by the probability of insolvency, ϕ. While it is impossible to guarantee with 100% certainty that the firm will not become insolvent in all possible states of the world and economy, the probability of insolvency should be constrained to be below some number, ϕ_1. The model can then be written as:

$$\text{Maximize } E(\pi) \text{ subject to } \phi \leq \phi_1.$$

Given an explicit expression for ϕ and given an admissible risk level, ϕ_1, a Lagrangian multiplier method can be used to solve the (nonlinear) problem after the probabilistic constraint has been transformed into a deterministic equivalent constraint.

As an example, define technical insolvency to be a loss of more than 30% of capital, C. The constraint can be specified in statistical terms as follows: $P(\pi \leq \pi_B) \leq 0.01$, where $\pi B = 0.3C$; that is, the probability of insolvency should be below 1%. Assuming that π follows a normal distribution and using the fact that the 99th percentile of the normal distribution is 2.33, the constraint can be transformed into a deterministic constraint as follows, $P(z \leq (-0.3C - E\pi)/\sigma_\pi) \leq 0.01$ so $E\pi \geq -0.3C + 2.33\sigma_\pi$. The CCP formulation has thus been converted to the deterministic problem as follows:

$$\begin{aligned} &\text{Maximize} \quad E\pi \\ &\text{subject to} \quad E\pi \geq -0.3C + 2.33\sigma_\pi; \\ &\qquad\qquad\quad 0 \leq S \leq 1.0\text{: } P \geq 0 \text{ and } k \geq 0, \end{aligned}$$

where C is the shareholder-supplied capital, P the price per standard exposure unit, k the average number of months elapsing between loss occurrences and loss payments, S the proportion of earning assets that can be invested in stocks, and σ_π the standard deviation of the profit. The variables P, S, and k are the decision variables. Clearly, different optimal solutions are obtained given different insolvency restrictions. Thus, the sensitivity analysis is possible.

Markle and Hofflander, in their quadratic programming portfolio approach, and Crum and Nye [85], in their network flow portfolio approach, both modeled the insurer's behavior at an operational level (as opposed to the more aggregated level used by McCabe and Witt [203]). The work of Brockett, Charnes, and Li [40] extended the McCabe and Witt CCP model to the micro level of analysis utilized by Markle and Hofflander [197] and by Crum and Nye [85].

In insurance, there are many cases in which the chance event can be expressed as a constraint. Events such as "becoming insolvent," "taking certain level of risk in business," "making a certain level (amount) of profit," and so on all involve a degree of uncertainty (randomness). These chance events can be formulated as constraints expressed in probabilistic terms (for example, the probability of being technically insolvent is no greater than

0.05). Techniques such as that illustrated previously can be used to transform the probabilistic constraint into its deterministic equivalent for solving via usual mathematical programming techniques. As mentioned above, sensitivity analysis can readily be conducted by varying the permissible chance levels. We also refer readers to Brockett et al. [42], for the transformation technique illustration.

2.8 Fuzzy Set Theory and Fuzzy Programming

Fuzzy set theory deals with ambiguity and imprecision in linguistic, reasoning, and decision-making. The applications of fuzzy set theory can be found extensively in linguistics, artificial intelligence, robotics, process control, decision analysis, and many other areas. Fuzzy set theory is, however, a rather new methodology to the actuarial and insurance communities. Lemaire [184] provided an introduction of fuzzy set theory and described how it might be used in insurance. Lemaire discussed three problems: the definition of a preferred policyholder in life insurance, the selection of an optimal excess of loss retention, and the computation of the fuzzy premium for a pure endowment policy. Some other applications can be found in Cummins and Derrig [87] and Derrig and Ostaszewski [100]. In addition, the Society of Actuaries has published a book by Ostaszewski [216], entitled *An Investigation into Possible Applications of Fuzzy Set Methods in Actuarial Science*, which further delineates the usefulness of fuzzy set theory to problems in actuarial science and insurance. Here we show how fuzzy set theory can be combined with cluster analysis and applied to risk and claim classification (Derrig and Ostaszewski [100]). We also illustrate a framework of fuzzy programming in insurance decision-making.

In a very fundamental way, there is an intimate relation between the theory of fuzzy sets and the theory of pattern recognition and classification. Acknowledging the fact that the boundaries between most insurance classes are fuzzy in nature (Kandel [160]). Derrig and Ostaszewski [100] applied fuzzy set theory to the clustering of rating territories and also to the classification of insurance claims according to their suspected level of fraud.

The fuzzy cluster method is developed from the so-called c-means cluster analysis method described below. Given n patterns, represented by p-dimensional vectors: $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, the goal of c-means cluster analysis is to divide these n patterns into c, $2 \leq c \leq n-1$ categorically homogeneous subsets (which are called clusters) such that the variances within clusters are minimized, while the variances between clusters are maximized.

Geometrically, each cluster is characteristically represented by its center point (which is also a p-vector) in the p-dimensional Euclidean space. Hence, the clustering problem is to find the c center points satisfying the above goals.

Clearly, the degree to which a particular pattern vector is believed to belong to a particular cluster is related to the distance between the pattern vector and the cluster mean. If the distance is zero, then clearly that pattern vector belongs to the corresponding cluster, while the degree of belief that the pattern vector belongs to the cluster decreases as the distance from the cluster increases. This degree of belief that the pattern belongs to the cluster, viewed as a function of the pattern vector $\mathbf{x}$, is called the membership function for the cluster. After normalization, each value of the membership function falls continuously between zero and one. When the fuzzy cluster algorithm is applied to classification of rating territories, the clusters are the risk classes, and the degree of belief that each territory belongs to a given cluster (risk class) is quantified as a real-valued number between zero and one. For a given territory, multiple clusters (risk classes) are possible. Such a classification methodology based on a membership function provides decision-makers with more information than does the crisp or nonfuzzy clustering, where the membership function can only take either zero (representing completely not-belonging-to) or one (showing perfectly-belonging-to). As a matter of fact, fuzzy cluster analysis can be transformed into crisp cluster analysis if we assign a pattern to a particular cluster and if the membership function value for that cluster is the largest among those for all the possible clusters. We refer readers to Derrig and Ostaszewski [100] for the details of the fuzzy cluster algorithm.

Fuzziness is to some extent similar to uncertainty, although they are different conceptually. Briefly speaking, fuzziness can arise because of our inability to describe the membership property of an object (ambiguity), whereas uncertainty occurs in situations in which the true membership exists but yet has not been fully revealed. Readers are referred to Zimmermann [289], and [290] for more information about fuzziness and to Dubois and Prade [104] for the foundation of fuzzy set theory and computation.

In the following, we show how a fuzzy linear programming is defined and given as an ordinary linear programming formulation by describing the technique for transforming the fuzzy programming to its crisp or nonfuzzy deterministic equivalent programming. Readers will find a similarity between the development of the deterministic equivalent programming problem in fuzzy programming and in CCP whereby probabilistic constraints are transformed into deterministic equivalent statements.

To make a decision is to achieve a set of goals while simultaneously being constrained by pertinent external and internal restrictions. Often goals and constraints are substitutable. In other words, there are trade-offs among goals and between goals and constraints. The trade-off between costs and benefits is a typical example. For example, an insurance company might be willing to pay higher commissions and increase other expenses to obtain greater premium growth. The decision-maker often prefers knowing how much of a gain on each goal can be obtained without sacrificing too much on other goal(s). Of course, the quantification "how much of a gain" or "sacrificing too much" is not a firm or crisp (nonfuzzy) process. Accordingly, rather than formulating the goal in a strict crisp form, a preferable formulation might be to use fuzzy formulation. The meaning of fuzziness in formulation is made clear in the illustration below.

To develop the notion of fuzzy linear programming, consider first the ordinary linear programming problem:

$$\text{Minimize} \quad C = \sum_{ij} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j} a_{ij} x_{ij} \geq b_i; \text{ and other constraints}$$

where x_{ij} is the decision variable; (suppose that) C denotes the total cost that is to be minimized; c_{ij} is the cost associated with x_{ij}; b_i the minimum level of some goal; and a_{ij} is the coefficient of x_{ij}.

