Bounds for Ruin Probabilities and Value at Risk

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July 29, 2007

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ACKNOWLEDGEMENTS

This research work was supported by the Actuarial Foundation, and the Casualty Actuarial Society. The last author was also partially supported by NSERC grant 290377.

BOUNDS FOR RUIN PROBABILITIES AND VALUE AT RISK

ABSTRACT. In many situations, complete information about a rare event is not available, meaning the underlying probability distribution is not completely specified. This paper finds the best one can do when the incomplete information consists of estimates of the first two moments of the distribution. These are called semiparametric lower and upper bounds. We consider value-at-risk (VaR) in the sense that we find bounds on probability of portfolio return less than some small value, given only the first two moments of the portfolio components. We also apply semiparametric bounds to a rare event hitting an insurer for which losses are extraordinary high and investment income is low. We refer to this as "ruin" although the company may survive; it is just a convenient way to describe a rare event that would threaten a company's solvency. In addition, we calculate bounds on insurance stop-loss payments. The payoff of a call or put option can be considered as a special case or a transform of the stop-loss payment. In order to numerically solve the semiparametric bounds considered here, we reformulate the corresponding semiparametric bound problem as a sum of squares (SOS) program. A SOS program is an optimization problem where the variables are coefficients of polynomials, the objective is a linear combination of the variable coefficients, and the constraints are given the polynomials being SOS. This form of reformulation allows us to use one of several readily available SOS programming solvers to solve the moment problem. For the stop-loss bound problem, Cox (1991)'s method is also investigated to confirm our SOS program solutions. Our numerical examples have shown that our technique works reasonably well.

1. INTRODUCTION

Sometimes, rare things happen and the least expected occurs. Indeed, some events occur once or twice in a lifetime — leaving little room to learn from experiences. In financial markets, extreme events, no matter how rare, could have a profound impact on a company or even the whole country (Liu, Pan, and Wang, 2005). One such example is the Asian currency crisis of 1997, largely attributed to over-expansion of corporate credit with un-hedged short-term borrowing from abroad; large amounts of unproductive capital investments; and speculation on overvalued assets and large trade deficits (Hong, 1998). In 1997, the value of Thai baht fell by 48.49%, Korean won dropped 47.46% and Malaysian ringgit fell by 35.36%.

Insurers are also not free from the impact of catastrophic large-scale extreme events. For example, the total loss of the tragic September 11 terrorist attacks exceeded \$80 billion with the insured losses amounting \$40.2 billion (Yu and Lin, 2007). As for mortality risks, a recent example of unanticipated catastrophe death losses is the devastating earthquake and tsunami across southern Asia and eastern Africa in December 26, 2004. The 2004 Indonesian population death index increased by 16.58% relative to the 2003 level (Cox, Lin, and Wang, 2006). The excess population mortality death rate is even higher for Sri Lanka, about 34%. Cummins and Doherty (1997) raise

Date: July 29, 2007.

concerns about the financial stability of the insurance industry before these recent catastrophic event, so the concern is even greater now.

As a result, with pervasive economic and financial revolutions sweeping our world and potential devastative catastrophes, the increasing interest in tail risk management is fuelled by practical issues, including investment downside risk and insurance catastrophe risk. Managing extreme losses caused by catastrophic events like U.S. stock market crash in 1929, hurricanes and earthquakes has been a major concern for market participants. Thus, developing statistical techniques to model extreme investment and insurance losses is certainly a major task for risk managers.

Unfortunately, our knowledge about the true distribution is limited. As such, increasing effort has been made to incorporate moment methodology into analysis without distribution assumptions. Among the first applications of the moment problem approach to *practical* problems were done by Scarf (1958) (inventory management) and Lo (1987) (mathematical finance). In particular, existing applications of moment theory in finance focus on option pricing to extend the well-known Black and Scholes (1973) formula (Merton, 1973; Perrakis and Ryan, 1984; Levy, 1985; Ritchken, 1985; Lo, 1987; Boyle and Lin, 1997; Bruckner, 2007; Gerber, Shiu, and Smith, 2007; Schepper and Heijnen, 2007) and other asset pricing and portfolio problems (Gallant, Hansen, and Tauchen, 1990; Hansen and Jagannathan, 1991; Ferson and Siegel, 2001, 2003). Brockett and Cox (1985); Cox (1991); Brockett, Cox, and Smith (1996) and Roos (2007) apply moment method in insurance. Bertsimas and Popescu (2005) give a review of the literature and historical perspective on this method, which covers developments from Chebyshev and Markov in the late 1800s to break-throughs in the last 10 years. However, very few papers use the moment method to study extreme financial and insurance events. Traditional statistical methods based on the estimation of the entire density are inappropriate for such tasks because these methods typically produce a good fit in those regions in which most of the data reside but at the expense of good fit in the tails (Hsieh, 2004). Therefore, the purpose of our paper is to apply moment methods to estimate the joint events such as concurrent extreme investment loss and insurance loss.

A novel aspect of this article is that it takes into account the correlation between different assets and insurance lines of business. Usually, models on risk-based capital and enterprise risk management decisions involve several random variables, such as losses, stock prices, interest rates, currency exchange rates and so on. There is an active interest in obtaining information on extremes of joint distributions of these random variables. For example, the insurer would like to know the probability of having a large loss payments exceeding a given threshold and a loss in their asset investment below a certain level at the same time. Therefore, the aim of the present paper is to explicitly solve the upper and lower bounds on the probability of such a joint event, given the first two sets of moments of the joint distribution. In other words it involves not only the variances of the individual asset returns and/or insurance margins but also their covariances.

In particular, suppose that X_1 and X_2 denote random variables in a model such as a random investment return and a random future insurance benefit payment. The variables may be dependent. For example, if the loss payment is subject to economic inflation, then it is correlated with investment return and the discount factor. In another example, the variables X_1 and X_2 may be security returns such as S&P 500 and Nikkei Index returns respectively. A risk manager might be interested in measuring the joint distribution of extreme values of X_1 and X_2. That is, X_1 and X_2 simultaneously take very high values. A third example comes from a stop-loss payment $\phi(X_1, X_2)$ with the form

$$\phi(X_1, X_2) = \begin{cases} b & \text{if } X_1 + X_2 \geq a + b \\ X_1 + X_2 - a & \text{if } a \leq X_1 + X_2 \leq a + b \\ 0 & \text{if } X_1 + X_2 \leq a. \end{cases} \tag{1}$$

Since the maximum claim amounts b will be paid when $X_1 + X_2 \geq a + b$, the reinsurer may want to know something about his expected payments. One way to estimate these measures is to use the observations of X_1 and X_2 to derive parameters of an assumed distribution (typically joint normal) and then reach an extreme measure of the joint distribution. In many instances, however, the low frequency of observations for X_1 and X_2 means that it is impossible to reach sound conclusions with the parametric approach. Even if a plenty of observations of X_1 and X_2 are available, for example, given day-to-day price observations, assuming a particular distribution for joint distribution of X_1 and X_2 might be perilous, specially when we are interested in estimating extreme joint distributions such as tail probabilities and value at risk (VaR). In fact, strong erroneous assumptions like this have lead to the failure of at least one hedge fund (e.g. the bankruptcy of the Long-Term Capital Management).

To address this problem, instead of assuming full knowledge of the distributions of the random variables of interest, to estimate extreme characteristics of the joint distribution, here, we show how to numerically compute upper and lower bounds on the probabilities $\Pr(w_1X_1 + w_2X_2 \leq a)$ and $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$ for some appropriate values of $t_1, t_2, w_1, w_2, a \in \mathbb{R}$, when assuming only up to the second order moment information (means, variances, and covariance) and the support of X_1 and X_2. Bounds on the stop-loss payment $\phi(X_1, X_2)$ are also computed for some levels of $a, b \in \mathbb{R}^+$ given certain supports and moments. These types of bounds are usually called semiparametric bounds (in recent related literature) or generalized Chebyshev inequalities (in classical probability theory).

The computation of semiparametric bounds is a classical probability problem (Karlin and Studden (1966); Vandenberghe, Boyd, and Comanor (2007); and Zuluaga and Peña (2005)). As a consequence, many related results come from different areas, such as finance, risk management, inventory theory, stochastic programming, supply chain management, and actuarial science. They are also widely used in other areas when complete information about the random variables of interest is unknown. For example, consider the work of Lo (1987); Grundy (1991); Boyle and Lin (1997); Bertsimas and Popescu (2002); Bertsimas and Sethuraman (2000); Cox (1991); Brockett et al. (1996); Bertsimas, Natarajan, and Teo (2006); Dokov and Morton (2005); Gallego and Moon (1993); Scarf (1958); Yue, Chen, and Wang (2006); and the references therein. Generally, semiparametric bounds are robust bounds that any reasonable model must satisfy. Moreover, they provide a mechanism for checking the consistency of models, as well as an initial estimate for cumulative probabilities regardless of any model specifications.

The remainder of the article is organized as follows. In Section 2, we formally state the semiparametric bound problems considered here. Furthermore, we outline the key well-known results that will be used in Section 3. Section 3 shows how the desired semiparametric bounds can be numerically computed with readily available optimization solvers. In Section 4, we present relevant numerical experiments to illustrate the application of our results. Section 5 is for our conclusions.

2. Preliminaries and Notation

Throughout the article, we focus on numerically solving joint semiparametric bound problems of the form:

$$\begin{array}{llll} \overline{p} \text{ (or } \underline{p}) = & \sup \text{(or inf)} & \mathbb{E}_\pi(\phi(X_1, X_2)) & \\ & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\ & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1, 2, \\ & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1, 2, \\ & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, & \\ & & \pi \text{ a probability distribution in } \mathcal{D}, & \end{array} \tag{2}$$

for relevant choices of the (given) function $\phi(X_1, X_2)$. In problem (2), μ_i, $\mu_i^{(2)}$, $i = 1, 2$, and μ_{12} denote the given first and second order non-central moments of the random variables X_1, X_2 (which can be readily obtained from mean, variance, and covariance information on X_1, X_2), and $\mathcal{D} \subseteq \mathbb{R}^2$ denotes the given support of X_1, X_2. Thus, problem (2) maximizes (or minimizes) the expected value $\mathbb{E}_\pi(\phi(X_1, X_2)) := \int_\mathcal{D} \phi(x_1, x_2) d\pi$ over all joint probability distributions π with support in $\mathcal{D} \subseteq \mathbb{R}^2$.

In particular, given $w_1, w_2, a \in \mathbb{R}$, we compute semiparametric bounds on $\Pr(w_1 X_1 + w_2 X_2 \leq a)$, for random variables X_1 and X_2, by setting $\phi(X_1, X_2) = \mathbb{I}_{\{w_1 X_1 + w_2 X_2 \leq a\}}$, and $\mathcal{D} = \mathbb{R}^2$; where $\mathbb{I}_\mathcal{S}$ is the indicator function of the set $\mathcal{S}$. Similarly, given $t_1, t_2 \in \mathbb{R}^+$ and non-negative random variables X_1 and X_2, we compute semiparametric bounds on the probability $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$, by setting $\phi(X_1, X_2) = \mathbb{I}_{\{X_1 \leq t_1 \text{ and } X_2 \leq t_2\}}$, and $\mathcal{D} = \mathbb{R}^{+2}$. Finally, given $a, b \in \mathbb{R}^+$, we compute semiparametric bounds on a stop-loss payment $\phi(X_1, X_2)$ in the form of equation (1), for non-negative random variables X_1 and X_2.

Let $\overline{p}$ ($\underline{p}$) be the optimal objective value of the sup (inf) version of problem (2). Notice that with the values of $\overline{p}$ and $\underline{p}$, we obtain a "100% confidence interval" $\underline{p} \leq \mathbb{E}_\pi(\phi(X_1, X_2)) \leq \overline{p}$ on the expected value of $\phi(X_1, X_2)$ for all models of the joint distribution of X_1, X_2 given some moments and support.

