Stochastic Analysis of Life Insurance Surplus

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Motivations

- How risky is the portfolio of life policies?
- How likely is the insurance company to become insolvent in any given year?
- Are premiums and level of initial surplus adequate to ensure high probability of solvency?
Framework

Surplus

Assets \rightarrow \text{Liabilities}

0 \rightarrow r \rightarrow n

issue date \rightarrow \text{valuation date} \rightarrow \text{maturity date}

view \rightarrow measure
Risks Facing Insurance Industry

- Mortality
- Investment
Risks Facing Insurance Industry

- Mortality
- Investment
- Expenses
- Persistency
- Other
**Decrements due to Mortality: Model**

- $K_x$: *curtate-future-lifetime* of a person aged $x$
  - number of *complete* years remaining until death

**Notation:**
- $P(K_x = k) = k\mid q_x$, $k = 0, 1, 2, ...$
- $P(K_x > n) = np_x$

- **Nonparametric** life table
  - Canada 1991, Age Nearest Birthday, Male, Aggregate, Population
\( \delta(k) \): force of interest in period \( (k - 1, k] \), \( k = 1, 2, \ldots, n \)

\( \delta_k \): realization of \( \delta(k) \)
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\( \delta_k \): realization of \( \delta(k) \)

\( I(s, r) \): force of interest accumulation function

\[
I(s, r) = \begin{cases} 
\sum_{j=s+1}^{r} \delta(j) & \text{if } s < r, \\
0 & \text{if } s = r.
\end{cases}
\]
Stochastic Rates of Return: Notation

- $I(s, r)$: force of interest accumulation function

$$I(s, r) = \begin{cases} \sum_{j=s+1}^{r} \delta(j) & \text{if } s < r, \\ 0 & \text{if } s = r. \end{cases}$$
Stochastic Rates of Return: Model

AR(1) model

\[ \delta(k) - \delta = \phi [\delta(k - 1) - \delta] + \epsilon(k), \]

where

- \( \epsilon(k) \sim N(0, \sigma^2) \)
- \( \delta: \text{long-term mean of the process} \)
- \( |\phi| < 1 \) (stationarity)
- conditional on starting value \( \delta(0) = \delta_0 \)
More Assumptions...

Assumptions

- Future lifetimes are i.i.d.
- Lifetimes are independent of rates of return
- Identical contracts (i.e., homogeneous portfolio)
Notation: Life Insurance Policy

- $n$: term of contract
- $x$: age at issue
- $b$: death benefit
  - payable at the end of the year of death
- $c$: pure endowment benefit
  - payable upon survival to time $n$
- $\pi$: premium
  - payable at the beginning of each year
Notation: Homogeneous Portfolio

\[ L_{i,j}(x) = \begin{cases} 
1 & \text{if policyholder } i \text{ aged } x \text{ survives for } j \text{ years}, \\
0 & \text{otherwise} 
\end{cases} \]

\[ L_j(x) = \sum_{i=1}^{m} L_{i,j}(x) \sim \text{BIN}(m, j \rho_x) \]

- number of policies in force at time \( j \)
Notation: Homogeneous Portfolio

\[ L_{i,j}(x) = \begin{cases} 
1 & \text{if policyholder } i \text{ aged } x \text{ survives for } j \text{ years}, \\
0 & \text{otherwise} 
\end{cases} \]

- \( (x) = \sum_{i=1}^{m} L_{i,j}(x) \sim \text{BIN}(m, jp_x) \)
- number of policies in force at time \( j \)

\[ D_{i,j}(x) = \begin{cases} 
1 & \text{if policyholder } i \text{ aged } x \text{ dies in year } j, \\
0 & \text{otherwise} 
\end{cases} \]

- \( (x) = \sum_{i=1}^{m} D_{i,j}(x) \sim \text{BIN}(m, j-1|q_x) \)
- number of deaths in year \( j, j \geq 1 \)
Retrospective Gain
Retrospective Gain

Definition

\[ RG_r = \sum_{j=0}^{r} RC_j^r \cdot e^{l(j,r)} \]

where

\( RC_j^r \): net cash flow at time \( j \) prior to time \( r \), \( 0 \leq j \leq r \)

\[ RC_j^r = \pi \cdot L_j(x) \cdot 1_{\{j<r\}} - b \cdot D_j(x) \cdot 1_{\{j>0\}} \]
Prospective Loss