Now, suppose the decision-maker is willing to be less precise and says that it is acceptable for the bound b_i to be decreased as low as $b_i - \lambda_i$ to achieve a better goal, that is, a lower cost in this case. For this illustration, let all λ_i be equal and denote this common value by λ for convenience. Define a membership function for the i-th constraint as follows:

$$\mu_i(z_i) = 1, \qquad \text{for } z_i \geq b_i;$$

$$\mu_i(z_i) = 1 - \frac{b_i - z_i}{\lambda}, \quad \text{for } b_i - \lambda \leq z_i < b_i; \text{ and}$$

$$\mu_i(z_i) = 0, \qquad \text{for } z_i < b_i - \lambda, \text{ (where } z_i = \Sigma_j\, a_{ij} x_{ij}).$$

Such a membership function of a constraint can be interpreted as follows: b_i is the perfectly satisfactory value of the constraint, while anything lower than $b_i - \lambda$ is a completely unsatisfactory value of the constraint. Between $b_i - \lambda$ and b_i, the satisfaction level with the constraint increases linearly as the actual value of the constraint increases from $b_i - \lambda$ to b_i. Thus, the membership function value increases continuously from 0 to 1.

When the decision-maker specifies the minimum acceptable satisfaction level θ for each of the constraints, a crisp equivalent programming can be derived as follows. Suppose that θ, $0 \leq \theta \leq 1$, is the minimum acceptable level of satisfaction for each of the constraints. Then we have

$$\mu_i(z_i) = 1 - (b_i - z_i)/\lambda \geq 0,$$

which is equivalent to $z_i - \lambda\theta \geq b_i - \lambda$, for constraint i. Accordingly, the fuzzy constraint $\mu_i(z) \geq \theta$, has been replaced by the deterministic equivalent constraint $z_i - \lambda\theta \geq b_i - \lambda$.

Similarly, if we can determine a target C_0 for the objective function, C (where $C_0 \geq C$ for the relaxation of the objective), then this leads to an inequality,

$$\sum_{ij} c_{ij} x_{ij} \leq C_0,$$

and we can treat the objective as a constraint. The discussion above also applies to the objective function using the membership function μ_0 as before (now linear between C and $C_0 - \lambda$). After inverting this constraint as described above, we obtain the complete crisp programming:

$$\begin{aligned} \text{Maximize} \quad & \theta \\ \text{subject to} \quad & \sum_{j} a_{ij} x_{ij} - \lambda\theta \geq b_i - \lambda; \\ & \sum_{ij} c_{ij} x_{ij} + \lambda\theta \leq C + \lambda; \text{ and} \\ & 0 \leq \theta \leq 1. \end{aligned}$$

The equivalent crisp or deterministic mathematical programming problem is specified once we are given the value of λ (which is the maximum permissible sacrifice level of the constraints and the objective). From the model, we can see that the larger the value of θ, the less the value (cost in this case) of the objective function and the larger the sacrifice value of the constraints. In other words, the crisp equivalent programming problem provides the trade-off between the objective function and the constraints. As previously described, for a multiple-objective optimization problem, some of the objectives can be formulated as constraints and one of them singled out as the objective function in a mathematical programming formulation. It is easy to see that the fuzzy programming method is well suited to the multiple-objective problem, where trade-offs among different objectives are established and the priorities of the various multiple goals are not self-evident.

Using intervals of possibilities to model vague and imprecise situations in insurance and other areas is another closely related approach. Instead of applying probability theory or fuzzy set theory, an interval rather than a single value is used to describe a vague or fuzzy or imprecise concept. Interval analysis as a branch of mathematics has found its applications to insurance issues, specifically to measuring and evaluating financial risk and uncertainty (Babad and Berliner [10], [11], and Berliner and Buehlman [25]).

3. OTHER OPERATIONS RESEARCH METHODS

The field of OR is constantly growing, and the applications of OR techniques in the area of insurance are expanding rapidly. The growing field of OR maintains substantial interactions with computer science, applied mathematics, engineering, finance, economics, and behavioral science. For example, game theory, which was originally developed for use in economics, is now also used in insurance. Portfolio analysis, which is widely used as an investment and risk management technique in finance, has been used in insurance, not only from the traditional investment perspective but also as an insurance composition design technique, by Markle and Hofflander [197], Crum and Nye [85], and others. Utility theory, decision analysis, and many other OR and management science methods have found use in the insurance industry. It is clearly difficult and space-consuming to discuss the numerous branches of OR and to delineate its applications in insurance. In the following sections, we concentrate mainly on game theory and three relatively new OR techniques: data envelopment analysis (DEA), expert systems (ES), and neural networks (NN). As these OR techniques illustrate, the continued expansion of OR methodologies owes greatly to the contributions of and interfaces with economics, applied mathematics, cognitive science, and computer science. Applications of these approaches in insurance is again demonstrated by examples.

3.1. Game Theory

Game theory is a formalized study of a kind of decision-making in which two or more competitors (called players) are involved and the decisions made by one player may affect the outcome of the other players. To achieve a goal, each player chooses a strategy. The final outcome or return from a player's strategic choice depends also on the strategic choice of all the other players. In other words, in such a decision-making context, a player must take this interdependence into account when choosing a strategy (making a decision). We refer readers to *Handbook of Game Theory with Economic Applications* Volume 1 [9] for more detailed formal information.

Game theory was suggested as a useful mathematical modeling technique for investigators involving insurance decision-making as early as the 1960s (Borch [28], Bragg [35], [36], and [264]). Since then game theory has been found useful in many insurance settings, such as cost allocation for an insurance company (Lemaire [183]), negotiation of insurance contracts (Kihlstrom and Roth [162]), optimal insurance purchasing in the presence of compulsory insurance and uninsurable risks (Schulenburg [254]), life insurance underwriting (Lemaire [182]), control of mutual insurance process (Tapiero [268]), out-of-court settlement of liability insurance claims (Fenn and Vlachonikolis [111]), unemployment policy (Zuckerman [292]), and so forth. This brief and incomplete list of applications demonstrates that game theory is applicable to purchasing, underwriting, management, liability claim settlement, and other areas in insurance or reinsurance. Game theory can also be used to explain the underwriting fluctuations as the result of rational behavior among interdependent firms in the industry (as opposed to viewing such underwriting fluctuations as merely irrational aberrations). In this paper, we review two applications: cost allocation and bargaining of liability claims.

Lemaire [183] proposed that a game theory methodology can be applied to the allocation of operating costs among the lines of business for an insurance company. Cost allocation in a large insurance company is extremely complex. For instance, a large Belgian company that operates in three lines of business—life, fire and accident—uses no less than 11 different criteria for cost allocation, including direct imputation; some operating costs are directly assigned to a class, while some costs, such as heating, water, and electricity, are assigned based on the basis of space occupied. Classical cost allocation methods, however, fail to satisfy certain important theoretical properties, as discussed by Lemaire [183]. Two of these properties are individual rationality and collective rationality in case of economies of scale. Lemaire shows that the cost allocation problem is identical to the problem of computing the value of a n-person cooperative game with transferable utilities. In other words, the cost allocation problem can be represented as a pair $[N, c(S)]$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $c(S)$ the characteristic (cost) function of the game. This characteristic or cost function is a super-additive set function that associates a real-valued number $c(S)$ to each coalition (subset) S of players with $c(S)+c(T) \geq c(S \cup T)$ for all S, $T \subset N$ and $S \cap T = \emptyset$. In most of the applications, economies of scale are sufficiently large that the game is convex; that is, $c(S) \div c(T) \geq c(S \cup T)+c(S \cap T)$ for all S, $T \subset N$. The solution space of a convex game is non-void. Further, Lemaire [183] showed that while the classical notions of "solutions of a game," such as the Shapley value, the nucleolus, and the disruptive nucleolus, are not appropriate for the cost allocation problem, the proportional nucleolus is a solution method that satisfies desirable properties for cost allocation and is therefore the best cost allocation method among the four solution methods discussed. The three theoretical properties considered to be desirable by Lemaire are collective rationality, monotonicity in costs, and additivity (see Lemaire [184] for a detailed discussion and justification of these three properties in an insurance cost allocation context).

Formally, let x_i denote a cost allocated to player i. The proportional nucleolus is obtained when the excess is defined by the formula

$$e(\bar{x},S) = \frac{c(S) - \sum_{i \in S} x_i}{c(S)}.$$

That is, the excess is the proportional gain obtained by players $i \in S$ acting as a coalition S rather than as individuals. Instead of granting the same amount to each proper coalition of N, a subsidy proportional to $c(S)$ is awarded. One has to solve the linear program

$$\begin{aligned} &\text{Maximize} \quad e \\ &\text{subject to} \quad \sum_{i \in S} x_i \leq c(S)(1 - e), \text{ for } \forall S \subset N,\ S \neq N,\ S \neq \emptyset; \\ &\qquad \sum_{i \in S} x_i = c(S), \text{ and} \\ &\qquad x_i \geq 0 \text{ for } \forall i. \end{aligned}$$

It can be shown that the philosophy of the decision expressed in the linear programming is to maximize the minimum excess according to the definition of the excess given above. In fact, many decision-makers take this conservative strategy when they face uncertainty.