In order to numerically solve the semiparametric bounds considered here, we will reformulate the corresponding semiparametric bound problem (2) as a sum of squares (SOS) program (cf. Prajna, Papachristodoulou, and Parrilo (2002) and the references therein). A detailed discussion about SOS programming is outside the scope of this article. However, let us mention that (informally) a SOS program is an optimization problem where the variables are coefficients of polynomials, the objective is a linear combination of the variable coefficients, and the constraints are given by the polynomials being SOS. A polynomial $p(x_1, \dots, x_n) := \sum_{i_1,\dots,i_n \in \mathbb{N}} y_{(i_1,\dots,i_n)} x_i^{i_1} \cdots x_n^{i_n}$ is said to be a SOS if $p(x_1, \dots, x_n) = \sum_i q_i(x_1, \dots, x_n)^2$ for some polynomials $q_i(x_1, \dots, x_n)$. The advantage of reformulating problem (2) as a SOS program is that the reformulation can be readily solved by some SOS programming solvers such as `SOSTOOLS` (cf. Prajna et al. (2002)), `GloptiPoly` (cf. Henrion and Lasserre (2003)), or `YALMIP` (cf. Löfberg (2004)). It is worth mentioning that any SOS program can be reformulated as a semidefinite program (SDP) (cf. Todd (2001), Parrilo (2000), and the references therein). In fact, SOS programming solvers work by reformulating the SOS program as a SDP, and then using SDP solvers such as SeDuMi (cf. Sturm (1999)). However, the SDP formulations of SOS programs can be fairly involved. To make it easy to reproduce our results, throughout the article we implement SOS programming tools instead of directly reformulating problem (2) as a SDP.

In order to obtain the desired SOS programming formulations, we will make use of the following well-known results about positive polynomials (cf. Prestel and Delzell (2001)).

Theorem 1 (Hilbert (1888))**.** *Let $p(x_1, \dots, x_n)$ be a quadratic polynomial. Then $p(x_1, \dots, x_n) \geq 0,\ \forall\, x_1, \dots, x_n \in \mathbb{R}$ if and only if $p(x_1, \dots, x_n)$ is a SOS polynomial.*

Theorem 2 (Diananda (1962))**.** *Let $p(x_1, \dots, x_n)$ be a quadratic polynomial. If $n \leq 3$, then $p(x_1, \dots, x_n) \geq 0,\ \forall\, x_1, \dots, x_n \geq 0$ if and only if $p(x_1^2, \dots, x_n^2)$ is a SOS polynomial.*

Notice that in both theorems above, we have chosen to present the results in a form that will be suitable for our purposes, instead of presenting them in their original form. In particular, the statement of Diananda's Theorem above means that to check if

$$p(x_1, x_2) = y_{00} + y_{10}x_1 + y_{01}x_2 + y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2$$

is positive for all $x_1, x_2 \geq 0$, one can check whether

$$p(x_1^2, x_2^2) = y_{00} + y_{10}x_1^2 + y_{01}x_2^2 + y_{20}x_1^4 + y_{02}x_2^4 + y_{11}x_1^2x_2^2$$

is a SOS. For a discussion about the equivalence between the original version of Diananda's Theorem, and Theorem 2 above, the reader is directed to Parrilo (2000), and Zuluaga (2004).

Another key result that will be used throughout the article is the fact the dual of problem (2) is (see, e.g., Karlin and Studden (1966); Bertsimas and Popescu (2002); and Zuluaga and Peña (2005)):

$$\begin{aligned} \overline{d} \text{ (or } \underline{d}) = \quad & \inf \text{ (or sup)} \quad y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} \quad p(x_1, x_2) \geq \text{ (or } \leq) \; \phi(x_1, x_2), \forall \; (x_1, x_2) \in \mathcal{D}, \end{aligned} \tag{3}$$

where the quadratic polynomial

$$p(x_1, x_2) := y_{00} + y_{10}x_1 + y_{01}x_2 + y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2.$$

It is not difficult to see that weak duality holds between (2) and (3); that is, $\overline{p} \leq \overline{d}$ (or $\underline{p} \geq \underline{d}$). More importantly, for the specific problems considered here, we will show that strong duality holds between (2) and (3); that is $\overline{p} = \overline{d}$ (or $\underline{p} = \underline{d}$), as long as (2) is feasible. Thus, in order to obtain the semiparametric bound $\overline{p}$ (or $\underline{p}$), we can solve (3). As we will see in the following sections, the constraint in the polynomial $p(x_1, x_2)$ in (3) is what leads to the use of results about positive polynomials such as Theorems 1 and 2 to solve (2) by using SOSTOOLS. This approach has been widely used to solve semiparametric bound problems in a number of areas (see, e.g., Karlin and Studden (1966); Bertsimas and Popescu (2002); Zuluaga and Peña (2005); Boyle and Lin (1997); Bertsimas et al. (2006); Lasserre (2002); Kemperman (1968); Kemperman (1965); and Vandenberghe et al. (2007)).

3. SOS Programming Formulations

In this section we formally present three semiparametric bound problems: VaR probability, joint probability and stop-loss payment of two random variables. Furthermore, we present their corresponding SOS programming formulations.

3.1. **VaR Probability Bounds.** We first consider the problem of finding sharp upper and lower bounds on $\Pr(w_1X_1 + w_2X_2 \leq a)$. Specifically, without making any assumption (other than moments) on the distribution of the random variables X_1, X_2, we solve for the probability that the portfolio $w_1X_1 + w_2X_2$ ($w_1, w_2 \in \mathbb{R}$) attains values lower than or equal to $a \in \mathbb{R}$, given up to the second order moment information (means, variances, and covariance) on X_1, X_2. The sharp upper and lower semiparametric bounds for this problem can be (respectively) formulated as the following optimization problems, obtained by setting problem (2) as $\phi(X_1, X_2) = \mathbb{I}_{\{w_1X_1+w_2X_2 \leq a\}}$, and $\mathcal{D} = \mathbb{R}^2$ (cf. Section 2). The upper bound is

$$
\begin{array}{llll}
\overline{p}_{\mathrm{VaR}} := & \sup & \mathbb{E}_\pi(\mathbb{I}_{\{w_1X_1+w_2X_2\le a\}}) & \\
 & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\
 & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1,2, \\
 & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1,2, \\
 & & \mathbb{E}_\pi(X_1X_2) = \mu_{12}, & \\
 & & \pi \text{ a probability distribution in } \mathbb{R}^2. &
\end{array}
\tag{4}
$$

And the lower bound is as follows:

$$
\begin{array}{llll}
\underline{p}_{\mathrm{VaR}} := & \inf & \mathbb{E}_\pi(\mathbb{I}_{\{w_1X_1+w_2X_2\le a\}}) & \\
 & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\
 & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1,2, \\
 & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1,2, \\
 & & \mathbb{E}_\pi(X_1X_2) = \mu_{12}, & \\
 & & \pi \text{ a probability distribution in } \mathbb{R}^2. &
\end{array}
\tag{5}
$$

Before obtaining the SOS programming formulation of these problems, let us state the well-known feasibility condition in terms of the moment parameters (Bertsimas and Sethuraman, 2000, Theorem 16.1.2).

Observation 1 (Feasibility)**.** *Problems* (4) *and* (5) *are feasible if and only if* Σ *is a positive semi-definite matrix (i.e., all eigenvalues are greater than or equal to zero), where* Σ *is the moment matrix:*

$$
\Sigma = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1^{(2)} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_2^{(2)} \end{bmatrix}.
$$

Proof. Follows from Diananda's Theorem (Theorem 2) and convex duality (cf. Rockafellar (1970)). □

Next we derive SOS programs to numerically compute $\overline{p}_{\mathrm{VaR}}$, and $\underline{p}_{\mathrm{VaR}}$ by using SOS programming solvers. To simplify the exposition, we will from now on assume without loss of generality that $w_1 = w_2 = 1$ in both (4), and (5).

3.1.1. *Upper bound.* We begin by stating the dual problem of (4):

$$
\begin{array}{lll}
\overline{d}_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
 & \text{such that} & p(x_1,x_2) \ge \mathbb{I}_{\{x_1+x_2\le a\}}, \forall\, x_1,x_2 \in \mathbb{R}.
\end{array}
\tag{6}
$$

As the following observation states, as long as problem (4) is feasible, we can obtain $\overline{p}_{\mathrm{var}}$ by solving problem (6).

Observation 2 (Strong Duality)**.** *Notice that the dual solution $y_{00} = 2$, and $y_{ij} = 0$ for $(i,j) \neq (0,0)$ strictly satisfies (i.e., with $>$) the constraint in* (6) *for all $x_1, x_2 \in \mathbb{R}$. Thus, if problem* (4) *is feasible, then $\overline{p}_{\mathrm{VaR}} = \overline{d}_{\mathrm{VaR}}$.*

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (6) as a SOS program, we proceed as follows. First notice that (6) is equivalent to:

$$(7)\qquad \begin{array}{rll} \overline{d}_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(x_1,x_2) \geq 1, \forall\, x_1, x_2 \text{ s.t. } x_1 + x_2 \leq a \\ & & p(x_1,x_2) \geq 0, \forall\, x_1, x_2 \in \mathbb{R}. \end{array}$$

Notice that we can directly express the second constraint in (7) as a SOS constraint by using Theorem 1. For the first constraint however, we need more work. Specifically, consider the transformation of the axes below:

(8)

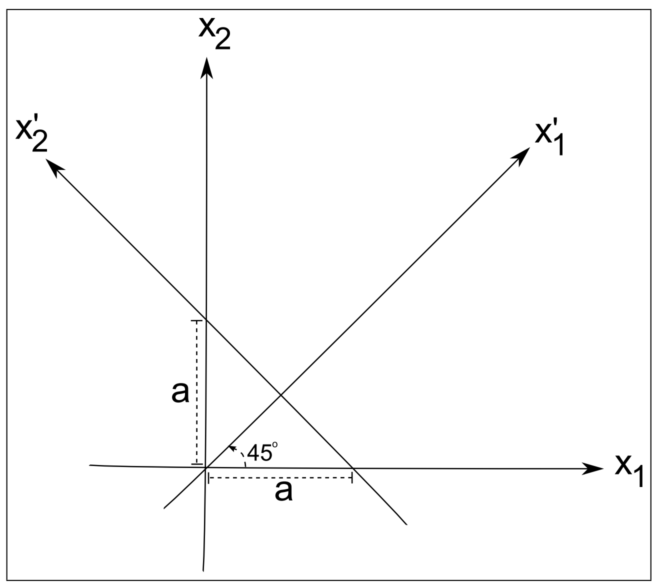
$$x_1' = x_1 \cos 45^0 + x_2 \sin 45^0 - a \cos 45^0$$
$$x_2' = -x_1 \sin 45^0 + x_2 \cos 45^0$$

that is:

$$x_1' = \tfrac{\sqrt{2}}{2}(x_1 + x_2 - a)$$
$$x_2' = \tfrac{\sqrt{2}}{2}(-x_1 + x_2)$$

or:

$$x_1 = \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' - x_2')$$
$$x_2 = \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' + x_2')$$



Applying the substitution $x_1 \to \frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1' - x_2')$, $x_2 \to \frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1' + x_2')$ to the first constraint of (7), we have that:

$$p(x_1,x_2) \geq 1, \forall\, x_1, x_2 \text{ s.t. } x_1 + x_2 \leq a$$
$$\Updownarrow$$
$$p(\tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' - x_2'), \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' + x_2')) \geq 1, \forall\, x_1' \leq 0, x_2' \in \mathbb{R}$$
$$\Updownarrow$$
$$p(\tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' - x_2'), \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' + x_2')) \geq 1, \forall\, x_1' \leq 0, x_2' \geq 0$$
$$p(\tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' - x_2'), \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1' + x_2')) \geq 1, \forall\, x_1' \leq 0, x_2' \leq 0.$$

By substituting $x_1' \to -x_1'$ in the second to last equation above and substituting $x_1' \to -x_1', x_2' \to -x_2'$ in the last equation, we obtain that (7) is equivalent to:

(9)
$$\begin{array}{lll} \overline{d}_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1' - x_2'), \frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1' + x_2')) - 1 \geq 0, \forall\, x_1' \geq 0, x_2' \geq 0 \\ & & p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1' + x_2'), \frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1' - x_2')) - 1 \geq 0, \forall\, x_1' \geq 0, x_2' \geq 0. \\ & & p(x_1, x_2) \geq 0, \qquad\qquad \forall\, x_1, x_2 \in \mathbb{R}. \end{array}$$

To finish, from Theorem 2 (applied to the first two constraints of (9)) and Theorem 1 (applied to the last constraint of (9)), it follows that (9) is equivalent to the following SOS program:

(10)
$$\begin{array}{lll} \overline{d}_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1^2 - x_2^2), \frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1^2 + x_2^2)) - 1 \text{ is a SOS polynomial} \\ & & p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1^2 + x_2^2), \frac{1}{2}a + \frac{1}{\sqrt{2}}(-x_1^2 - x_2^2)) - 1 \text{ is a SOS polynomial} \\ & & p(x_1^2, x_2^2) \text{ is a SOS polynomial.} \end{array}$$

Notice that above we drop the primes in the variable labels (they are just variable labels). Also, we do not go through the details of $q(x_1, x_2) = p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1 - x_2), \frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1 + x_2)) - 1$ and the SOS constraint $q(x_1^2, x_2^2) = p(\frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1^2 - x_2^2), \frac{1}{2}a + \frac{1}{\sqrt{2}}(x_1^2 + x_2^2)) - 1$. The algebraic expressions of the polynomials in (10) are left out for brevity purposes. In fact, with current SOS solvers it is not even necessary to provide the expanded algebraic expression of these polynomials.

