Optimal Consumption Strategy in the Presence of Default Risk: Discrete-Time Case

K. C. Cheung and Dr. Hailiang Yang
Department of Statistics and Actuarial Science
The University of Hong Kong

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Regime-Switching Model

- Market situation may change $\Rightarrow$ distribution of asset's return will change over time

- Regime-Switching model: market environment may switch among different regimes in a Markovian manner $\Rightarrow$ distribution of asset's return will change over time in a Markovian manner

Regime-Switching Model

- Options: Di Masi et al. (1994), Buffington and Elliott (2001), Guo (2001), Hardy (2001)

- Optimal Trading Rules, Optimal Portfolio: Zhang (2001), Zhou and Yin (2003), Cheung and Yang (2004, 2004)

Model

Discrete-time setting: investor can decide the level of consumption, c_n at time $n = 0, 1, 2, \ldots, T$

After consumption, all the remaining money will be invested in a risky asset

The random return of the risky asset in different time periods will depend on the state of a time-homogeneous Markov chain $\{\xi_n\}_{0 \le n \le T}$ with state space $\mathcal{M} = \{1, 2, \ldots, M\}$ and transition probability matrix $\boldsymbol{P} = (p_{ij})$

Absorption State — Default Risk

Assume that state M of the Markov Chain is an absorbing state:

$$\begin{aligned} p_{Mj} &= 0 \qquad j = 1, 2, \ldots, M-1, \\ p_{MM} &= 1. \end{aligned}$$

Default occurs at time n if $\xi_n = M$. In this case, the investor can only receive a fraction, δ, of the amount that he/she should have received.

The recovery rate δ is a random variable, valued in $[0, 1]$

$\{W_n\}_{0\leq n\leq T}$: wealth process of the investor

$$W_{n+1}=\begin{cases}(W_n-c_n)R_n^{\xi_n}(1_{\{\xi_{n+1}\neq M\}}+\delta 1_{\{\xi_{n+1}=M\}}) & \text{if } \xi_n\neq M,\\ W_n-c_n & \text{if } \xi_n=M,\end{cases}$$

$n=0,1,\ldots,T-1$, where $1_{\{\cdots\}}$ is the indicator function.

R_n^i is the return of the risky asset in the time period $[n,n+1]$, given that the Markov chain is at regime i at time n.

Assumptions

1. The random returns $R_0^i, R_1^i, \ldots, R_{T-1}^i$ are i.i.d. with distribution F_i; they are strictly positive and integrable

2. R_n^i is independent of R_m^j, for all $m \neq n$

3. The Markov chain $\{\xi\}$ is stochastically independent to the random returns in the following sense:

$$\mathbb{P}(\xi_{n+1} = i_{n+1}, R_n^{i_n} \in B \mid \xi_0 = i_0, \ldots, \xi_n = i_n) = p_{i_n i_{n+1}} \mathbb{P}(R_n^{i_n} \in B)$$

for all $i_0, \ldots, i_n, i_{n+1} \in \mathcal{S}, B \in \mathfrak{B}(\mathbb{R})$ and $n = 0, 1, \ldots, T-1$

Assumptions

4. $0 \leq c_n \leq W_n$ (Budget constraint)

5. The recovery rate δ is stochastically independent of all other random variables

Given that the initial wealth is W_0 and the initial regime is $i_0 \in \mathcal{M}^* := \mathcal{M} \setminus \{M\}$, the objective of the investor is to

$$\max_{\{c_0,\ldots,c_T\}} \mathbb{E}_0 \left[\sum_{n=0}^{T} \frac{1}{\gamma} (c_n)^\gamma \right]$$

over all admissible consumption strategies. Here $0 < \gamma < 1$.

Admissible consumption strategy: a feedback law $c_n = c_n(\xi_n, W_n)$ satisfying the budget constraint

Optimal Consumption Strategy: $\widehat{C} = \{\widehat{c_0}, \ldots, \widehat{c_T}\}$

Definition 1 *For* $n = 0, 1, \ldots, T$, *the* value function $V_n(\xi_n, W_n)$ *is defined as*

$$V_n(\xi_n, W_n) = \max_{\{c_n, c_{n+1}, \ldots, c_T\}} \mathbb{E}_n \left[\sum_{k=n}^{T} \frac{1}{\gamma} (c_k)^\gamma \right].$$