Notice that the total number of possible coalitions of N players is 2^N-1, which increases exponentially with the size of N. Thus, the number of constraints in the linear programming process should also increase exponentially. The other three solution concepts for theory models (Sharpley value, nucleolus, and disruptive nucleolus) also suffer the same computation-complexity problems. It is, in fact, a disadvantage of the proposed game model for cost allocation. However, for N small, the above computation (and cost allocation) is quite possible.

In yet another insurance application, Fenn and Vlachonikolis [111] model a liability insurance bargaining problem as a game problem. When a liability insurance claim is filed, the lawyers for the defendant are charged with the task of responding to the claim: by persuading the claimant to withdraw, by taking the case to court for adjudication, or by agreeing to an acceptable out-of-court settlement of the claim. This process raises a number of issues. From the insurer's point of view, the predictability of the length of time to settle, as well as the size of the eventual settlement amount, are factors of actuarial importance. However, there is an uncertain relationship between the settlement amount and the actual loss incurred, and this uncertainty may raise questions about the adequacy and equity of the resulting compensation to the plaintiff. If the settlement process is indeterminable and capricious, it may even raise the moral hazard problem.

It is assumed that the lawyer for the insurer is a repeat-playing specialist acting for an insurance company with a large diversified portfolio of risks. The plaintiff, on the other hand, is usually more of a one-time player with a considerable sum at stake relative to his or her wealth and might even possibly be acting with nonspecialized legal advice. The lawyer for the insurer offers an amount for claim compensation based upon an estimate of the minimum "ask value" of the plaintiff. If the offer is greater than the plaintiff's minimum "ask value," the plaintiff will accept the offer. The offer will be rejected in the contrary case. If the offer is rejected, the lawyer for the insurer may either make another greater offer or take the case to court. This process is repeated until either the offer exceeds the minimum "ask value" and the case is settled outside the court, or the minimum "ask value" by the plaintiff is greater than the maximum "willing to be offered" by the insurer and the case has no bargaining outlet. In addition, the insurer's lawyer may not be willing to make too many offers (and risk the loss of reputation as a hard bargainer or incur the multiple fixed costs associated with the offers).

The above bargaining process can be modeled as in a game theoretic manner. Let A denote the minimum ask value of the plaintiff, B the maximum offer value of the insurer, C the amount of costs involved in litigation, and D the amount of damages that would be awarded were the case to be taken to trial. Then: $A=E_p(D)-E_p(C)$, and $B=E_d(D)+E_d(C)$, where E_p and E_d represent the expectation of the plaintiff and defendant (insurer), respectively. Suppose that both parties are risk averse, and let R_p and R_d denote the discount adjustment factors for risk or uncertainty used by the plaintiff and defendant, respectively. Then: $A=E_p(D)-E_p(C)-R_p$, and $B=E_d(D)+E_d(C)+R_d$. To model the bargaining process, the insurer's estimate of the plaintiff's minimum ask value, denoted by A^*, is: $A^*=E^*(D)-E^*(C)-R^*$. Let O denote the offer, then the process is formalized as follows:

Initially, the minimum ask value of the plaintiff is estimated by the defendant (insurer) as the mean of the subjective probability distribution of the minimum ask value. An offer is made at this value if the estimated value obtained by this process is lower than the maximum willing offer level; that is, $O_1=A^*$ if $A^*<B$, and $O_1=0$ otherwise.

The offer is accepted if it exceeds the minimum ask A. If the offer is rejected, the minimum ask should be greater than the estimated value, and the distribution is truncated. The new estimate is based on the truncated distribution: $O_2=O_1+E[\varepsilon|\varepsilon>0]$ if $A>O_1$, and $O_2=0$ otherwise, where ε is a stochastic error, with $\varepsilon\sim N(0,\sigma^2)$. The process is repeated until either the plaintiff accepts the offer at some point or the lawyer cannot or is not willing to offer more. The result of the former is the settlement of the case, whereas that of the latter is the trial at court. More detailed mathematical treatment can be found in Fenn and Vlachonikolis [111].

In this model, the subjective probability distribution should be consistent with the empirical liability settlement data. Whether the truncated distribution retains the identical properties of the original distribution function is another issue. As indicated by Danzon [90], the assumption of not learning by the plaintiff is questionable in that 89% of plaintiffs are represented by a trade union or other solicitor, in which case both parties can act to some extent strategically. In other words, not only does the defendant estimate the minimum ask of its distribution subjectively and also from the previous rejection by the plaintiff, but also the plaintiff adjusts the ask for compensation based on the estimate of the maximum offer of the defendant. In this case the game becomes more complicated and a modified game model is required.

The offer is accepted if it exceeds the minimum ask A. If the offer is rejected, the minimum ask value must be greater than this estimated value, and hence the subjective probability distribution for the minimum ask value can be recalibrated with a lower truncated value equal to the new rejected offer value. The new estimate of the minimum ask value is then based on this newly truncated distribution: $O_2=O_1+E[\varepsilon|\varepsilon>0]$ if $A>O_1$, and $O_2=0$ otherwise, where ε is a stochastic error term, which for computational purposes is assigned to be normally distributed with mean zero. The above process is repeated until either the plaintiff accepts the offer at some point or the lawyer for the insurer cannot or is not willing to offer more. The result of the former is the settlement of the case, whereas that of the latter is a court trial. More detailed mathematical treatment can be found in Fenn and Vlachonikolis [111].

In this model, the subjective probability distribution should be consistent with the empirical liability settlement data. Whether the truncated distribution retains the identical properties of the original distribution function is another issue. As indicated by Danzon [90], the assumption that the plaintiff does not learn (and hence change the minimum ask value) is questionable, because 89% of plaintiffs are represented by a trade union or other solicitor, in which case both parties can act to some extent strategically or in a game theoretic method. In other words, not only does the defendant estimate the minimum ask of its distribution subjectively and also from the previous rejection by the plaintiff, but also the plaintiff adjusts the ask price for compensation based on the estimate of the maximum offer of the defendant. In this case the game becomes even more complicated and a modified game theoretic model is required.

3.2 Data Envelopment Analysis (DEA)

Data envelopment analysis (DEA), invented by Charnes et al. [72], is an approach for comparing the relative efficiency of decision-making units (DMU), such as hospitals, schools, insurance agencies, and similar instances, in which there is a relatively homogeneous set of decision-making units with multiple inputs and multiple outputs. In other words, rather than using some absolute norms or standards, DEA evaluates the relative efficiency of each DMU within this homogeneous set or comparable DMUs. Accordingly, DEA does not assume any prespecified production function (such as Cobb-Douglas function) as the norm or standard in an absolute productivity efficiency evaluation. Rather, DEA is a nonparametric methodology.

Similar to the engineering notion of efficiency being the ratio of output to input, the measure of relative efficiency used in DEA models can be simply characterized as:

$$\text{efficiency} = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}$$

where the technique allows for multiple inputs and multiple outputs. The ability of DEA models to handle multiple outputs is an important and distinctive feature. Indeed, this is an important feature in insurance company efficiency comparisons because different companies may stress different outputs or inputs in their management strategy or business plan. Charnes et al. [72] proposed that the efficiency of a target decision-making unit j_0 can be obtained by solving the following model:

$$\text{Maximize} \quad h_0 = \frac{\sum_{r=1}^{t} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}}$$

$$\text{subject to} \quad \frac{\sum_{r=1}^{t} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \text{ for } 1 \leq j \leq n,$$

$$u_r, v_i \geq e, \text{ for all } r \text{ and } i$$

where y_{rj}=amount of output r obtained by DMU j; x_{ij}=amount of input i used by DMU j; u_r=the weight (or virtual multiplier) given to output r; v_i=the weight (or virtual multiplier) given to input i; n, t, m are the number of DMUs, outputs and inputs; and ε is a small positive number.

Essentially the virtual multipliers u_r and v_i are selected by the DMU in such a manner as to frame their own production performance in the best possible (most efficient) light (hence the maximization). The only constraint is that they cannot pick a weighting scheme for inputs and outputs that makes another DMU appear to be "super-efficient" (hence the ≤ 1

constraint). If a DMU is inefficient (has an objective value less than 1) even when it has chosen the virtual multipliers u_r and v_i to put its own efficiency in the best possible light, then it is indeed inefficient since another DMU (or combination of DMU's) using this same "strategic weighting" of inputs and outputs can take the same virtual input x_{ij} and produce higher virtual output y_{ij} than the designated inefficient DMU. Thus, the inefficient DMU is dominated in a Pareto-Koopmans economic efficiency manner.