The SOS program (10) can be readily solved with a SOS programming solver. Thus, if problem (4) is feasible (cf. Observation 1), it follows from Observation 2 that we can numerically obtain the VaR semiparametric upper bound $\overline{p}_{\mathrm{VaR}}$ by solving problem (10) with a SOS solver.

3.1.2. *Lower bound.* We start with a semiparametric bound closely related to problem (5) as follows:

$$\begin{array}{lll} \underline{p}^c_{\mathrm{VaR}} := & \sup & \mathbb{E}_\pi(\mathbb{I}_{\{w_1X_1 + w_2X_2 \geq a\}}) \\ & \text{such that} & \mathbb{E}_\pi(1) = 1, \\ & & \mathbb{E}_\pi(X_i) = \mu_i, \qquad i = 1, 2, \\ & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, \qquad i = 1, 2, \\ & & \mathbb{E}_\pi(X_1X_2) = \mu_{12}, \\ & & \pi \text{ a probability distribution in } \mathbb{R}^2. \end{array} \tag{11}$$

Notice that in (11) we are computing the upper semiparametric bound on the complement of the VaR probability $\Pr(w_1X_1 + w_2X_2 \geq a)$. Thus, it clearly follows that $\underline{p}_{\mathrm{VaR}} = 1 - \underline{p}^c_{\mathrm{VaR}}$ (cf. (5)). The feasibility of problem (11) is also characterized by the moment matrix condition of Observation 1; and as it turns out, it is much easier to reformulate (11) as a SOS program.

As in previous sections, we begin by stating the dual of (11):

$$
\begin{array}{llll}
 & \underline{d}^c_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
(12) & & & \\
 & & \text{such that} & p(x_1, x_2) \geq \mathbb{I}_{\{x_1+x_2 \geq a\}}, \forall\, x_1, x_2 \in \mathbb{R}.
\end{array}
$$

Analogous to Section 3.1.1 (see Observation 2), strong duality between (11) and (12) follows if (11) is feasibe (i.e., if the condition in Observation 1 is satisfied).

Following the analogous steps to those taken in Section 3.1.1 for problem (6), we obtain that problem (12) is equivalent to the SOS program below:

$$
\begin{array}{llll}
 & \underline{d}^c_{\mathrm{VaR}} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
(13) & & \text{such that} & p(\tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1^2 - x_2^2), \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1^2 + x_2^2)) - 1 \text{ is a SOS polynomial} \\
 & & & p(\tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1^2 + x_2^2), \tfrac{1}{2}a + \tfrac{1}{\sqrt{2}}(x_1^2 - x_2^2)) - 1 \text{ is a SOS polynomial} \\
 & & & p(x_1^2, x_2^2) \text{ is a SOS polynomial.}
\end{array}
$$

The SOS program (13) can be readily solved with a SOS programming solver. Thus, if problem (5) is feasible (cf. Observation 1), it follows that we can numerically obtain the VaR semiparametric lower bound $\underline{p}_{\mathrm{VaR}} = 1 - \underline{d}^c_{\mathrm{VaR}}$ by solving problem (13) with a SOS solver.

It follows the probability $\Pr(X_1 + X_2 \leq a) = 1 - \Pr(X_1 + X_2 \geq a)$. As long as we know the upper and lower bounds on $\Pr(X_1 + X_2 \geq a)$, the bounds $\Pr(X_1 + X_2 \leq a)$ can be easily obtained.

3.2. **Probability Bounds.** We consider the problem of finding sharp upper and lower bounds on the probability $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$ of two non-negative random variables X_1, X_2, attaining values lower than or equal to $t_1, t_2 \in \mathbb{R}^+$ respectively, without making any assumption on the distribution of the random variables X_1, X_2. Finding the sharp upper and lower semiparametric bounds for this problem can be obtained by setting problem (2) as $\phi(X_1, X_2) = \mathbb{I}_{\{X_1 \leq t_1 \text{ and } X_2 \leq t_2\}}$ and $\mathcal{D} = \mathbb{R}^{+2}$ (cf. Section 2), given up to the second order moment information (means, variances, and covariance) on X_1, X_2:

$$
\begin{array}{lllll}
 & \overline{p} := & \sup & \mathbb{E}_\pi(\mathbb{I}_{\{X_1 \leq t_1 \text{ and } X_2 \leq t_2\}}) & \\
 & & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\
(14) & & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1, 2, \\
 & & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1, 2, \\
 & & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, & \\
 & & & \pi \text{ a probability distribution in } \mathbb{R}^{+2}, &
\end{array}
$$

and

$$
\begin{array}{llll}
& \underline{p} := & \inf & \mathbb{E}_\pi(\mathbb{I}_{\{X_1 \le t_1 \text{ and } X_2 \le t_2\}}) \\
& & \text{such that} & \mathbb{E}_\pi(1) = 1, \\
(15) & & & \mathbb{E}_\pi(X_i) = \mu_i, \qquad i = 1,2, \\
& & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, \qquad i = 1,2, \\
& & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, \\
& & & \pi \text{ a probability distribution in } \mathbb{R}^{+2}.
\end{array}
$$

Before obtaining the SOS programming formulation of these problems, we discuss their feasibility in terms of the moment information.

Observation 3 (Feasibility)**.** *Problems* (14) *and* (15) *are feasible if and only if* Σ *is a positive semidefinite matrix (i.e., all eigenvalues are greater than or equal to zero) and all elements of* Σ *are non-negative, where* Σ *is the moment matrix:*

$$
\Sigma = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1^{(2)} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_2^{(2)} \end{bmatrix}.
$$

Next we derive SOS programs to numerically compute $\overline{p}$, and $\underline{p}$ by using SOS programming solvers.

3.2.1. *Upper bound.* We begin by stating the dual problem of (14):

$$
\begin{array}{llll}
& \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
(16) & & \text{such that} & p(x_1, x_2) \ge \mathbb{I}_{\{x_1 \le t_1 \text{ and } x_2 \le t_2\}}, \forall\, x_1, x_2 \ge 0.
\end{array}
$$

As the following observation states, as long as problem (14) is feasible, we can obtain $\overline{p}$ by solving problem (16).

Observation 4 (Strong Duality)**.** *Notice that the dual solution* $y_{00} = 2$*, and* $y_{ij} = 0$ *for* $(i, j) \ne (0, 0)$ *strictly satisfies (i.e., with* $>$*) the constraint in* (16) *for all* $x_1, x_2 \ge 0$*. Thus, if problem* (14) *is feasible, then* $\overline{p} = \overline{d}$*.*

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (16) as a SOS program, we proceed as follows. First notice that (16) is equivalent to:

$$
\begin{array}{llll}
& \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
(17) & & \text{such that} & p(x_1, x_2) \ge 1, \forall\, 0 \le x_1 \le t_1, 0 \le x_2 \le t_2 \\
& & & p(x_1, x_2) \ge 0, \forall\, x_1, x_2 \ge 0.
\end{array}
$$

Although the second constraint of (17) can be handled directly, the first constraint is difficult to reformulate as a SOS constraint. That is, there is no linear transformation from $0 \le x_1 \le t_1, 0 \le x_2 \le t_2$ to $\mathbb{R}^{+2}$ or to $\mathbb{R}^2$ (that would allow the use of Theorems 1 and 2). Thus, we change the problem to end up with a SOS program that either solves or approximates problem (17). Specifically, consider the following problem related to (17):

$$(18)\qquad \begin{array}{rll} \overline{d}' = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(x_1, x_2) \ge 1, \forall\, x_1 \le t_1, x_2 \le t_2 \\ & & p(x_1, x_2) \ge 0, \forall\, x_1 \ge 0, x_2 \ge 0. \end{array}$$

Notice that (18) is less constrained than (17) (the first constraint of (18) includes more values of x_1 and x_2). Thus, $\overline{d}'$ is a upper bound on $\overline{d}$; that is $\overline{d}' \ge \overline{d}$ (in fact, our intuition suggests that $\overline{d}' = \overline{d}$).

After we apply the substitution $x_1 \to t_1 - x_1, x_2 \to t_2 - x_2$ to the first constraint of (18), problem (18) is equivalent to:

$$(19)\qquad \begin{array}{rlll} \overline{d}' = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} & \\ & \text{such that} & p(t_1 - x_1, t_2 - x_2) - 1 \ge 0, & \forall\, x_1, x_2 \ge 0 \\ & & p(x_1, x_2) \ge 0, & \forall\, x_1, x_2 \ge 0. \end{array}$$

If we let $q(x_1, x_2) = p(t_1 - x_1, t_2 - x_2) - 1$, i.e.

$$\begin{aligned} q(x_1, x_2) = \ & (y_{00} + y_{10}t_1 + y_{01}t_2 + y_{20}t_1^2 + y_{02}t_2^2 + y_{11}t_1t_2 - 1) \\ & -(y_{10} + 2t_1y_{20} + y_{11}t_2)x_1 \\ & -(y_{01} + 2t_2y_{02} + y_{11}t_1)x_2 \\ & +y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2. \end{aligned}$$

The first constraint of (19) can be replaced by $q(x_1, x_2) \ge 0, \forall\, x_1, x_2 \ge 0$. To finish, from Theorem 2, it follows that (19) (with the first constraint written in terms of $q(x_1, x_2)$) is equivalent to the following SOS program:

$$(20)\qquad \begin{array}{rll} \overline{d}' = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & q(x_1^2, x_2^2) \text{ is a SOS polynomial} \\ & & p(x_1^2, x_2^2) \text{ is a SOS polynomial.} \end{array}$$

The SOS program (20) can be readily solved with a SOS programming solver. Thus, if problem (14) is feasible (cf. Observation 3), it follows from Observation 4 that we can numerically obtain a semiparametric bound $\Pr(X_1 \le t_1, X_2 \le t_2) \le \overline{d}'$ by solving problem (20) with a SOS solver.

3.2.2. *Lower bound.* We begin with stating the dual problem of (15):

$$\underline{d} = \sup \quad y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$
$$\text{such that} \quad p(x_1, x_2) \leq \mathbb{I}_{\{x_1 \leq t_1 \text{ and } x_2 \leq t_2\}}, \forall\, x_1, x_2 \geq 0. \tag{21}$$

As the following observation states, as long as problem (15) is feasible, we can obtain $\underline{p}$ by solving problem (21).