-π -π ... -π b

0 1 ... r r+1 ... Kx Kx+1 ... n
Prospective Loss

Definition

\[ PL_r = \sum_{j=0}^{n-r} PC^r_j \cdot e^{-I(r,r+j)} \]

where

\( PC^r_j \): net cash flow \( j \) time units after time \( r \), \( 0 \leq j \leq n - r \)

\[ PC^r_j = b \cdot D_{r+j}(x) \cdot 1_{\{j>0\}} + c \cdot L_n(x) \cdot 1_{\{j=n-r\}} - \pi \cdot L_{r+j}(x) \cdot 1_{\{j<n-r\}} \]
Stochastic Surplus

\[ S_{r}^{stoch} = RG_r - PL_r \]
Stochastic Surplus

\[ S^{stoch}_r = RG_r - PL_r \]

Accounting Surplus

\[ S^{acct}_r = RG_r - rV \]

where \( rV \equiv rV(L_r, \delta(r)) \) is the actuarial reserve at time \( r \)

- different ways to calculate \( rV \)
- \( rV = E[PL_r|L_r, \delta(r)] \)
Recall: $S_r^{acct} = RG_r - r V(L_r, \delta(r))$

**Observation**

- Given values of $L_r$ and $\delta(r)$, $r V(L_r, \delta(r))$ is constant

$\Rightarrow$ cdf of $S_r^{acct}$ can be obtained from cdf $RG_r$ via

$$P[S_r^{acct} \leq \xi \mid L_r = m_r, \delta(r) = \delta_r] =$$

$$= P[RG_r \leq \xi + r V(m_r, \delta_r) \mid L_r = m_r, \delta(r) = \delta_r]$$
Let $G_t = \sum_{j=0}^{t} RC_j \cdot e^{l(j,t)}$, $0 \leq t \leq r$

Note: $G_r = RG_r$

It can be shown: $G_t = G_{t-1} \cdot e^{\delta(t)} + RC_t$

Consider a function $g_t(\lambda, m_t, \delta_t)$ given by

$$g_t(\lambda, m_t, \delta_t) = P[G_t \leq \lambda | L_t = m_t, \delta(t) = \delta_t] \times P[L_t = m_t] \times f_{\delta(t)}(\delta_t)$$
Recursive Formula for $g_t(\lambda, m_t, \delta_t)$

Result

For $1 < t \leq r,$

$$g_t(\lambda, m_t, \delta_t) =$$

$$= \int_{-\infty}^{\infty} \left( \sum_{m_{t-1}=m_t}^{m} \mathbb{P}[\mathcal{L}_t = m_t \mid \mathcal{L}_{t-1} = m_{t-1}] \cdot g_{t-1} \left( \frac{\lambda - \eta_t}{e^{\delta_t}}, m_{t-1}, \delta_{t-1} \right) \right) \times$$

$$\times f_{\delta(t)}(\delta_t \mid \delta(t - 1) = \delta_{t-1}) \, d\delta_{t-1}$$

where $\eta_t$ is the realization of $RC_t^r$ for given values of $m_{t-1}$ and $m_t,$

$$\eta_t = \begin{cases} 
\pi \cdot m_t - b \cdot (m_{t-1} - m_t), & 1 \leq t \leq r - 1, \\
-b \cdot (m_{t-1} - m_t), & t = r
\end{cases}$$

with the starting value given by

$$g_1(\lambda, m_1, \delta_1) = \begin{cases} 
\mathbb{P}[\mathcal{L}_1(x) = m_1] \cdot f_{\delta(1)}(\delta_1) & \text{if } G_1 \leq \lambda \\
0 & \text{otherwise.}
\end{cases}$$
Distribution Function of Accounting Surplus

It is easy to show:

\[ P[S_{acct}^r \leq \xi] = \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} g_r(\xi + r V(m_r, \delta_r), m_r, \delta_r) \, d\delta_r \]
Figure: CDF of Accounting Surplus (1000 5-yr Temporary Policies)
Figure: CDF of Accounting Surplus (Limiting Portfolio)
Future Work

- Under the same assumptions ...
  - Probability of solvency over all years
  - Distribution of stochastic surplus
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- Under the same assumptions ...
  - Probability of solvency over all years
  - Distribution of stochastic surplus

- Extend model to ...
  - general portfolio
  - include expenses, lapses