The above fractional programming problem can be converted into an equivalent linear programming model (termed the CCR model), as follows:

$$\text{Maximize} \quad h_0 = \sum_{r=1}^{t} u_r y_{rj_0}$$

$$\text{subject to} \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1,$$

$$\sum_{r=1}^{t} u_r v_{ri} - \sum_{i=1}^{m} v_i x_{ij} \leq 1, \text{ for } 1 \leq j \leq n,$$

$$u_r,\ v_i \geq e, \text{ for all } r \text{ and } i,$$

Its dual linear program is:

$$\text{Minimize} \quad Z_0 - \varepsilon \sum_{r=1}^{t} s_r^- - \varepsilon \sum_{i=1}^{m} s_i^-$$

$$\text{subject to} \quad x_{ij_0} Z_0 - s_i^- - \sum_{j=1}^{n} x_{ij} \lambda_j = 0,\ 1 \leq i \leq m;$$

$$- s_r^+ + \sum_{j=1}^{n} y_{rj} \lambda_j = y_{rj_0},\ 1 \leq r \leq t;$$

$$Z_0 \text{ unrestricted; and } \lambda_j,\ s_i^-,\ s_r^+ \geq 0, \text{ for all } j,\ r, \text{ and } i.$$

From this dual LP (DLP), we can see that DMU j_0 is efficient if and only if all slack variables are equal to zero and Z_0 is equal to one. Conversely, if DMU j_0 is inefficient, then Z_0 is less than one and/or some slacks are positive. The optimal values of λ_j can be used to construct a composite DMU (or a linear combination of DMU's) with exactly the same inputs as the evaluated DMU but with an output that is larger than that obtained by DMU j_0. This then can provide a set of targets for benchmarking purpose by the inefficient DMU j_0 (that is, which utilize the same inputs to produce strictly more outputs). Z_0 represents the maximum proportion of the input levels that DMU j_0 should be expending to secure at least its current output level. See Brockett et al. [44] for a detailed example of this methodology used for benchmarking purpose.

The CCR model assumes constant returns to scale. However, when DMUs vary in returns to scale, the efficiency measure given by the CCR model may be complicated by the varying returns to scale factor. Banker et al. [17] proposed an extension that can decompose the overall or aggregate efficiency given by the CCR model into its technical and scale efficiency components. The modified model, termed the BCC model, is as follows (cf., the dual formulation of the CCR model):

$$\text{Minimize} \quad h - \varepsilon \sum_{r=1}^{t} s_r^+ - \varepsilon \sum_{i=1}^{m} s_i^-$$

$$\text{subject to} \quad x_{ij_0} h - s_i^- - \sum_{j=1}^{n} x_{ij}\lambda_j = 0, \; 1 \le i \le m;$$

$$- s_r^- + \sum_{j=1}^{n} y_{rj}\lambda_j = y_{rj_0}, \; 1 \le r \le t;$$

$$\sum_{j=1}^{n} \lambda_j = 1; \text{ and } \lambda_j, s_i^-, s_r^+ \ge 0, \text{ for all } j, r, \text{ and } i.$$

The BCC model differs from the CCR model only by the addition of a single constraint on the multipliers, that is, that the summation of all multipliers is equal to unity. This ensures that the BCC model yields a measure of the pure technical efficiency of DMU j_0 (cf., Banker et al. [17]).

Applications of the DEA approach to insurance problems can be found in Bjurek and Hjalmarsson [27b], Rousseau [238], and Mahajan [194]. In this paper, we introduce two of them: a DEA model for assessing the relative efficiency of the insurance-selling function (Mahajan [194]) and a DEA model for detecting troubled or potentially insolvent insurance companies (Rousseau [238]).

Mahajan [194] used the BCC model for assessing the relative efficiency of sales units. The model simultaneously incorporates multiple sales outcomes, controllable and uncontrollable resources, and environmental factors.

The model enables comparisons among a reference set of sales units engaged in selling the same product-service by deriving a single summary measure of relative sales efficiency. Conditions under which the sales unit has additional control over resources are explored, and the effects on relative efficiency are examined. The proposed model is illustrated by applying it to data collected from the branch operations of 33 insurance companies.

Rousseau [238] illustrated the role of DEA in detecting financially troubled insurance companies. The data for this efficiency study came from the National Association of Insurance Commissioners (NAIC) data set on a sample of 111 Texas domestic stock companies for 1987, 1988, and 1989. The DEA analysis was conducted by The Magellan Group, a division of MRCA Information Services. Although DEA can accommodate both financial and nonfinancial variables, only financial variables were selected in the prototype study.

The DEA study can provide the overall performance efficiency rating across all companies, the potential improvement in each input or output factor for an individual company, and the time trend of the overall performance rating of an individual company. If the efficiency rating of a company significantly deteriorates over time, the company is indicated to be in trouble and an early warning for that company is released. In addition, the factor performance analysis (which is unique to the DEA method, as compared to the more widely used regression analysis) provides information on the input or output factors that need to be improved the most to promote the overall efficient performance of the company. These effects are shown in Rousseau [238].

3.3 Expert Systems

An expert system (ES) usually has two components, a knowledge base (facts and rules) and an inference engine (interpreter and scheduler). Domain knowledge comprises the facts and a set of rules that use those facts as the basis for decision-making. The inference engine contains an interpreter that decides how to apply the logical rules to infer new knowledge and a scheduler that decides the order in which the rules should be applied. In an expert system, the knowledge is explicitly represented and accessible. This means that expert system approach is appropriate for those areas in which the knowledge that is to be used in the decision-making can be explicitly expressed. Once built, an expert system provides the high-level expertise to aid in problem-solving and has predictive modeling power; that is, it provides the output when given a situation as the input.

The use of expert systems in the insurance industry is not a new phenomenon. Financial underwriting applications, as well as life insurance applications, have been developed. Both life-health and property-casualty insurers are developing expert systems to aid in the underwriting process. Systems are also being developed to assist in claims management and investment planning. Personal financial planning, loss prevention, risk assessment, and product design are all areas in which expert system development is currently under consideration.

A product of expert systems, called Smart Systems, is becoming a component in insurers' strategic and competitive underwriting and claim systems. For instance, Connecticut Mutual Life Insurance Company expects to realize a 35% productivity gain in underwriting by using image and expert systems. Travelers Insurance Company uses expert systems to detect unusual or illogical patterns in health providers' behavior and claims to thwart fraud, and Erie Insurance Group uses expert systems to combat property and casualty fraud. In this paper, we briefly introduce two applications of expert systems: monitoring health-related expenditures by detecting unusual claims (Martin and Harrison [199]) and auditing workers' compensation insurance premiums (Koster and Raafat [170]).

Firms have often turned to self-insurance to control health care costs. In 1987 nearly 60% of all employees who had health care coverage were enrolled in a plan with some aspect of self-insurance. One reason for the lack of success in controlling health care costs, however, is that most firms do not have the expertise to properly monitor health-related expenditures. This is in fact reason to turn to third-party administers (TPAs) for claims-handling. The claims audit is one method used to monitor the administrator's performance and to help in controlling costs.

Martin and Harrison [199] described an expert system for claim monitoring and fraud detection for such a self-insured company. The expert system monitor's primary task is to review claim payments and identify opportunities for reducing health care expenditures. Cost reduction typically results from recovering improper payments and preventing similar mistakes from occurring in the future. The expert system functions as an initial filter since it reviews claims and identifies potential errors (that is, unjustified payments) by grouping payments according to likely errors and then estimating the likely value of the total error for each group. The production system consists of a set of if-then rules to determine what (if any) error is made on each payment. The knowledge base comprises approximately 50 rules that are mostly independent of each other, and these rules are used to identify 32 different types of errors. For each type of error, there exists a rule that specifies the probability of error, the value of the error, and the time required to further investigate the error (since the database is too large to investigate every claim). The validation experiments demonstrate that the system can screen claims in a manner comparable to human experts in the field (Martin and Harrison [199]).

Koster and Raafat [170] presented another expert system for auditing premium computations for workers compensation insurance. The purpose of premium auditing is to ascertain that the activities of the business are as recorded and to determine whether the employees are complying with the regulations of the insurance governing body in each state. This task is complicated, time-consuming, and error prone. The expert systems described in their paper assist insurance carriers and businesses not only by increasing compliance with statutory requirements but also by improving premium estimation accuracy, reducing auditing errors, and saving auditing time. This system is considered a prototype because it does not encompass all the rules in the workers compensation insurance manual. With this package, however, a user can: (1) solicit advice and justification for a claim, (2) request definitional clarifications, (3) ask for help, (4) request a session trace, and (5) solicit "what if" analysis and improvement suggestions. Koster and Raafat [170] suggested that insurance companies and state insurance agencies would be prime beneficiaries of such a system. The system also could be integrated and could share the same database with other business software.