Observation 5 (Strong Duality)**.** *Notice that the dual solution* $y_{00} = -1$*, and* $y_{ij} = 0$ *for* $(i,j) \neq (0,0)$ *strictly satisfies (i.e., with* $<$*) the constraint in* (21) *for all* $x_1, x_2 \geq 0$*. Thus, if problem* (15) *is feasible, then* $\underline{p} = \underline{d}$*.*

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). ☐

Now, problem (21) is equivalent to

$$\begin{aligned} \underline{d} = \sup \quad & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ \text{such that} \quad & p(x_1, x_2) \leq 1, \forall\, 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2 \\ & p(x_1, x_2) \leq 0, \qquad \forall\, x_1 \geq t_1, x_2 \geq 0, \\ & p(x_1, x_2) \leq 0, \qquad \forall\, x_1 \geq 0, x_2 \geq t_2. \end{aligned} \tag{22}$$

Using the similar approximation as the upper bound to the first constraint and applying $x_1 \to t_1 + x_1, x_2 \to t_2 + x_2$ to the second and third constraints respectively, we have that (22) can be approximated by solving:

$$\begin{aligned} \underline{d}' = \sup \quad & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ \text{such that} \quad & 1 - p(t_1 - x_1, t_2 - x_2) \geq 0, \forall\, x_1, x_2 \geq 0 \\ & -p(t_1 + x_1, x_2) \geq 0, \qquad \forall\, x_1, x_2 \geq 0, \\ & -p(x_1, t_2 + x_2) \geq 0, \qquad \forall\, x_1, x_2 \geq 0, \end{aligned} \tag{23}$$

where $\underline{d}' \leq \underline{d}$. Similar to the upper bound problem, we now let:

$$\begin{aligned} q_1(x_1, x_2) &= 1 - p(t_1 - x_1, t_2 - x_2) \\ q_2(x_1, x_2) &= -p(t_1 + x_1, x_2) \\ q_3(x_1, x_2) &= -p(x_1, t_2 + x_2); \end{aligned}$$

that is,

$$
\begin{aligned}
q_1(x_1,x_2) = \;& 1-(y_{00}+y_{10}t_1+y_{01}t_2+y_{20}t_1^2+y_{02}t_2^2+y_{11}t_1t_2)\\
& +(y_{10}+2t_1y_{20}+y_{11}t_2)x_1+(y_{01}+2t_2y_{02}+y_{11}t_1)x_2\\
& -y_{20}x_1^2-y_{02}x_2^2-y_{11}x_1x_2\\
q_2(x_1,x_2) = \;& -(y_{00}+y_{10}t_1+y_{20}t_1^2)\\
& -(y_{10}+2y_{20}t_1)x_1-(y_{01}+y_{11}t_1)x_2\\
& -y_{20}x_1^2-y_{02}x_2^2-y_{11}x_1x_2\\
q_3(x_1,x_2) = \;& -(y_{00}+y_{01}t_2+y_{02}t_2^2)\\
& -(y_{10}+y_{11}t_2)x_1-(y_{01}+2y_{02}t_2)x_2\\
& -y_{20}x_1^2-y_{02}x_2^2-y_{11}x_1x_2.
\end{aligned}
$$

To finish, from Theorem 2, it follows that (23) (with the three constraints written in terms of $q_i(x_1,x_2)$, $i=1,2,3$) is equivalent to the following SOS program:

$$
\begin{aligned}
\underline{d}' = \quad & \sup \quad && y_{00}+y_{10}\mu_1+y_{01}\mu_2+y_{20}\mu_1^{(2)}+y_{02}\mu_2^{(2)}+y_{11}\mu_{12}\\
& \text{such that} && q_1(x_1^2,x_2^2) \text{ is a SOS polynomial}\\
& && q_2(x_1^2,x_2^2) \text{ is a SOS polynomial}\\
& && q_3(x_1^2,x_2^2) \text{ is a SOS polynomial.}
\end{aligned}
\tag{24}
$$

The SOS program (24) can be readily solved with a SOS programming solver. Thus, if problem (15) is feasible (cf. Observation 3), it follows from Observation 5 that we can numerically approximate the ruin probability semiparametric lower bound $\underline{d}$ by solving problem (24) with a SOS solver. Furthermore, notice that by solving (20) and (24) we obtain a "100% confidence interval" $\underline{d}' \le \Pr(X_1 \le t_1 \text{ and } X_2 \le t_2) \le \overline{d}'$ on the value of the probability when given only up to the second order moment information on the non-negative random variables X_1, X_2.

Following the same technique, we also derive the upper and lower bounds on the joint probability $\Pr(X_1 \ge t_1 \text{ and } X_2 \ge t_2)$ of two non-negative random variables X_1, X_2. See Appendix A for details.

3.3. **Bounds on Stop-Loss payments.** Stop-loss payments we consider here have two loss components X_1 and X_2. For example, a homeowner's policy covers both property losses X_1 and liability losses X_2. Similarly, X_1 could be hospital room and board costs and X_2 be surgical expenses in health insurance. We find the upper and lower bounds on the aggregate loss $Z = X_1 + X_2$, given the mean, variance and covariance of X_1 and X_2. This time our function $\phi(X_1, X_2)$ in problem (2) is defined as follows:

$$
\phi(X_1,X_2) = \begin{cases} b & \text{if } X_1+X_2 \ge a+b\\ X_1+X_2-a & \text{if } a \le X_1+X_2 \le a+b\\ 0 & \text{if } X_1+X_2 \le a. \end{cases}
\tag{25}
$$

Suppose the function $\phi(X_1, X_2)$ represents the benefits a direct insurer pays to a reinsurer, given losses of X_1 and X_2. Under this contract, when the total losses are less than a, the direct insurer retains all losses. When the sum exceeds the threshold a, the reinsurer pays the excess up to a maximum of b. If the total losses exceed $a + b$, the part higher than b will be retained or ceded to other reinsurers by the direct insurer. Compared with the previous problems, bounds on stop-loss coverage is relatively easy to compute since X_1 and X_2 always appear in the form of $X_1 + X_2$ in the objective function (25). Therefore, this problem can be considered as a one variable problem by setting $Z = X_1 + X_2$ and calculating the moments of Z as follows:

$$\mu_z = \mu_1 + \mu_2 \text{ and } \mu_z^{(2)} = \mu_1^{(2)} + \mu_2^{(2)} + 2\mu_{12}.$$

With this transformation, the objective function (25) can be written as:

$$\phi(Z) = \begin{cases} b & \text{if } Z \geq a + b \\ Z - a & \text{if } a \leq Z \leq a + b \\ 0 & \text{if } Z \leq a. \end{cases} \tag{26}$$

Cox (1991) provides an explicit solution to a transformed problem of (26).[1] We first solve this problem numerically with a SOS program and then compare its results with those obtained from Cox (1991)'s method to test the robustness of the SOS approach.

3.3.1. *SOS program.* Given problem (25) and $\mathcal{D} = \mathbb{R}^{+2}$, the upper and lower semiparametric bounds for this problem are formulated as the following optimization problems:[2]

$$\begin{array}{rlll} \overline{p}_{\text{StopLoss}}(\text{or } \underline{p}_{\text{StopLoss}}) = & \sup\,(\text{or } \inf) & \mathbb{E}_\pi(\phi(X_1 + X_2)) & \\ & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\ & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1, 2, \\ & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1, 2, \\ & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, & \\ & & \pi \text{ a probability distribution in } \mathbb{R}^{+2}. & \end{array} \tag{27}$$

[1] Only few bound problems have explicit solutions, but many of them can be solved by SOS programs.

[2] In general, this problem has a support $\mathcal{D} = \mathbb{R}^2$. But if X_1 and X_2 stand for losses (as in our example), they are nonnegative numbers.

Letting $Z = X_1 + X_2$, problem (27) is transferred to a one-variable bound problem. Its upper bound is expressed as:

$$\begin{aligned} \overline{p}_{\text{StopLoss}} = \quad & \sup & & \mathbb{E}_\pi(\phi(Z)) \\ & \text{such that} & & \mathbb{E}_\pi(1) = 1, \\ & & & \mathbb{E}_\pi(Z) = \mu_z \\ & & & \mathbb{E}_\pi(Z^2) = \mu_z^{(2)} \\ & & & \pi \text{ a probability distribution in } \mathbb{R}^+, \end{aligned} \tag{28}$$

and its lower bound is as follows:

$$\begin{aligned} \underline{p}_{\text{StopLoss}} = \quad & \inf & & \mathbb{E}_\pi(\phi(Z)) \\ & \text{such that} & & \mathbb{E}_\pi(1) = 1, \\ & & & \mathbb{E}_\pi(Z) = \mu_z \\ & & & \mathbb{E}_\pi(Z^2) = \mu_z^{(2)} \\ & & & \pi \text{ a probability distribution in } \mathbb{R}^+. \end{aligned} \tag{29}$$

Before obtaining the SOS programming formulation of the primal problems (28) and (29), we discuss their feasibility in terms of their moment parameters.

Observation 6 (Feasibility)**.** *When the feasibility of stop-loss bounds is considered, we should go back to the two-variable problem with the moment matrix Σ expressed as follows. Similar to probability bounds problem in Section 3.2, problems* (28) *and* (29) *are feasible if and only if Σ is a positive semidefinite matrix (i.e., all eigenvalues are greater than or equal to zero) and all elements of Σ are non-negative.*

$$\Sigma = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1^{(2)} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_2^{(2)} \end{bmatrix}.$$

Furthermore, when two-variable feasibility is satisfied, one-variable feasibility is also met automatically. That is, Σ_z is a positive semidefinite matrix and all elements of Σ_z are non-negative.

$$\Sigma_z = \begin{bmatrix} 1 & \mu_z \\ \mu_z & \mu_z^{(2)} \end{bmatrix}.$$

Next we derive SOS programs to numerically compute $\overline{p}_{\text{StopLoss}}$, and $\underline{p}_{\text{StopLoss}}$ by using SOS programming solvers.

Upper bound. We begin with stating the dual problem of (28) as follows:

$$\begin{aligned} \overline{d}_{\text{Stoploss}} = \quad & \inf & & y_0 + y_1\mu_z + y_2\mu_z^{(2)} \\ & \text{such that} & & p(z) \geq \phi(z), \forall\, z \geq 0, \end{aligned} \tag{30}$$

where $p(z) = y_0 + y_1 z + y_2 z^2$.

As the following observation states, as long as problem (28) is feasible, we can obtain $\overline{p}_{\text{Stoploss}}$ by solving problem (30).

Observation 7 (Strong Duality)**.** *Notice that the dual solution* $y_0 = 2$*, and* $y_1 = y_2 = 0$ *strictly satisfies (i.e., with* $>$*) the constraint in* (30) *for all* $z \geq 0$*. Thus, if problem* (28) *is feasible, then* $\overline{p}_{\text{Stoploss}} = \overline{d}_{\text{Stoploss}}$.

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (30) as a SOS program, we rewrite the inequality constraint in (30) as three simultaneous inequalities. Problem (30) is equivalent to:

$$\begin{array}{llll} \overline{d}_{\text{StopLoss}} = & \inf & y_0 + y_1\mu_z + y_2\mu_z^{(2)} & \\ & \text{such that} & p(z) - b \geq 0, \quad \forall z \in [a+b, \infty) & \\ & & p(z) - z + a \geq 0, \quad \forall z \in [a, a+b] & \\ & & p(z) \geq 0, \quad \forall z \in [0, a]. & \end{array} \tag{31}$$

The univariate SOS program (31) can be readily solved with a SOS programming solver. Thus, if problem (28) is feasible (cf. Observation 6), it follows from Observation 7 that we can numerically obtain the semiparametric upper bound $\overline{p}_{\text{Stoploss}}$ by solving problem (31) with a SOS solver.

Lower bound. We begin with staring the dual problem of (29):

$$\begin{array}{lll} \underline{d}_{\text{Stoploss}} = & \sup & y_0 + y_1\mu_z + y_2\mu_z^{(2)} \\ & \text{such that} & p(z) \leq \phi(z), \forall\, z \geq 0. \end{array} \tag{32}$$

As the following observation states, as long as problem (29) is feasible, we can obtain $\underline{p}_{\text{Stoploss}}$ by solving problem (32).

Observation 8 (Strong Duality)**.** *Notice that the dual solution* $y_0 = -1$*, and* $y_1 = y_2 = 0$ *strictly satisfies (i.e., with* $>$*) the constraint in* (32) *for all* $z \geq 0$*. Thus, if problem* (29) *is feasible, then* $\underline{p}_{\text{Stoploss}} = \underline{d}_{\text{Stoploss}}$.