Bellman's Equation:

$$\begin{cases} V_n(\xi_n, W_n) = \max_{0 \le c_n \le W_n} \mathbb{E}_n [U(c_n) + V_{n+1}(\xi_{n+1}, W_{n+1})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, \ldots, T-1 \\ V_T(\xi_T, W_T) = \frac{1}{\gamma} W_T^\gamma \end{cases}$$

Define some symbols recursively:

$$
\begin{aligned}
M^{(i)} &= \{\mathbb{E}[(R^i)^\gamma]\}^{\frac{1}{1-\gamma}}, \qquad i \in \mathcal{M}^*, \\
L_0^{(i)} &= 0, \qquad i \in \mathcal{M}, \\
L_n^{(i)} &= M^{(i)} K_n^{(i)} 1_{\{i \neq M\}} + n 1_{\{i=M\}}, \qquad i \in \mathcal{M}, n = 1, 2, \ldots, T, \\
K_1^{(i)} &= [1 - p_{iM} + p_{iM}\mathbb{E}(\delta^\gamma)]^{\frac{1}{1-\gamma}}, \qquad i \in \mathcal{M}^*, \\
K_n^{(i)} &= \left\{ \sum_{j=1}^{M-1} p_{ij}(1 + L_{n-1}^{(j)})^{1-\gamma} + p_{iM}\mathbb{E}(\delta^\gamma)(1 + L_{n-1}^{(M)})^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}, \\
& \quad i \in \mathcal{M}^*, n = 2, \ldots, T.
\end{aligned}
$$

Note that $K_\cdot^{(M)}$'s are not defined. $M^{(i)}$ is well-defined since we have assumed that R^i is integrable.

Theorem 1 *For* $n = 0, 1, \ldots, T$*, the value functions are given by*

$$V_{T-n}(i, w) = \frac{1}{\gamma} w^{\gamma} (1 + L_n^{(i)})^{1-\gamma},$$

and the optimal consumption strategy $\widehat{C}$ *is given by*

$$\widehat{c}_{T-n}(i, w) = \frac{w}{(1 + L_n^{(i)})}.$$

From Theorem 1, we see that if we are now at time $T - n$, and at regime i, then we should consume a fraction of our wealth which is equal to

$$\frac{1}{1 + L_i^{(n)}}.$$

Thus our optimal consumption strategy depends heavily on the current regime and the remaining investment time through the function L.

Proposition 1 *(a) For fixed $i \in \mathcal{M}$, $L_n^{(i)}$ is increasing in n:*

$$0 = L_0^{(i)} \le L_1^{(i)} \le \ldots \le L_T^{(i)}.$$

(b) For fixed $i \in \mathcal{M}^*$, $K_n^{(i)}$ is increasing in n:

$$0 \le K_1^{(i)} \le K_2^{(i)} \le \ldots \le K_T^{(i)}.$$

The monotonicity of L implies at the same regime, we should consume a larger fraction of our wealth when we are closer to the maturity.

This strategy is quite reasonable. If we are closer to the maturity, a short-term fluctuation in the return of the risky asset will bring a loss to us that we may not have enough time to cover. Therefore, we should consume more and invest less.

Next, we may guess that at any time period, say $T-n$, if we are at a “better” regime, then we should consume less and invest more.

Need two ingredients:

1. A criterion to compare the distributions of the returns in different regimes $\Longrightarrow$ **second order stochastic dominance**

2. Market has to “regular” enough $\Longrightarrow$ **stochastically monotone transition matrix**

Definition 2 *Suppose that* X *and* Y *are two random variables satisfying*

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$$

for any increasing and concave function g *such that the expectations exist, then we say* X is dominated by Y in the sense of second order stochastic dominance *and it is denoted by* $X \leq_{SSD} Y$*.*

Definition 3 *Suppose* $P = (p_{ij})$ *is an* $m \times m$ *stochastic matrix. It is called stochastically monotone if*

$$\sum_{l=k}^{m} p_{il} \leq \sum_{l=k}^{m} p_{jl}$$

for all $1 \leq i < j \leq m$ *and* $k = 1, 2, \ldots, m$.