Qualitative reasoning is often used as the inference mechanism by an expert system. Qualitative reasoning is realized by using logic induction, deduction, comparison, and certain other techniques. Examples of such reasoning includes the mathematical statements, "if A is true then B is true," "A is equivalent to B," and "A implies B," and so on, where A and B are propositions. Hence, the inference rules in expert systems are various rules expressed as "if-then" models. One may well wonder how those reasoning activities are accomplished by a computer since, as we know, computers complete every job by executing a series of binary operations. Below, we introduce a technique that may help elucidate this question. The technique described in the sequel formulates the qualitative reasoning process explicitly and equivalently by discrete mathematical programming.

By using atomic propositions or statements, compound propositions are created by using so-called propositional calculus. The propositional or logical operators (called connectives) include negation, conjunction, disjunction, implication, and equivalence. It is possible to define all propositional connectives in terms of a smaller subset of them so that any given expression can be converted into a "normal form" such as conjunctive normal form (CNF) (the subset of connective includes only negation and conjunction in this case) or disjunctive normal form (DNF) (the subset includes only negation and disjunction in this case). Also, these two normal forms can be converted mutually by using De Morgan's laws. De Morgan's laws, combined with other equivalence transformations, can convert any logical expression to a conjunction or disjunction of clauses by using equivalent statements. This was all shown in Hadjiconstantinou and Mitra [128].

The next step of systematically relating qualitative reasoning to discrete mathematical programming is to express these conjunctive or disjunctive statements as linear constraints involving only binary variables. We illustrate below the variable transformation process in two cases: "*A* implies *B*" and "either *A* or *B*." Other relationships can be found by referring to Hadjiconstantinou and Mitra [128]. Define x and y as follows: $x=1$ if *A* is true, 0 otherwise, while $y=1$ if *B* is true, 0 otherwise. Then, "either *A* or *B*" can be equivalently expressed as "$x+y\geq 1$." Also, "*A* implies *B*" is actually equivalent to "either not *A* or *B*," the equivalent inequality is "$x-y\leq 0$."

In a manner similar to that illustrated above, each of the rules in an expert system can be numerically formulated as a set of (in)equalities, so that the problem-solving procedure of the expert system is, accordingly, transformed into an integer programming solution process. Since, as discussed previously, integer programming is a rapidly developing area of operations research, such an equivalent transformation provides a promising way for optimally solving qualitative reasoning problems in insurance. In addition, this connection between integer programming and qualitative reasoning provides a bridge connecting qualitative reasoning to quantitative calculation. In addition, this transformation also clearly illustrates what we previously expressed about the expert systems: that expert systems are an alternative modeling technique that utilizes qualitative knowledge rather than explicit numerical computation.

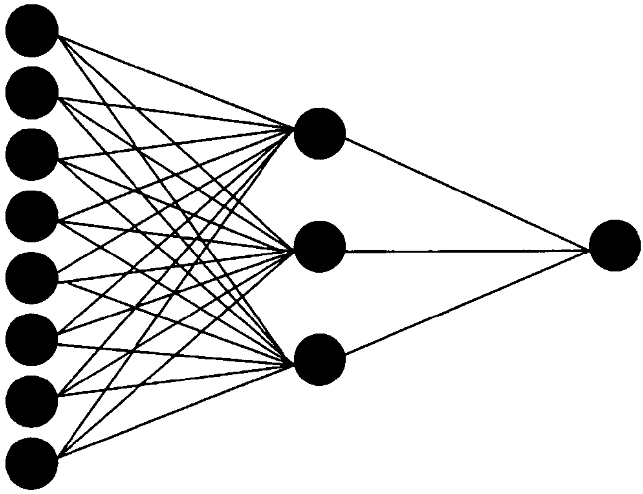
3.4 Neural Networks

In contrast to the expert system, the neural network model is a relatively new methodology for the insurance community. Inspired by the neurophysiological structure of the brain, the neural network model can be structurally represented as a massively parallel interconnection of many simple processing units, similar to the interconnection of individual neurons in the brain. Mathematically, the neural network emulates the relationship between inputs and outputs in which outputs are produced as some transformed and weighted composition of the inputs. The formation of proper weights needs a learning process, according to which the neural network is classified as an artificial intelligence approach. The two typical learning strategies (algorithms) are supervised learning and unsupervised learning. The difference between the two types results from the characteristics of the patterns in the example or training data set. If each pattern in the example or training set also contains the observed output values, then a supervised learning algorithm suffices. In the contrary situation, an unsupervised learning strategy is necessary. The back-propagation algorithm (Rumelhart et al. [239]) is the most widely used supervised training algorithm based on (multiple) layer feed-forward networks. Kohonen's self-organizing feature map (Kohonen [167], [168]) is a very popular unsupervised learning method. The two applications of the neural networks introduced below belong to these two distinct categories. Specifically, one is applied to constructing an index or ranking of insurance company insolvency (and thus the creation of an early warning system (Brockett et al. [44]), and the second application is directed towards detecting bodily injury claims fraud (Brockett et al. [48]).

Brockett et al. [47] used a three-layer feed-forward neural network to develop an early warning system for insurers to years prior to insolvency (see Figure 2 for an illustration). The basic building block of the neural network is the mathematical construct known as the single neural-processing unit. This unit takes the multitude of individual inputs, determines (through the learning algorithm) the connection weights that are appropriate (or most effective) for these inputs, and applies a combining or aggregation function to the derived connection weighted inputs to concatenate the multitude of individual inputs into a single value. An activation function, which is then applied, takes the aggregated weighted values for the individual neural unit and produces an individual output for the neural unit. This process is repeated at the individual neural unit, resulting in the use of as many single neural-processing units as desired, connected in whatever fashion (such as the feed-forward layered structure shown in Figure 2) is needed to produce a well functioning global neural network. The combined (weighted aggregate) result, $z = \Sigma w_i x_i$, is then "interpreted" by the network through the use of an activation function. The logistic

FIGURE 2

A THREE-LAYERED FEED-FORWARD NETWORK



function, $F(z)=1/[1+\exp(-\eta-z)]$, is the usual choice for the activation function because of its desirable properties and its simplicity in analytic representation.

In this particular three-layer network for an early warning model, the eight input units correspond to eight financial input variables that differ significantly between the insolvent and solvent firm. The output of the analysis is the probability of insolvency. Each connection between a unit in a layer output and a unit in the next layer of the neural network is associated with a weight. The learning strategy is the back-propagation algorithm, which is used to find the optimal weights based on minimizing the disparity between the predicted outcome and the observed outcome for the available examples. The data set used in the analysis was 60 U.S. property-casualty insurance companies that became insolvent and 183 companies that remained solvent over the period 1991–93. The data set is divided into three subsets: training sample, stopping rule sample, and testing sample, consisting of 60%, 20%, and 20% of the data, respectively. Their result showed that, for predicting insolvency, the neural network approach outperformed statistical methods such as discriminant analysis and did far better than the A. M. Best ratings and the National Association of Insurance Commissioners IRIS system.