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (32) as a SOS program, we rewrite the inequality constraint in (32) as three simultaneous inequalities. Problem (32) is equivalent to:

$$\begin{array}{llll} \underline{d}_{\text{StopLoss}} = & \sup & y_0 + y_1\mu_z + y_2\mu_z^{(2)} & \\ & \text{such that} & b - p(z) \geq 0, \quad \forall z \in [a+b, \infty) & \\ & & (z - a) - p(z) \geq 0, \quad \forall z \in [a, a+b] & \\ & & -p(z) \geq 0, \quad \forall z \in [0, a]. & \end{array} \tag{33}$$

The univariate SOS program (33) can be readily solved with a SOS programming solver. Thus, if problem (29) is feasible (cf. Observation 6), it follows from Observation 8 that we can numerically obtain the semiparametric lower bound $\underline{p}_{\text{Stoploss}}$ by solving problem (33) with a SOS solver.

In addition, the lower bound of stop-loss payment $\underline{p}(\phi)$ can be obtained by solving upper bound of a transformed problem with objective function $\psi(Z)$ where $\psi(Z) = Z - \phi(Z)$.

$$\psi(Z) = \begin{cases} Z - b & \text{if } Z \geq a + b \\ a & \text{if } a \leq Z \leq a + b \\ Z & \text{if } Z \leq a. \end{cases} \tag{34}$$

If the moment matrix Σ satisfies the feasibility requirement (cf. Observation 6), we can numerically obtain the semiparametric upper bound $\overline{p}(\psi)$ by solving the following dual problem (35) with a SOS solver:

$$\begin{aligned} \overline{d}(\psi) = \quad & \inf \qquad y_0 + y_1\mu_z + y_2\mu_z^{(2)} \\ & \text{such that} \quad p(z) - (z - b) \geq 0, \quad \forall z \in [a + b, \infty) \\ & \qquad\qquad\quad p(z) - a \geq 0, \quad \forall z \in [a, a + b] \\ & \qquad\qquad\quad p(z) - z \geq 0, \quad \forall z \in [0, a]. \end{aligned} \tag{35}$$

Apparently, the upper bound of $\psi(Z)$, $\overline{p}(\psi) = \sup\{\mathrm{E}_\pi[\psi(Z)]\}$ given the same moment information, equals μ_z minus the lower bound of $\phi(Z)$. That is, $\overline{p}(\psi) = \mu_z - \underline{p}(\phi)$. Similarly, the upper bound of stop-loss payment $\overline{p}(\phi)$ can be obtained from the relation $\underline{p}(\psi) = \mu_z - \overline{p}(\phi)$ after we solve $\underline{p}(\psi)$.

Proof. See Appendix B. □

3.3.2. *Cox (1991)'s Method.* Suppose a direct insurer purchases a reinsurance policy and his overall claim payment $\psi(Z)$ follows equation (34). Cox (1991) develops an explicit solution to the bounds of the expected claim payment $\mathrm{E}[\psi(Z)]$ of the direct insurer, given mean and variance. $\overline{p}(\psi)$, the upper bound on $\mathrm{E}[\psi(Z)]$, is described as follows: For values of a satisfying $0 \leq a < \mu_z$,

$$\overline{p}(\psi) = \begin{cases} \dfrac{(\mu_z - b)(\mu_z - a)^2 + \mu_z\sigma_z^2}{(\mu_z - a)^2 + \sigma_z^2} & \text{if } a \leq a + b \leq \dfrac{\sigma_z^2 + \mu_z^2 - a^2}{2(\mu_z - a)} \\ a + \dfrac{1}{2}\left[\mu_z - a - b + \sqrt{(a + b - \mu_z)^2 + \sigma_z^2}\right] & \text{if } a + b > \dfrac{\sigma_z^2 + \mu_z^2 - a^2}{2(\mu_z - a)}, \end{cases}$$

where $\sigma_z = \sqrt{\mu_z^{(2)} - \mu_z^2}$.

When $a \geq \mu_z$, the upper bound $\overline{p}(\psi) = \mu_z$.

The lower bound on $\mathrm{E}[\psi(Z)]$, $\underline{p}(\psi)$, is described as follows: For values of $a + b$ satisfying $0 \leq a + b \leq \mu_z$,

$$\underline{p}(\psi) = \mu_z - b.$$

If $\mu_z \leq a+b \leq \mu_z + \dfrac{\sigma_z^2}{\mu_z}$,

$$\underline{p}(\psi) = \frac{a\mu_z}{a+b}.$$

When $a+b \geq \mu_z + \dfrac{\sigma_z^2}{\mu_z}$,

$$\underline{p}(\psi) = \begin{cases} \dfrac{a\mu_z^2}{\sigma_z^2+\mu_z^2} & \text{if } 0 \leq a \leq \dfrac{\mu_z}{2} + \dfrac{\sigma_z^2}{2\mu_z} \\ \dfrac{1}{2}\left[\mu_z + a - \sqrt{(\mu_z-a)^2+\sigma_z^2}\right] & \text{if } \dfrac{\mu_z}{2} + \dfrac{\sigma_z^2}{2\mu_z} < a \leq \dfrac{(a+b)^2-\mu_z^2-\sigma_z^2}{2(a+b-\mu_z)} \\ \dfrac{\mu_z(a+b-\mu_z)^2+(\mu_z-b)\sigma_z^2}{(a+b-\mu_z)^2+\sigma_z^2} & \text{if } \dfrac{(a+b)^2-\mu_z^2-\sigma_z^2}{2(a+b-\mu_z)} \leq a \leq a+b. \end{cases}$$

After the upper and lower bounds $\overline{p}(\psi)$ and $\underline{p}(\psi)$ are calculated, the bounds on the stop-loss payment $\phi(Z) = Z - \psi(Z)$ can be found by the relations $\overline{p}(\phi) = \mu_z - \underline{p}(\psi)$ and $\underline{p}(\phi) = \mu_z - \overline{p}(\psi)$.

4. Numerical Analysis

To understand the extent to which extreme events affect our decision, we apply the moment methods to the insurance and financial markets with three examples. The first example is an application of VaR probability bounds we derive in Section 3.1; the second one is for probability bounds (See Section 3.2); the bounds on stop-loss payments derived in Section 3.3 are illustrated in example three.

4.1. **Example of VaR Probability Bounds.** The VaR problem is to find the upper and lower bounds on a where $\Pr(w_1X_1 + w_2X_2 \leq a) = 0.05$, subject to the moment information on X_1 and X_2. We connect this to a semiparametric probability problem by finding bounds on $\Pr(w_1X_1 + w_2X_2 \leq a)$ for enough values of a to solve the inverse problem.

In Section 3.1, we find bounds for the special case $\Pr(X_1 + X_2 \leq a)$. We can easily convert $\Pr(w_1X_1 + w_2X_2 \leq a)$ to $\Pr(X_1 + X_2 \leq a)$ by adjusting the moments of X_1 and X_2. Let $X_1' = w_1X_1$ and $X_2' = w_2X_2$. Then we have the following relationships:

$$\begin{aligned} &\mathrm{E}(X_i') = \mathrm{E}(w_iX_i) = w_i\mu_i, && i=1,2 \\ &\mathrm{E}(X_i'^2) = \mathrm{E}(w_i^2X_i^2) = w_i^2\mu_i^{(2)}, && i=1,2 \\ &\mathrm{E}(X_1'X_2') = \mathrm{E}(w_1X_1w_2X_2) = w_1w_2\mu_{12}. \end{aligned} \tag{36}$$

That is, we can rescale a problem in the form $w_1X_1 + w_2X_2 \leq a$ to the form $X_1 + X_2 \leq a$.

To show how to solve the bound on VaR, we study a possible extreme scenario in the international stock markets. That is, what may happen if the stock indices of two countries both reach some very low levels. Specifically, we analyze the tail joint probability of total return of a portfolio investing in the S&P500 and Nikkei indices.

First, we calculate the moments of the S&P500 annualized return (denoted $\mathbf{r}_{\text{sp}}$) and that of the Nikkei (denoted $\mathbf{r}_{\text{nk}}$) based on the monthly historical data from 1984 to 2006. There are 276 observations in our sample. Their moments are as follows:

$$\begin{aligned}
&\mathrm{E}(X_1) &&= 0.1107 = \mathrm{E}(\mathbf{r}_{\text{sp}}) = \mu_1 && \mathrm{E}(X_1^2) = 0.0349\\
&\mathrm{E}(X_2) &&= 0.0473 = \mathrm{E}(\mathbf{r}_{\text{nk}}) = \mu_2 && \mathrm{E}(X_2^2) = 0.0554\\
&\mathrm{Var}(X_1) &&= 0.0227 = \mathrm{Var}(\mathbf{r}_{\text{sp}}) && \rho \qquad = 0.4190\\
&\mathrm{Var}(X_2) &&= 0.0531 = \mathrm{Var}(\mathbf{r}_{\text{nk}}) &&\\
&\mathrm{Cov}(X_1, X_2) &&= 0.0145 = \mathrm{Cov}(\mathbf{r}_{\text{sp}}, \mathbf{r}_{\text{nk}}). &&
\end{aligned}$$

On average, the S&P500 annualized return (0.1107) is higher than that of Nikkei (0.0473) but the S&P500 is less volatile $(\mathrm{Var}(\mathbf{r}_{\text{sp}}) < \mathrm{Var}(\mathbf{r}_{\text{nk}}))$. Moreover, they have a positive correlation 0.4190. This relatively high correlation reflects the impact of economic globalization, thus weakening the diversification effect.

Second, suppose we invest 50% of our assets in the S&P500 and 50% in Nikkei, i.e. $0.5X_1 + 0.5X_2 = 0.5\mathbf{r}_{\text{sp}} + 0.5\mathbf{r}_{\text{nk}}$. We calculate the upper and lower bounds for the probability that this portfolio return falls below the level a, i.e. $\Pr(0.5\mathbf{r}_{\text{sp}} + 0.5\mathbf{r}_{\text{nk}} \le a)$. The upper and lower lines in Figure 1 respectively represent the upper and lower bounds of joint probabilities with different values of a. The upper and lower bounds include all possible joint probabilities, including the bivariate normal joint probability shown as the middle line. This means that although we know only the moments of order 1 and 2, we can be sure the probability of this rare event is between the upper and lower bounds.

Finally, we use Figure 1 to obtain the upper and lower bounds of the VaR, a, given a tail probability. Popular left tail levels usually are 1% and 5%. Figure 1 gives us an idea how likely the return of this portfolio will be lower than a over a year under different conditions. For example, if we focus on the 5%-VaR, the upper bound a_L tells us that there is a 5% chance the portfolio return would fall below -0.70 and the lower bound a_U suggests the VaR equals to 0.10 with the same probability.

4.2. **Example of Probability Bounds.** What makes the moment methods valuable for our analysis is that, they depend on much less restrictive assumptions to compute default risk and ruin probability. We show how to estimate the joint probability of extreme events, regardless of the specific choice of distributions. We detail a simple calibration exercise to compute the bounds of a joint probability as follows.

Consider the American International Group (AIG), a publicly-traded insurance company. It, as other insurers, faces the same problem of managing extreme events, i.e. unexpectedly high claims due to catastrophic events like Hurricane Katrina in 2005 and simultaneously suffering unanticipated poor asset returns caused by financial market crashes. This leads us to calculate the

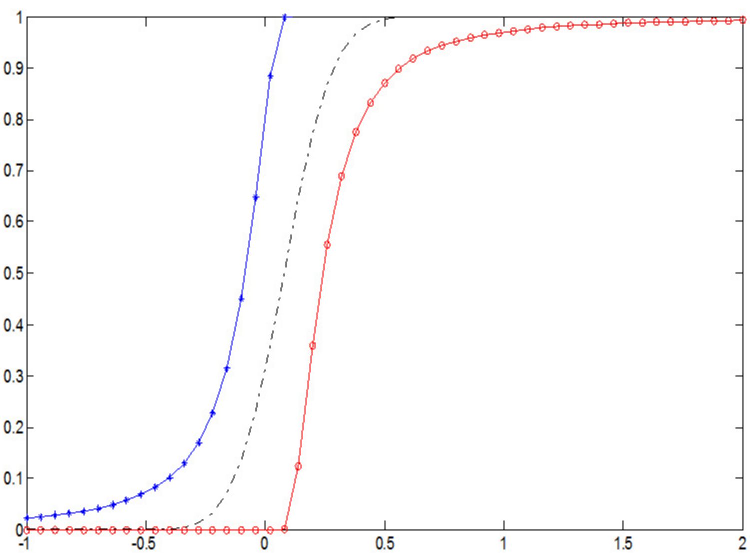


FIGURE 1. Upper and lower bounds for the probability $\Pr(0.5\mathbf{r}_{\text{sp}} + 0.5\mathbf{r}_{\text{nk}} \leq a)$ where $\mathbf{r}_{\text{sp}}$ is the monthly annualized return on the S&P500 index and $\mathbf{r}_{\text{nk}}$ is that of the Nikkei index. The vertical axis is the probability and the horizontal axis stands for different values of a. The 5% $\text{VaR}_{0.05}$ of the normal distribution equals to $a = -0.20$. It falls between the semiparametric lower bound a_L and upper bound a_U. That is, $a_L < \text{VaR}_{0.05} < a_U$.

bounds on $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$ given moment information, where $\mathbf{r}$ is AIG's return on its invested assets and $\mathbf{m}$ is the margin on its insurance business.