Suppose $\boldsymbol{P}$ is a $M \times M$ matrix. Let $e_k = (1, \ldots, 1, 0, \ldots, 0)'$ (i.e. first k coordinates are 1, the rest are 0) for $k = 1, 2, \ldots, M$. Let $\mathbb{D}_M = \{(x_1, \ldots, x_M)' \in \mathbb{R}^M \mid x_1 \geq \cdots \geq x_M\}$ and $\boldsymbol{P}_D = \{y \in \mathbb{D}_M \mid \boldsymbol{P}y \in \mathbb{D}_M\}$.

Lemma 1 *The following statements are equivalent:*

1. $\boldsymbol{P}$ *is stochastically monotone*
2. $\boldsymbol{P}_D = \mathbb{D}_M$
3. $e_k \in \boldsymbol{P}_D$ *for all* $k = 1, 2, \ldots, M$

Proposition 2 *Suppose that the transition probability matrix* P *is stochastically monotone and*

$$R^1 \geq_{SSD} R^2 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}.$$

Assume further that

$$M^{(i)} K_1^{(i)} \geq 1 \qquad \forall i \in \mathcal{M}^*.$$

Then we have for $n = 1, 2, \ldots, T$

$$L_n^{(1)} \geq L_n^{(2)} \geq \cdots \geq L_n^{(M-1)} \geq L_n^{(M)},$$

as well as

$$K_n^{(1)} \geq K_n^{(2)} \geq \cdots \geq K_n^{(M-1)}.$$

Meaning of $R^1 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}$

Preference of investor: increasing and concave utility function

$+$

Return of the risky asset in regime i: R^i

$+$

Definition of SSD order

$\Downarrow$

The $M-1$ regimes are ranked according to their favorability to the risk-averse investor: regime 1 is the most favorable, regime $M-1$ is the most unfavorable

Meaning of P being stochastically monotone:

For $1 \leq i < j \leq M-1$ (regime i is more favorable to regime j)

- $\sum_{l=k}^{M} p_{il}$ is the probability of switching to the worst $m-k+1$ regimes from regime i

- $\sum_{l=k}^{M} p_{jl}$ is the probability of switching to the worst $m-k+1$ regimes from regime j

Intuitively, if the market is "regular" enough, we should have

$$\sum_{l=k}^{M} p_{il} \leq \sum_{l=k}^{M} p_{jl}$$

for all possible k. This precisely means that $\boldsymbol{P}$ is stochastically monotone.

Meaning of $M^{(i)}K_1^{(i)} \geq 1 \qquad \forall i \in \mathcal{M}^*$:

If \$1 is invested today (regime i), then $M^{(i)}K_1^{(i)}$ is the expected utility of the amount one period later, allowing for default risk.

$M^{(i)}K_1^{(i)} \geq 1 \quad \forall \in \mathcal{M}^*$ means that the risk-averse investor would prefer the risky asset to a risk-free asset (risk-free interest rate is zero) in any regimes.

Corollary 1 *Suppose that the transition probability matrix* P *is stochastically monotone and*

$$R^1 \geq_{SSD} R^2 \geq_{SSD} \cdots \geq_{SSD} R^{M-1}.$$

Assume further that

$$M^{(i)} K_1^{(i)} \geq 1 \qquad \forall i \in \mathcal{M}^*.$$

Then for $w > 0$ *and* $n = 0, 1, \ldots, T$*,*

$$c_n(1, w) \leq c_n(2, w) \leq \cdots \leq c_n(M, w).$$

Effect of Recovery Rate

Proposition 3 *Suppose that* δ_1 *and* δ_2 *are two* $[0,1]$*-valued random variables that are independent of the Markov chain* $\{\xi\}$ *and all the random returns. If*

$$\mathbb{E}[\delta_1^\gamma] \le \mathbb{E}[\delta_2^\gamma],$$

then

$$c_n(i, w; \delta_1) \ge c_n(i, w; \delta_2).$$

Example

- $\delta_1 \sim U(0,1) \longrightarrow \mathbb{E}(\delta_1^\gamma) = 1/(1+\gamma)$

- $\delta_2 \equiv 1/2 \longrightarrow \mathbb{E}(\delta_2^\gamma) = 1/2^\gamma$

It is not difficult to show that

$$\frac{1}{1+\gamma} \leq \frac{1}{2^\gamma}$$

for $0 < \gamma < 1$, i.e.

$$\mathbb{E}(\delta_1^\gamma) \leq \mathbb{E}(\delta_2^\gamma),$$

hence

$$c_n(i, w; \delta_1) \geq c_n(i, w; \delta_2).$$

THE END
THANK YOU