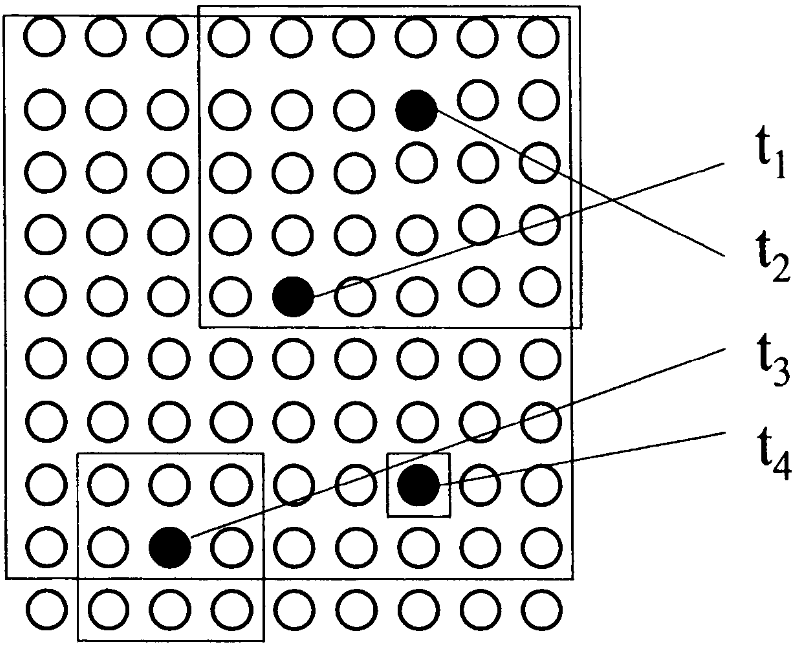
Brockett et al. [48] also applied Kohonen's feature map (Kohonen [167], [168]) to address the problem of uncovering bodily injury claims fraud. Kohonen's feature map is a two-layered and fully connected network, with output units arranged in some topographical form such as squares (as used in the study), rectangles or hexagons. This means that every output unit is associated with a weight vector whose dimensionality is equal to the number of input units or input variables (because of the full connection). As previously mentioned, this technique utilizes an unsupervised learning strategy; in other words, the pattern of each example (bodily injury claim record variables) contains only claims' recorded input variables without presupposing knowledge about the ultimate conclusion on the fraudulence of the claim. The task is to determine whether a claim is fraudulent and to determine the level of suspicion of fraud associated with the claim record file. Rather than using a dichotomous scale, that is, either fraudulent or perfectly valid, an increasing scaled measure of suspicion of fraudulence is used. Hence, the described detection system is designed to provide a fraudulence suspicion level for each claim, and each claim is uniquely classified according to its suspicion level. The Massachusetts Bureau of Automobiles provided a database comprising 127 claims, each of which has 65 objective indicators or input variables about the claim, the accident, and the claimant, such as "were there any witnesses?"

A learning algorithm is used to adjust weights to obtain improved results for classification. The learning process of the feature map can be briefly described as follows.

Each set of prototypical weight vectors is initialized with random numbers before learning begins. Within an epoch or training period, each pattern in the training sample is selected (either randomly or in a fixed order) and fed into the network once as the input vector. The program then computes the distance between this input pattern and each of the prototypical weight vectors and finds an output unit (the best-matching unit), whose prototypical weight vector is the smallest distance to this given input vector (or pattern). The value of the prototypical weight vector is then adjusted to better imitate the current input pattern. This is the "learning" feature. The uniqueness of Kohonen's self-organizing feature map is that updating occurs not only on the weight vector of the best-matching unit but also on the weight vectors of the units "neighboring" the best-matching unit. At the very beginning of the learning process (epoch 1), the neighborhood in which this adjustment takes place is relatively large. The radius of the neighborhood then sequentially decreases to zero, until finally, after a sufficiently large number of epochs or training periods, the neighborhood includes only the best-matching unit. The best-matching unit generally differs across the different input patterns within the training sample and for any particular given pattern may even vary from epoch to epoch. Figure 3 depicts the output units, which are arranged as a square. The figure also captures four snapshots, at t_1, t_2, t_3, and t_4, of the learning process along the time horizon. As shown in the figure, the size of neighborhoods decreases as the learning time proceeds. Kohonen [167], [168] presented a detailed description of the algorithm and the neurophysiological foundation.

FIGURE 3

UPDATING WEIGHTS IN NEIGHBORHOOD SET



The entire data set of bodily injury fraud claims used in the study was divided into a training sample comprising 77 claims and a testing sample comprising the remaining 50 claims. Each claim, or pattern, was a 65-dimensional binary vector reflecting various claims characteristics.

It is assumed that if two claims have common or similar characteristics (pattern), they should result in approximately equivalent suspicion levels (a continuity assumption). Euclidean geometric distance is used to measure the similarity of two vectors. Consequently, the two input vectors that are close together (have similar claim indicator input variables) should be assigned similar output values (fraud suspicion levels). It is further assumed that each claim indicator is of equal importance in determining the explaining suspicion level for a claim.

As described previously, each claim vector has a corresponding best-matching output unit. Accordingly, when the learning process is terminated, we have obtained a mapping from the input claim vectors to the output space (which in this case is a square). Moreover, because of the topographical arrangement of the output units, the mapping effect can be displayed by a planar map. This planar map shows the correspondence between the claims vectors and the weights vectors (or the output units). By construction and due to the learning process, any two claims that have similar input vectors should be mapped onto geographically close units in the output space. By partitioning the square output space into regions, a topographical division of the map is obtained in which fraudulent claims tend to be mapped onto one area of the square and valid claims tend to be mapped onto a very different area of the square.

In applying this methodology, Brockett et al. [48] found that the feature map learning algorithm outperforms human experts, that is, claims adjusters and investigators, in assessing the suspicion level of bodily injury (BI) claims based on a BI claims sample from Massachusetts. A similar methodology can be used for detecting fraud in Medicare claims as well.

As a final comment, neural network modeling is a nonparametric approach. Without prespecifying any underlying functional form of the relationship between the inputs and the output, a structure is determined and a learning process is applied in order to predict. The described neural network approach can also be related to various statistical methods. In fact, the multilayered feed-forward neural network methodology can be viewed in the context of a constrained nonlinear regression analysis, in which various neural networks differ by the structure and the algorithms. Because of the characteristic of learning intrinsic in algorithms, the neural network approach is better categorized as an artificial intelligence approach.

4. CONCLUSION AND DISCUSSION

Both OR methodologies and insurance industry research are experiencing rapid theoretical and technical developments. In OR, various new algorithms, new modeling techniques, and even new approaches are being developed very rapidly. New methods have been proposed to solve larger and even more complex real-world problems. Moreover, because of rapid advances in computer methodology, operations research techniques are becoming much more easily implemented and on a much larger scale.

For any particular problem, network optimization and their applications are often the cooperative work of scientists and practitioners from various areas such as mathematics, computer science, engineering, and OR. In addition, expert systems and neural network models, which were originally developed by computer scientists, are now found to be useful OR approaches and are being applied in the solution of insurance problems. Because the development of OR depends greatly on algorithmic design and computational implementation, its rapid growth is, to a very large extent, a result of the fast-paced development of the computer industry. Many large-scale problems, which previously could be solved only on supercomputers (or simply not be solved within an acceptable time), can now be solved on desktop PCs.

As mentioned previously, the boundaries between the different OR methods are becoming increasingly blurred. For example, qualitative reasoning processes can be formulated as integer programming models, and this technique also provides a bridge between expert systems and mathematical programming models. Fuzzy programming is, to some extent, similar to robust optimization methods and to CCP in decision-making philosophy. Fuzzy programming is also related to goal programming if certain goals can be treated as constraints in the formulation.

In fact, in the real world, an absolute distinction between various goals often cannot be delineated to determine which goals are to be formulated as belonging to the objective function and which goals are to be formulated as more properly belonging to the constraint set. In practice, of course, these distinctions depend primarily upon the circumstances and the preferences of decision-makers. The fact that a portfolio problem in finance can be formulated as a linear programming problem, a nonlinear programming problem, a network optimization problem, a goal-programming problem, a chance-constrained programming problem, or a dynamic programming problem demonstrates the intrinsic applicability of numerous OR techniques to practical insurance problems.

A further example of asset portfolio-modeling techniques was discussed in Hiller and Schaack [142]. In that paper, four different structured bond portfolio models were presented, although they arise from different situations and need different quantitative considerations in modeling, implementation, and computational solution.

As stated earlier, nonlinear relationships often prevail between variables. Accordingly, one might reasonably look to advances in nonlinear function theory and computation for future OR methods that will be applicable to insurance research. Scientists have developed very efficient algorithms, such as the simplex algorithm, for solving linear programming problems. However, for nonlinear programming problems (except for certain special structured nonlinear programming problems like quadratic programming), conventional algorithms often stop with local optima rather than finding a global optimum. Recently, researchers have been inspired by the knowledge and experience obtained from nonlinear dynamic, neural networks and other classical or new methodologies to further investigate this problem. One highly touted method is called simulated annealing (SA) (Kirkpatrick, Gelatt, and Vecchi [163]). SA algorithms have been theoretically proved to be convergent to the global optimum. Romeo and Sangiovanni-Vincentelli [233] provided a theoretical review on SA algorithms. The solution-searching process, however, is in practice highly dependent on the parameter design (schedule). Many empirical studies, Goffe, Ferrier, and Rogers [125], for instance, confirm that the SA algorithm finds the global optimum with a much higher probability, but it also runs more slowly when compared with various widely used optimization algorithms such as conjugated gradient methods.