The return $\mathbf{r}_i$ of asset i in the portfolio is equal to $\text{P}_{i,t}/\text{P}_{i,t-1} - 1$ where $\text{P}_{i,t-1}$ and $\text{P}_{i,t}$ denote the prices of asset i at the beginning and the end of the period. If we focus on the price ratio, the condition $\mathbf{r} \leq t_1$ changes to

$$X_{1i} = \mathbf{r}_i + 1 = \frac{\text{P}_{i,t}}{\text{P}_{i,t-1}} \leq t_1', \tag{37}$$

where $t_1' = t_1 + 1$. As for AIG's portfolio, $\mathbf{r}$ is the weighted average return of 6 assets: stocks, government bonds, corporate bonds, real estates, mortgages and cash & short-term investments $(i = 1, 2, \ldots, 6)$:

$$\mathbf{r} = \sum_{i=1}^{6} w_i X_{1i} - 1 = X_1 - 1,$$

where w_i is the weight of asset i in the portfolio. Indeed, we calculate the bounds for $\Pr(X_1 \leq t_1') = \Pr(\sum_{i=1}^{6} w_i X_{1i} \leq t_1')$ which is equivalent to $\Pr(\mathbf{r} \leq t_1)$. We make this shift from asset returns to price ratios to apply our SOS results because we need non-negative random variables.

The margin on insurance business $\mathbf{m}$ is defined as

$$\mathbf{m} = 1 - \mathbf{LR},$$

where $\mathbf{LR}$ is the economic loss ratio. Following a standard measure in the insurance literature (Cummins, 1990; Phillips, Cummins, and Allen, 1998; Yu and Lin, 2007), we calculate the economic loss ratio as follows:

$$\mathbf{LR} = \frac{\sum_{k=1}^{12} \mathrm{PVF}_k \times \mathrm{NLI}_k}{\sum_{k=1}^{12} \mathrm{NPE}_k}.$$

We classify AIG's business into twelve categories ($k = 1, 2, \ldots, 12$).[3] The present value factor PVF_k is calculated from the industry liability payout factor for loss category k ($k = 1, 2, \ldots, 12$) and the discount rates. The discount rates are the risk-free rates estimated from the U.S. Treasury spot-rate yield curves.[4] The variable NLI_k is the net loss incurred for category k for AIG. The variable NPE_k is its net premium earned for category k. See Cummins (1990) for calculation details. Using the actual premium in the denominator and the riskless present value of losses in the numerator allows us to capture changes in loss ratios due to insurance shocks. In order to reformulate the condition $\mathbf{m} \leq t_2$ so that the condition fits our SOS results, similar to the asset return case, we replace $\mathbf{m} \leq t_2$ with $X_2 \leq t_2'$ where $X_2 = \mathbf{m} + 1$ and $t_2' = t_2 + 1$. It clearly follows that $\Pr(\mathbf{m} \leq t_2)$ is equivalent to $\Pr(X_2 \leq t_2')$.

The weights of different asset categories (w_i) are calculated from the quarterly data of the National Association of Insurance Commissioners (NAIC). The quarterly AIG and industry losses and premiums are also obtained from the NAIC. We use the quarterly annualized returns of the Standard & Poor's 500 (S&P500), the LB IT government bond index, the domestic high-yield corporate bond index, the NAREIT-All index, the ML mortgage index and the U.S. 30 Day T-Bill as proxies for AIG's stock returns, government bond returns, corporate bond returns, real estate returns, mortgage returns and cash & short-term investment returns respectively. In sum, we have 52 quarterly observations from 1991 to 2003. Here are their moments:

$$\begin{array}{llll}
\mathrm{E}(X_1) & = 1.0442 = \mathrm{E}(\mathbf{r}) + 1 = \mu_1 & \mathrm{E}(X_1^2) & = 1.0967 \\
\mathrm{E}(X_2) & = 1.3393 = \mathrm{E}(\mathbf{m}) + 1 = \mu_2 & \mathrm{E}(X_2^2) & = 1.8287 \\
\mathrm{Var}(X_1) & = 0.0063 = \mathrm{Var}(\mathbf{r}) & \rho & = 0.1244 \\
\mathrm{Var}(X_2) & = 0.0350 = \mathrm{Var}(\mathbf{m}) & & \\
\mathrm{Cov}(X_1, X_2) & = 0.0019. & &
\end{array}$$

On average, AIG's margin on its insurance business ($\mathrm{E}(\mathbf{m}) = 0.3393$) is higher than its asset return ($\mathrm{E}(\mathbf{r}) = 0.0442$) while the margin is more volatile ($\mathrm{Var}(\mathbf{m}) > \mathrm{Var}(\mathbf{r})$). Moreover, the asset

[3]Following the NAIC classifications, our twelve insurance business categories include farmowners and homeowners multiple peril; private passenger auto liability; workers' compensation; commercial multiple peril; medical malpractice; special liability; special property; automobile physical damage; fidelity and surety; other; financial guarantee and mortgage guarantee; and other liability and product liability.

[4]Data source: the Federal Reserve Bank of St. Louis' Federal Reserve Economic Data (FRED).

return and insurance margin are positively correlated (0.1244). This implies that generally AIG's insurance business and investment performances moderately move in the same direction.

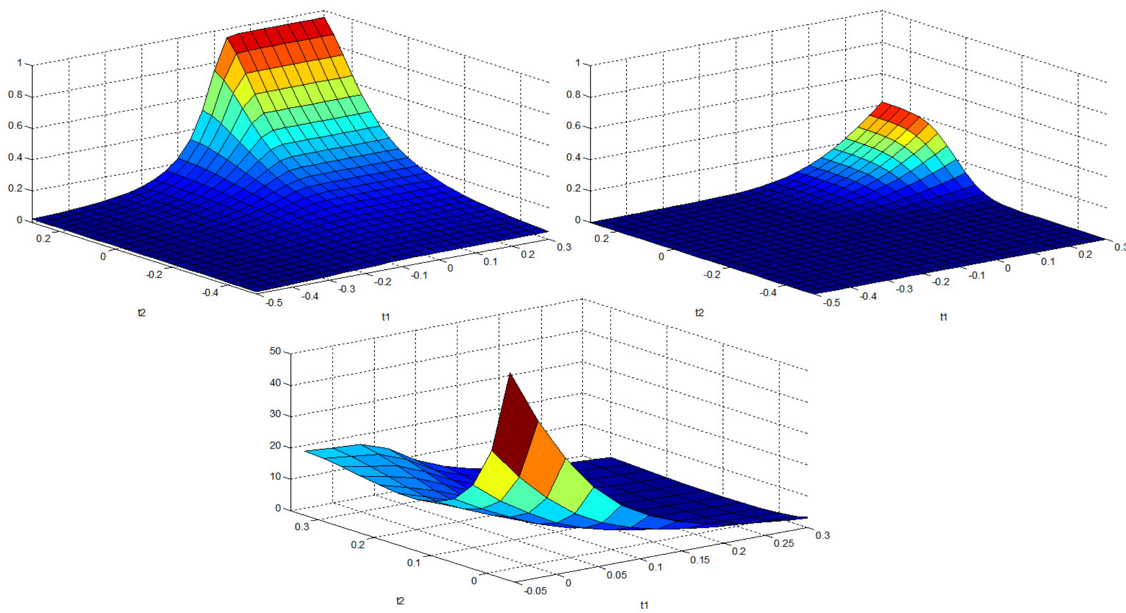


FIGURE 2. The upper left plot shows the upper bound of the joint probability $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$ where $\mathbf{r}$ is invested asset return and $\mathbf{m}$ is insurance business margin of AIG. The upper right one is the bivariate normal cumulative probabilities with the same moments for AIG. The ratio of the upper bound to the bivariate normal cumulative joint probabilities is shown in the third graph. The vertical axis of the graphs is the probability. It is the ratio in the third graph. The two axes at the bottom in all three graphs represent the return t_1 and the insurance margin t_2.

To examine the tail-risk implication of our model, we start with the SOS programming to solve $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$. Then we compare it to the bivariate normal cumulative joint probability with the same set of moments. The upper left 3-dimensional (3D) plot in Figure 2 shows the upper bounds of the joint probability $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$ with different values of t_1 and t_2 and the upper right one is the bivariate normal cumulative joint probabilities with the same moments for AIG. The lower bound is always zero. The ratios of the upper bounds to the bivariate normal cumulative joint probabilities are shown in the third graph. We can see that the ratios are always above 1. This means that the upper bound probabilities are always higher than those of the bivariate normal. Their difference is much larger when t_1 and t_2 are low. For example, when $t_1 = 0$ and $t_2 = 0$, the upper bound of $\Pr(\mathbf{r} \leq 0, \mathbf{m} \leq 0)$ is about 45 times higher than the cumulative joint probability of the bivariate normal. That is, the upper bound has a much fatter tail.

Next, we explore the upper bound implication for the joint probabilities across different values of one t_i given the other t_j is unchanged $(i = 1$ or 2 and $i \neq j)$. Specifically, we are interested in how one value (e.g. asset return t_1) changes the joint tail probability if the other value (e.g.