Another technique, terminal repeller unconstrained subenergy tunneling (TRUST), which was developed at the Jet Propulsion Laboratory at the California Institute of Technology (Cetin, Barhen, and Burdick [55]), is applied to neural network (back-propagation) training to avoid stopping at local minima.[7] This method seems to perform satisfactorily (Cetin, Burdick, and

[7]Finding the optimal weights for a multilayer feed-forward back-propagation network is difficult because of many potential local minima. In other words, it is hard to find the best weights for the neural network to extract as much information as possible from the training sample. Hence, an ad-hoc training schedule very likely ends up with finding suboptimal weights for the network model instead of the optimal weights.

Barhen [55]). One might expect that TRUST can also be used for other situations with a local minima problem.

The genetic algorithms (or evolutionary algorithms) have also been suggested as providing a potential methodology for avoiding local optima. Interestingly, genetic algorithms are currently being used to find models for predicting performance of financial instruments (such as the pricing of stocks and derivatives). We believe, sooner or later, a practical global optimization technique will be developed.

For the insurance industry, as indicated in Haehling von Lanzenauer and Wright [131], a few trends appear to manifest themselves. One trend is the ability to simultaneously consider a variety of problem dimensions and to explicitly model their interactions, often in a dynamic environment. These trends in insurance studies and decision-making practice strive to obtain much more realistic methods for dealing with real-world problems while retaining the ability to actually compute solutions in well specified cases. Decisions by senior management are becoming increasingly dependent on the analytical support from OR or management science. On the other hand, new issues in insurance never stop occurring. These include environmental pollution, the liability insurance crisis, underwriting cycles, regulation and pricing, availability and admissibility, equity and discrimination, and many other legal, financial, economic, and technological issues. These topics should challenge the OR community for some time to come. Globalization of the insurance market also presents new problems. Recent developments also show that financial risks may turn out to be the most dominant cause for risk management techniques (as has already been observed in finance).

Finally, we would like to share our view on optimization software and information technology. For those who have experienced inconvenience in interacting with various software packages such as Lindo, GAMS, GRG2 (Lasdon [173]), Lasdon et al. [174], [175], and other general-purpose optimization software, the good news is that a very user-friendly optimization problem-solver based on the GRG2 algorithm has been available since MS-Excel version 4.0. The solver incorporated in Excel is much easier to use. In addition, as new generations of PCs continue to run faster and faster, the problems of solving large-scale optimization problems will decrease. Another recent advance in developing optimization software is the use of C++ language and object-oriented methodology to build reusable libraries and friendly interfaces, which will likely lower the costs of software development and optimization. The information age is rapidly becoming a reality as more and more people are turning to Internet and other global real-time information superhighways. The impact of information technology on OR and insurance may well be beyond our imagination.

Operations research never stops providing insightful solutions and contributing to the healthy development of the insurance industry. The development of OR methodologies is both challenged and motivated by the complexity of most real-world problems, including those in the insurance industry. Practitioners and students in the insurance industry will quite possibly find that OR techniques can provide a powerful, flexible, accessible, and promising tool for insurance research.

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6. BIBLIOGRAPHY

A bibliography is important for a review paper, so we have built three cross-reference lists, each for a particular purpose. These are organized (1) by author, (2) by insurance area involved, and (3) by the OR techniques used. In the first list, all the references are presented in alphabetical order by the last name of the first author. Readers can answer such questions as which author has done what work. Clearly, readers are able to dig deeper in literature through this detailed reference list. The number is then used in the two subsequent reference lists.

In the second list, the references are organized according to the insurance areas that each paper addresses. This insurance area classification scheme was designed so that the references could be distributed into the different areas in a relatively balanced way. Certain areas, such as graduation and environmental risk management, are listed separately rather than embedded in their higher level areas (actuarial science and risk management, respectively). In addition, while we realize that duration-matching and immunization is a very active research in actuarial science and that to insurance companies, it is a vital part of their investment strategy, we still list investment and duration/immunization as separate topics to emphasize that duration matching and immunization in the scheme are more insurance-related, while the references listed under investment are more diversified. Also, duration-matching and immunization can also be considered as a technique used in asset/liability management (ALM). Hence, we have not tried to present a perfect insurance/actuarial science classification. Instead, the scheme is designed to help researchers and practitioners who face a research or practical problem and need to find relevant references and methodologies for solutions.

In the third cross-classification, the references are distributed according to OR method, assuming there are circumstances in which researchers, practitioners, actuarial students, or teachers are studying OR and trying to find relevant applications in insurance and actuarial science.

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REFERENCES CLASSIFIED BY INSURANCE AREAS OF APPLICATION

Accounting (Cost Allocation, Break-Even, etc.), see also Capital Budgeting, Finance
64, 107, 174

Asset/Liability Management, see also Cash Flow, Duration Matching and Immunization, Insolvency, Ruin and Reserve Issues
52, 53, 81, 91, 114, 171, 213, 223, 246

Business Planning, see also Strategic Thinking and Planning, Capital Budgeting, General Management, Human Resources
30, 52, 53, 138, 159, 164

Capital Budgeting, see also Business Planning, Cash Flow, Duration Matching and Immunization
68, 69, 70, 126, 176, 179, 224, 265.

Claim Distribution (Claim Size, Arrival Process), see also Ratemaking, Underwriting
95, 145, 268

REFERENCES CLASSIFIED BY OPERATIONS RESEARCH METHODS USED

TRANSACTIONS OF SOCIETY OF ACTUARIES
1995 VOL. 47

DISCUSSION OF PRECEDING PAPER

TZONG-HWA WU*:

Dr. Brockett and Dr. Xia are to be congratulated for this excellent review paper listing nearly 300 references. As illustrated in the paper, operations research (OR) models have been formulated to solve a wide variety of problems in the insurance industry. Actuaries and management scientists have constructed stochastic financial models to assist insurers in determining the future financial impact of insured events. As pointed out in the paper, the areas in which these models are applied include determination of insurance premiums, calculation of benefit reserves, estimation of insurance fund solvency, measurement of uncertainty and risk in investments, and asset/liability management (ALM). The purpose of this discussion is to supplement this fine paper by reviewing two ALM models that are perhaps overlooked in the actuarial literature. The first one is due to Bradley and Crane [2], [3], and the second one is developed by Mulvey and his colleagues [5]–[8].

ALM has evolved over the last 20 years in response to the growth of financial markets, the problem of interest rate risk, and the availability of new analytic tools and information systems. The unpredictable path of financial innovation has shaped the development of ALM and poses new challenges for the evolution of current systems. These challenges are not only technical but also organizational. Successful financial institutions need to retain operational flexibility in spite of an ever increasing number of regulatory constraints. This target is especially problematic to achieve for institutions rooted in traditions different from the ones from which current ALM techniques originated.

The Bradley and Crane Model

The stochastic decision tree model depends upon the development of economic scenarios that are intended to include all possible outcomes. The scenarios can be viewed as a tree diagram for which each element (economic conditions) in each path has a set of cash flows and interest rates. The problem is formulated as a linear program whose objective is the maximization of expected terminal wealth of the firm. There are four types of constraints:

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(a) Cash flows constraint. The firm cannot purchase more assets than it has funds available.
(b) Inventory balance constraint. This ensures that the firm cannot sell and/or hold more of an asset at the end of a period than it held at the beginning.
(c) Initial holdings constraint. We set the values of the variables $h^k_{0,0}(e_0)$, which refer to the holdings of securities in the initial portfolio, to these amounts.
(d) Non-negativity constraint. The non-negativity of the variables implies that short sales are not permitted.