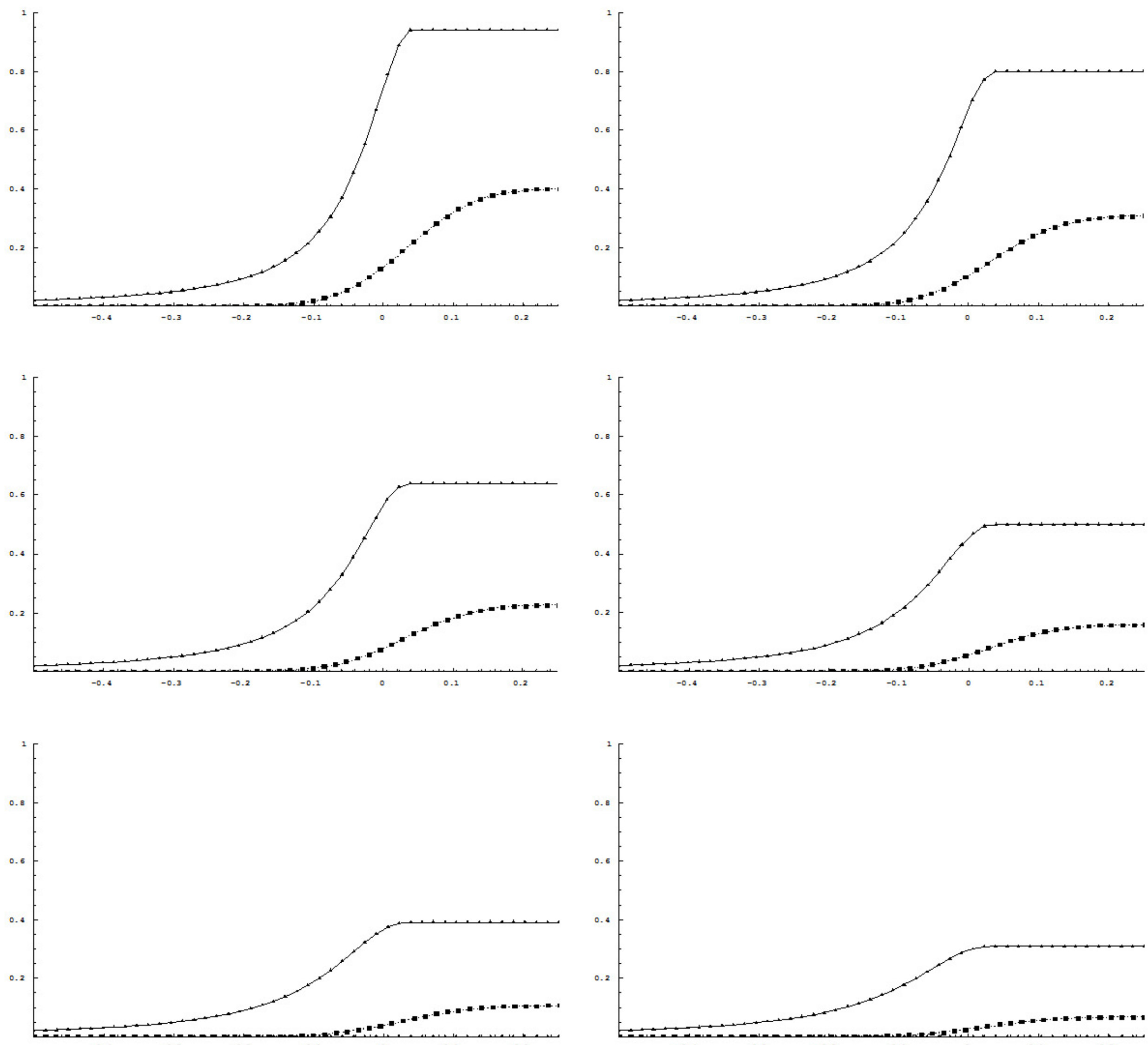


FIGURE 3. Each plot shows the upper bound on the joint probability $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$ (the upper curve in each graph) and the bivariate normal cumulative probability with the same moments (the lower curve) for AIG. They are a function of asset return t_1 given an insurance margin level t_2. Six graphs fix t_2 at different values: $\mathrm{E}(\mathbf{m}) - 0.25\sqrt{\mathrm{Var}(\mathbf{m})} = 0.2925, \mathrm{E}(\mathbf{m}) - 0.50\sqrt{\mathrm{Var}(\mathbf{m})} = 0.2457, \mathrm{E}(\mathbf{m}) - 0.75\sqrt{\mathrm{Var}(\mathbf{m})} = 0.1989, \mathrm{E}(\mathbf{m}) - \sqrt{\mathrm{Var}(\mathbf{m})} = 0.1521, \mathrm{E}(\mathbf{m}) - 1.25\sqrt{\mathrm{Var}(\mathbf{m})} = 0.1053$ and $\mathrm{E}(\mathbf{m}) - 1.50\sqrt{\mathrm{Var}(\mathbf{m})} = 0.0585$. The upper left graph corresponds to the case with $\mathrm{E}(\mathbf{m}) - 0.25\sqrt{\mathrm{Var}(\mathbf{m})}$ and the upper right one is for $\mathrm{E}(\mathbf{m}) - 0.50\sqrt{\mathrm{Var}(\mathbf{m})}$, and so on. The vertical axis is the probability and the horizontal axis is the return on asset t_1.

insurance margin t_2) is a constant. As an illustration, we fix the insurance margin t_2 and then solve the upper bound of joint probability $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$ by changing t_1. The first graph in Figure 3 shows its upper bound given $t_2 = \mathrm{E}(\mathbf{m}) - 0.25\sqrt{\mathrm{Var}(\mathbf{m})} = 0.2925$. We also compare this result with the bivariate normal case. As we expect, the upper bound is above the bivariate normal curve. That is, the upper bound has a fatter tail which suggests a higher ruin probability.

Furthermore, we set the variable t_2 (insurance margin) at five different levels based on 0.5, 0.75, 1, 1.25 and 1.50 standard deviations lower than the mean: 0.2457, 0.1989, 0.1521, 0.1053 and 0.0585. Then we draw their upper bounds and bivariate normal curves (the last five graphs in Figure 3). The trend of these graphs are consistent with our expectation. As t_2 decreases, the cumulative joint probability levels out at a lower value. For example, when $t_2 = 0.2925$, the upper bound of cumulative joint probability stays at 0.95 after it reaches this level. However, the stable level is only about 0.35 when $t_2 = 0.0585$. Intuitively, a lower value is associated with a lower cumulative probability. Again, the bivariate normal curve is below the upper bound in all graphs.

4.3. **Example of Stop-loss Payments.** In this section, we find the upper and lower bounds on the expected payment of a stop-loss contract written by a reinsurance company. Suppose AIG sells \$1 million new homeowners insurance and \$1 million new private passenger auto liability policies this year. It reinsures claim costs in excess of a million arising from these two businesses to Swiss Re. Swiss Re pays part of AIG's claims only if the threshold or deductible a is reached, subject to a policy limit b million. The upper and lower bounds on the expected payment of Swiss Re is examined here following Section 3.3.

The quarterly data of AIG from 1991 to 2004 are obtained from the NAIC. There are 56 observations from which we calculate the moments of AIG loss payments per \$1 million premium earned, respectively, for its homeowners insurance ($\mathbf{L}_{\text{HO}}$) and its private passenger auto liability insurance ($\mathbf{L}_{\text{PPA}}$). Their moments of loss amounts in million dollars are summarized as follows:

$$\begin{array}{lll} \mathrm{E}(X_1) & = 0.6370 = \mathrm{E}(\mathbf{L}_{\text{HO}}) = \mu_1 & \mathrm{E}(X_1^2) = 0.9364 \\ \mathrm{E}(X_2) & = 0.6844 = \mathrm{E}(\mathbf{L}_{\text{PPA}}) = \mu_2 & \mathrm{E}(X_2^2) = 0.5073 \\ \mathrm{Var}(X_1) & = 0.5306 = \mathrm{Var}(\mathbf{L}_{\text{HO}}) & \rho \qquad = 0.1647 \\ \mathrm{Var}(X_2) & = 0.0390 = \mathrm{Var}(\mathbf{L}_{\text{PPA}}) & \\ \mathrm{Cov}(X_1, X_2) & = 0.02369. & \end{array}$$

On average, the expected claim payments of these two lines of business are similar although the homeowners insurance is much more volatile since the homeowners business is more vulnerable to catastrophes and other weather-related claims.

Since the stop-loss payment depends on given levels of the deductible a and the policy limit b, we only change one parameter (e.g. b) and fix the other (e.g. a) to show our upper and lower bounds. Figure 4 illustrates the upper and lower bounds with different policy limits b given a certain deductible level of a. The upper and lower solid lines in each graph stand for the upper and lower bounds numerically solved by the SOS program and the bubble lines are the upper and lower bounds computed from Cox (1991)'s method. The upper left graph in Figure 4 shows the expected stop-loss payment of Swiss Re to AIG with no deductible ($a = 0$). When the policy limit b equals

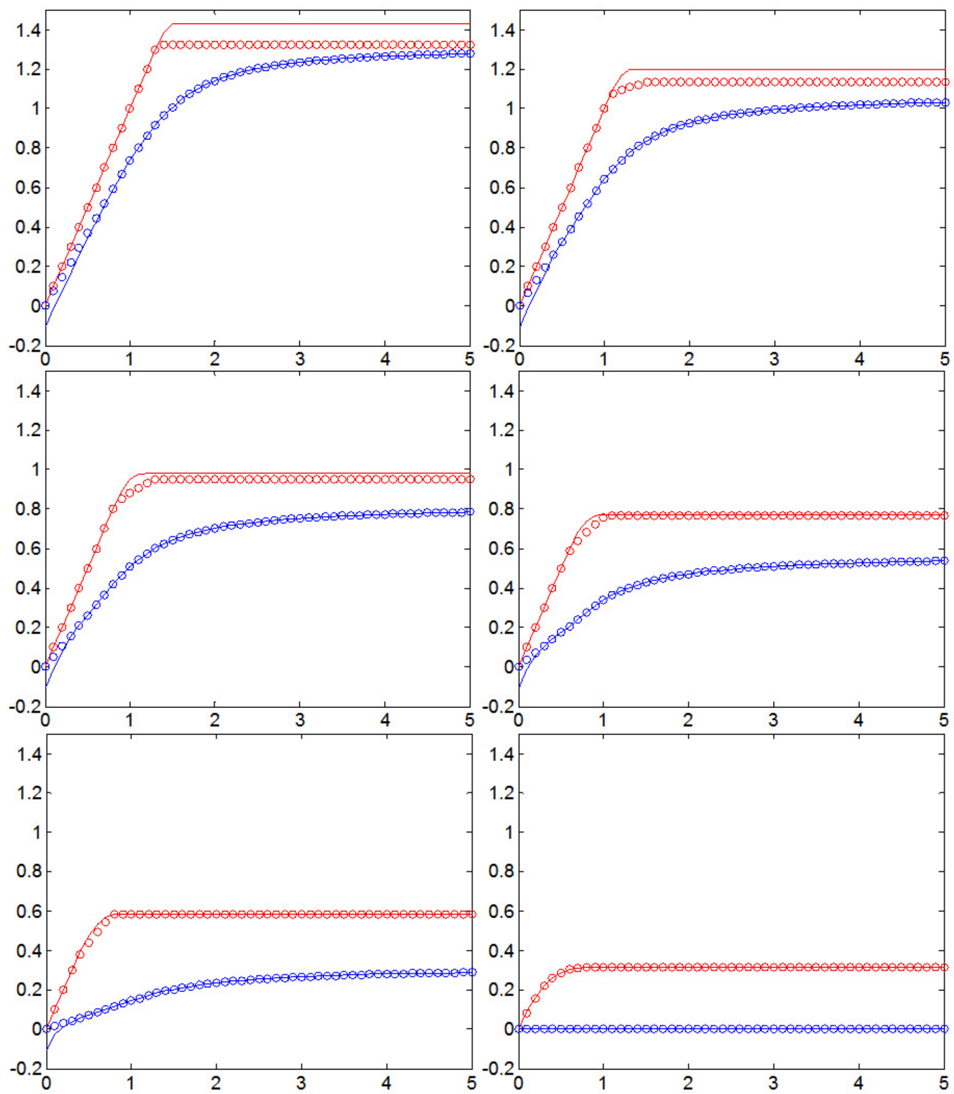


FIGURE 4. Each plot shows the upper (the top curve in each graph) and lower bounds (the curve in the bottom) on the expected stop-loss payment. They are a function of the policy limit b given a level of the deductible a. The solid lines are the upper and lower bounds obtained from the SOS programs. The bubble lines show the upper and lower bound solutions based on the Cox (1991)'s explicit formula. Six graphs fix a at $0, 0.25, 0.5, 0.75, 1$ and 1.5 million dollars respectively, with $a = 0$ on the upper left and running to the right then down. The vertical axis is the expected payment and the horizontal axis is the policy limit b, both in million dollars.

to \$1 million, both methods obtain the same upper and lower bounds, \$1 million and \$0.7 million respectively; when $b > 1.5$, the lower bound of these two methods matches pretty well while the upper bound of the SOS program levels out at a relatively higher value (\$1.4 million) than that of Cox (1991)'s method (\$1.3 million). For the cases that the deductible a is fixed at the level 0.5, 0.75, 1 or 1.5, the solutions from the SOS and from the explicit formula of Cox (1991)'s method are almost identical. It suggests that the SOS program works pretty well for this stop-loss payment problem. In addition, we should note that SOS program can be flexibly applied to more problems, most of which cannot be explicitly solved.