The basic formulation is

$$\text{maximize} \sum_{e_N \in E_N} p(e_N) \sum_{k=1}^{K} \left\{ \sum_{m=0}^{N-1} [y^k_m(e_m) + v^k_{m,N}(e_N)] h^k_{mN}(e_N) + [y^k_N(e_N) + v^k_{Nn}(e_N)] b^k_N(e_N) \right\}$$

subject to

(a) Cash Flows $$\sum_{k=1}^{N} b^k_n(e_n) - \sum_{k=1}^{K} \left[\sum_{m=0}^{n-2} y^k_m(e_m) h^k_{m,n-1}(e_{n-1}) + y^k_{n-1}(e_{n-1}) b^k_{n-1}(e_{n-1}) \right] - \sum_{k=1}^{K} \sum_{m=0}^{n-1} [1 + g^k_{m,n}(e_n)] s^k_{m,n}(e_n) = f_n(e_n)$$

(b) Inventory Balance $-h^k_{m,n-1}(e_n - 1) + s^k_{m,n}(e_n) + h^k_{m,n}(e_n) = 0,$

$m = 0, \ldots, n - 2$

$-b^k_{n-1}(e_{n-1}) + s^k_{n-1,n}(e_n) + h^k_{n-1,n}(e_n) = 0,$

(c) Initial Holdings $h^k_{0,0}(e_0) = h^k_0$

(d) Non-negativity $b^k_{m,n}(e_n) \geq 0,\ s^k_{m,n}(e_n) \geq 0,\ h^k_{m,n}(e_n) \geq 0,$

$m = 1, \ldots, n - 1$

where

e_n	$\in E_n$, $n = 1, \ldots, N$; $k = 1, \ldots, K$
e_n	is an economic scenario from period 1 to n having probability $p(e_n)$
E_n	is the set of possible economic scenario from 1 to n
K_i	is the number of assets of type i, and K is the total number of assets
N	is the number of time periods
$y^k_m(e_m)$	is the income yield per dollar of purchase price in period m of asset of asset k, conditional on e_m
$v^k_{m,N}(e_N)$	is the expected terminal value per dollar of purchase price in period m of asset k held at the horizon (period N), conditional on e_N
$b^k_n(e_n)$	is the dollar amount of asset k purchased in period n, conditional on e_n
$h^k_{m,n}(e_n)$	is the dollar amount of asset k purchased in period m and held in period n, conditional on e_n
$s^k_{m,n}(e_n)$	is the dollar amount of asset k purchased in period m and sold in period n, conditional on e_n
$f_n(e_n)$	is the incremental increase (decrease) of funds available for period n.

The Mulvey Approach

Stochastic programming provides an ideal framework for modeling financial decisions and investment strategies over time. In financial planning via multistage stochastic programs, Mulvey [6] uses the following equations to determine interest rate scenarios:

Short rate: $$dr_t = a(r_0 - r_t)dt + b\sqrt{r_t}dZ_1$$

Long rate: $$dl_t = c(l_0 - l_t)dt + e\sqrt{l_t}dZ_2$$

where r_t and l_t represent the short and long interest rates at time t, respectively; a and c are drift coefficients; b and e are instantaneous volatility coefficients; v_0 and l_0 are mean reverting levels. The random coefficients, dZ_1 and dZ_2, depict correlated Wiener terms. These two diffusion equations provide the building blocks for the remaining spot interest rates and then full yield curves.

A fundamental issue in carrying out a financial modeling effort is to settle on the choice of an objective function and the underlying preference structure. There are numerous possibilities. In the basic model, the proposed objective maximizes the investor's wealth at the beginning of period τ, subject to the payout of intermediate cash outflows (liabilities) under each of the $s \in S$ scenarios. The investor's true wealth at the horizon τ equals the following

$$\text{wealth}_\tau^s = \sum_i x_{i,\tau}^s - PV(\text{liab}_{\tau,T}^s) - \text{prin}_\tau^s,$$

where the primary decision variable, $x_{i,\tau}^s$, denotes the amount of investment in asset category i at the beginning of time period τ under scenario s; $\text{liab}_{\tau,T}^s$ is the liability stream from period τ to period T; and prin_τ^s depicts the amount of loans outstanding at time period τ.

There are various alternative objective functions. One possibility is to employ the classical mean-variance function:

$$\max \exp(\text{wealth}_\tau) - \rho \text{ variance } (\text{wealth}_\tau),$$

where ρ indicates the relative importance of variance as compared with the expected value. This objective leads to an efficient frontier of wealth at period τ by varying ρ.

An obvious alternative to mean-variance is the Von Neumann-Morgenstern (VM) expected utility (EU) of wealth at period τ. Here, the objective becomes

$$\max \sum_s \text{prob}_s \text{ utility}(\text{wealth}_\tau^s),$$

where prob_s is the probability of scenario s, and utility(wealth) is the VM utility function as derived via certainty equivalence and risk premium questions. A general objective function for this problem is as follows,

$$\max \sum_s \text{prob}_s \text{ utility}(\text{wealth}_1^s, \text{wealth}_2^s, \ldots, \text{wealth}_\tau^s).$$

In the approach of Mulvey [6], the primary decision variable, $x_{i,t}^s$, denotes the amount of investment in asset category i at the beginning of time period t under scenario s. The x vector depicts the state of the system after the rebalancing decisions have been made in the previous period. At that time the investor's total assets are equal to:

$$\sum_i x_{i,t}^s = \text{assets}_t^s,\ s \in S,\ t \in T.$$

The uncertain return, $r^s_{i,t}$, for the asset categories—for asset i, time t, and scenario s—are projected by the stochastic modeling subsystem. Each scenario is internally consistent. Thus, $v^s_{i,t}$, the wealth accumulated at the end of the t-th period before rebalancing in asset i, is

$$x^s_{i,t}\left(\frac{1 + r^s_{i,t}}{100}\right) = v^s_{i,t}, \forall\ i \in I, t \in T, s \in S.$$

Rebalancing decisions are rendered at the end of each period. Purchases and sales of assets are accommodated by the variables $ybuys^s_{i,t}$ and $ysells^s_{i,t}$ with transaction costs defined via the coefficients t_s assuring symmetry in the transaction costs.

Using the terminology of robust optimization (Mulvey, Vanderbei, and Zenios [9]), the relationships of the various investment categories are constructed at each period as structural constraints. The flow balance constraint for each asset category and time period is

$$x^s_{i,t+1} = v^s_{i,t} + ybuys^s_{i,t-1}(1 - t_i) - ysells^s_{i,t-1}, \forall\ i \in I, t \in T, s \in S.$$

This equation restricts the cash flows at each period to be consistent. It is assumed that dividends and interest are forthcoming simultaneously with the rebalancing decisions. Thus, the $ysell$ variables consist of two parts corresponding to the involuntary cash outflow—dividend or interest—and a voluntary component for the cash flow—the amount actively sold (*sales*). The requisite equation is

$$ysells^s_{i,t} = div^s_{i,t} + sales^s_{i,t}, \forall\ i \in I, t \in T, s \in S,$$

where $div^s_{i,t} = x^s_{i,t}(divp)^s_i$ and $divp$ is the dividend payout percentage ratio for asset i under scenario s. The cash node at each period t also requires a flow balancing equation

$$\begin{aligned} cash^s_t &= cashin^s_{t-1} + \Sigma_i\,[(sales)^s_{i,t-1}(1 - t_i) + div^s_{i,t-1}] \\ &\quad - \Sigma_i\,(ybuys^s_{i,t} + bor^s_{i,t-1}) + cash^s_{t-1} - liab^s_{t-1} \\ &\quad + bor^s_{i,t} - prin^s_{t-1}, \forall\ t \in T, s \in S, \end{aligned}$$

with two new decision variables: $bor^s_{i,t}$ corresponding to the amount of borrowing in each period t; and $liab^s_t$ corresponding to committed liabilities other than borrowing. The variable $prin^s_t$ represents the reduction in borrowed funds that occurs during period t under scenario s. The liability decisions may be dependent upon the state of the world, as depicted by scenario s.

One assumes that all borrowing is done on a single-period basis. (This assumption can be avoided by adding new decision variables for each category of multiperiod borrowing.) Initial wealth at the end of period 0 equals $v_{i,0}$, for all scenarios s.

In practice, investors restrict their investments in asset categories for a diversity of purposes such as company policy, legal, and historical rules and considerations. These policy constraints may take any form, but we keep the structure to a set of linear restrictions as specified by

$$A^s x^s = b^s, \forall s \in S,$$

where A is an $(m \times n)$ matrix with coefficients that depend upon scenario s.

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(AUTHORS' REVIEW OF DISCUSSION)

PATRICK L. BROCKETT AND XIAOHUA XIA:

We thank Mr. Wu for his comments on our article. In the interest of being concise, several applications of OR techniques to actuarial science and insurance were not dealt with fully (or at all in some cases) in our article. Already the paper was quite long. We tried to emphasize techniques learned by SOA students or new techniques that they should learn. In some cases, other articles have provided adequate references and details. We appreciate Mr. Wu providing more applications to asset-liability matching since it touched on a methodology (stochastic optimization) that we did not detail.

Finally, we would like to note also that the Mulvey et al. approach [9] is similar in many ways to that obtained using chance constrained programming as outlined in Brockett, Charnes, and Sun (ref. [41] in the paper). However, the stochastic calculus approach to interest rate term structure was not used and the constraints were not assumed to hold under every conceivable scenario (that is, with probability one) but rather with very high probability. In this sense the chance constrained method gives a "policy" rather than a "hard and fast rule" methodology to asset-liability matching.