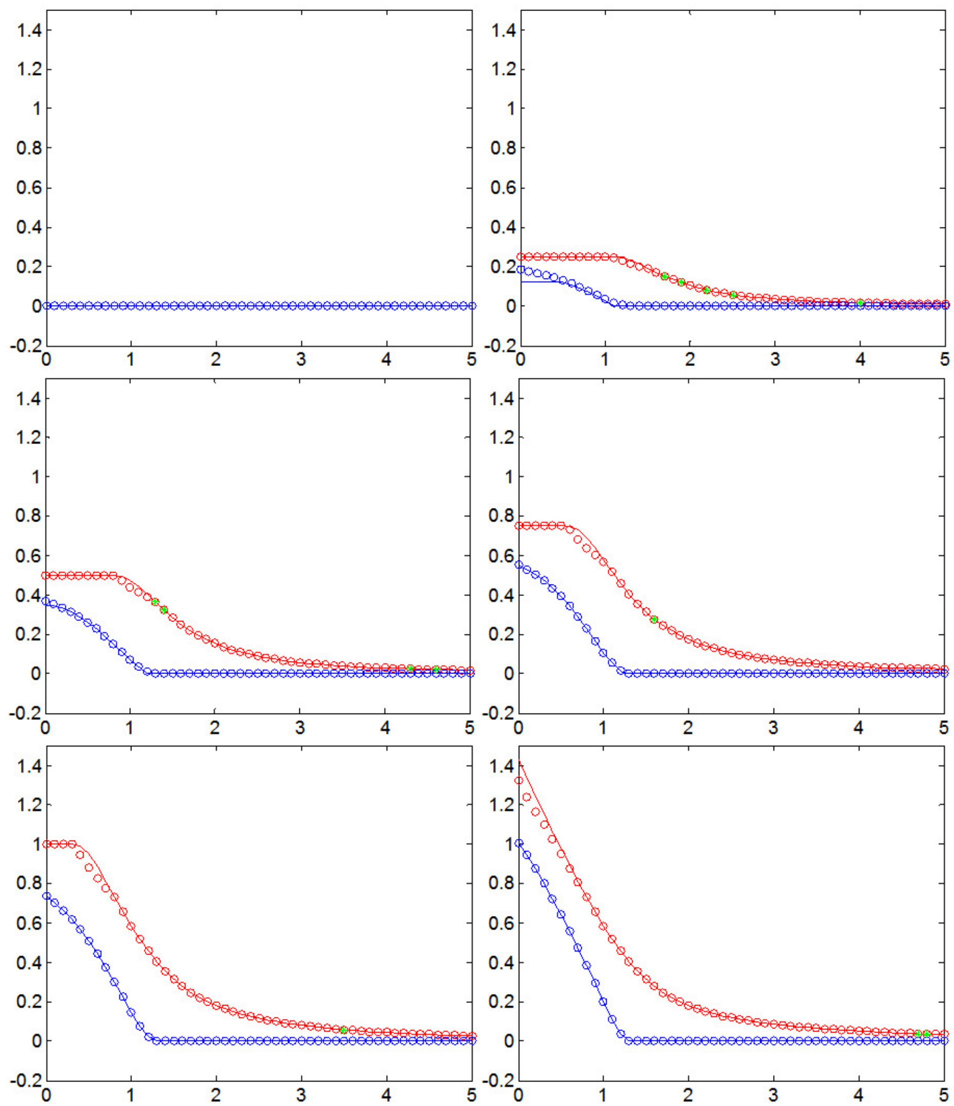


FIGURE 5. Each plot shows the upper (the top curve in each graph) and lower bounds (the curve in the bottom) on the expected stop-loss payment. They are a function of the deductible a given a level of the policy limit b. The solid lines are the upper and lower bounds obtained from the SOS programs. The bubble lines show the upper and lower bound solutions based on the Cox (1991)'s explicit formula. Six graphs fix b at $0, 0.25, 0.5, 0.75, 1$ and 1.5 million dollars respectively, with $b = 0$ on the upper left and running to the right then down. The vertical axis is the expected payment and the horizontal axis is the deductible a, both in million dollars.

To further show the robustness of SOS program solutions, we consider the upper and lower bounds of a Swiss Re stop-loss policy paying up to a fixed level b while AIG could select different deductibles a. Each graph in Figure 5 shows the upper and lower bounds given a certain policy limit b with different deductibles a. As we expect, the bounds on Swiss Re's expected payments increase as the fixed value b increases (i.e. the stop-loss policy covers more losses). Again, bounds calculated from the SOS program and the Cox (1991)'s method remain qualitatively similar.

5. CONCLUSION

We have extended the application of classical moment problems (or semiparametric methods) to finance, insurance and actuarial science in three ways, all taking into account the correlation between different random variables. The first finds bounds on the sum of two variables, given up to second moment information. The second allows us to put "100% confidence intervals" on the joint probability of two extreme events. The third one computes bounds on the expected payment of a stop-loss policy, given only the moments of the loss components.

In each case the moment information may be based on historical observations or judgements from scenario analysis. We provide examples to illustrate the potential usefulness of moment methods in assessing probability of rare events. There are other applications where our approach could be useful. For example, this approach can be used to estimate the default probability of fixed-income securities, under incomplete knowledge of the enterprise and economic factors driving the credit risk. In other areas such as inventory and supply chain management, this approach can be applied to find inventory policies that will be robust to different (unknown) demand distributions in the future. Even when the distributions of the random variables are assumed to be known, this approach can be implemented to measure sensitivity of the given joint probabilities, VaR and expected benefits to model misspecification (Lo, 1987; Hobson, Laurence, and Wang, 2005).

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Appendix A: Probability bounds on $\Pr(X_1 \geq t_1, X_2 \geq t_2)$

We consider the problem of finding sharp upper and lower bounds on the probability $\Pr(X_1 \geq t_1 \text{ and } X_2 \geq t_2)$ of two non-negative random variables X_1, X_2, attaining values higher than or equal to $t_1, t_2 \in \mathbb{R}^+$ respectively, given up to second order moment information (means, variances, and covariance) on X_1, X_2, without making any other assumption on the distribution of the random variables X_1, X_2. Finding the sharp upper and lower semiparametric bounds for this problem can be (respectively) formulated as the following optimization problems, obtained by setting in problem (2) $\phi(X_1, X_2) = \mathbb{I}_{\{X_1 \geq t_1 \text{ and } X_2 \geq t_2\}}$, and $\mathcal{D} = \mathbb{R}^{+2}$ (cf. Section 2):

$$
\begin{array}{llll}
\overline{p} := & \sup & \mathbb{E}_\pi(\mathbb{I}_{\{X_1 \geq t_1 \text{ and } X_2 \geq t_2\}}) & \\
 & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\
 & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1, 2, \\
 & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1, 2, \\
 & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, & \\
 & & \pi \text{ a probability distribution in } \mathbb{R}^{+2}, &
\end{array}
\tag{38}
$$

and

$$
\begin{array}{llll}
\underline{p} := & \inf & \mathbb{E}_\pi(\mathbb{I}_{\{X_1 \geq t_1 \text{ and } X_2 \geq t_2\}}) & \\
 & \text{such that} & \mathbb{E}_\pi(1) = 1, & \\
 & & \mathbb{E}_\pi(X_i) = \mu_i, & i = 1, 2, \\
 & & \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, & i = 1, 2, \\
 & & \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, & \\
 & & \pi \text{ a probability distribution in } \mathbb{R}^{+2}. &
\end{array}
\tag{39}
$$

Before obtaining the SOS programming formulation of these problems, we discuss its feasibility in terms of their moments.

Observation 9 (Feasibility)**.** *Problems* (38) *and* (39) *are feasible if and only if* Σ *is a positive semidefinite matrix (i.e., all eigenvalues are greater than or equal to zero) and all elements of* Σ *are non-negative, where* Σ *is the moment matrix:*

$$
\Sigma = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1^{(2)} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_2^{(2)} \end{bmatrix}.
$$

Proof. Follows from Diananda's Theorem (Theorem 2) and convex duality (cf. Rockafellar (1970)). $\square$

Next we derive SOS programs to numerically compute $\overline{p}$, and $\underline{p}$ by using SOS programming solvers.

Upper bound. We begin by stating the dual problem of (38):

$$\begin{array}{lll} \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(x_1,x_2) \geq \mathbb{I}_{\{x_1 \geq t_1 \text{ and } x_2 \geq t_2\}}, \forall\, x_1, x_2 \geq 0. \end{array} \tag{40}$$

As the following observation states, as long as problem (38) is feasible, we can obtain $\overline{p}$ by solving problem (40).

Observation 10 (Strong Duality)**.** *Notice that the dual solution* $y_{00} = 2$*, and* $y_{ij} = 0$ *for* $(i,j) \neq (0,0)$ *strictly satisfies (i.e., with* $>$*) the constraint in* (40) *for all* $x_1, x_2 \geq 0$*. Thus, if problem* (38) *is feasible, then* $\overline{p} = \overline{d}$*.*

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (40) as a SOS program, we proceed as follows. First notice that (40) is equivalent to:

$$\begin{array}{lll} \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(x_1,x_2) \geq 1, \forall\, x_1 \geq t_1, x_2 \geq t_2 \\ & & p(x_1,x_2) \geq 0, \forall\, x_1, x_2 \geq 0. \end{array} \tag{41}$$

Applying the substitution $x_1 \to x_1 + t_1, x_2 \to x_2 + t_2$ to the first constraint of (41), we have that (41) is equivalent to:

$$\begin{array}{lll} \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & p(x_1+t_1,x_2+t_2) - 1 \geq 0, \forall\, x_1, x_2 \geq 0 \\ & & p(x_1,x_2) \geq 0, \qquad\qquad\quad \forall\, x_1, x_2 \geq 0. \end{array} \tag{42}$$

Now let $q(x_1,x_2) = p(x_1+t_1,x_2+t_2) - 1$; that is

$$\begin{aligned} q(x_1,x_2) = \;& (y_{00} + y_{10}t_1 + y_{01}t_2 + y_{20}t_1^2 + y_{02}t_2^2 + y_{11}t_1t_2 - 1) \\ & + (y_{10} + 2t_1y_{20} + y_{11}t_2)x_1 \\ & + (y_{01} + 2t_2y_{02} + y_{11}t_1)x_2 \\ & + y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2, \end{aligned}$$

so that the first constraint of (42) can be replaced by $q(x_1,x_2) \geq 0, \forall\, x_1,x_2 \geq 0$. To finish, from Theorem 2, it follows that (42) (with the first constraint written in terms of $q(x_1,x_2)$) is equivalent to the following SOS program:

$$\begin{array}{lll} \overline{d} = & \inf & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\ & \text{such that} & q(x_1^2,x_2^2) \text{ is a SOS polynomial} \\ & & p(x_1^2,x_2^2) \text{ is a SOS polynomial.} \end{array} \tag{43}$$

The SOS program (43) can be readily solved with a SOS programming solver. Thus, if problem (38) is feasible (cf. Observation 9), it follows from Observation 10 that we can numerically obtain the probability semiparametric upper bound $\overline{p}$ by solving problem (43) with a SOS solver.

Lower bound. We begin by stating the dual problem of (39):

$$
\begin{aligned}
\underline{d} = \quad & \sup \quad && y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{such that} && p(x_1, x_2) \le \mathbb{I}_{\{x_1 \ge t_1 \text{ and } x_2 \ge t_2\}}, \forall\, x_1, x_2 \ge 0.
\end{aligned} \tag{44}
$$

As the following observation states, as long as problem (39) is feasible, we can obtain $\underline{p}$ by solving problem (44).

Observation 11 (Strong Duality)**.** *Notice that the dual solution $y_{00} = -1$, and $y_{ij} = 0$ for $(i,j) \neq (0,0)$ strictly satisfies (i.e., with $<$) the constraint in* (44) *for all $x_1, x_2 \ge 0$. Thus, if problem* (39) *is feasible, then $\underline{p} = \underline{d}$.*

Proof. Follows from convex duality (cf. (Zuluaga and Peña, 2005, Proposition 3.1)). □

To formulate problem (44) as a SOS program, we proceed as follows. First notice that (44) is equivalent to:

$$
\begin{aligned}
\underline{d} = \quad & \sup \quad && y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{s.t.} && p(x_1, x_2) \le 1, \forall\, x_1 \ge t_1, x_2 \ge t_2 \\
& && p(x_1, x_2) \le 0, \forall\, x_1 \ge 0, 0 \le x_2 \le t_2, \\
& && p(x_1, x_2) \le 0, \forall\, 0 \le x_1 \le t_1, x_2 \ge 0.
\end{aligned} \tag{45}
$$

Although the first constraint of (45) can be handled as in the upper bound problem (cf. Section 3.2.1), the last two constraints are difficult to reformulate as SOS constraints. That is, there is no linear transformation from $x_1 \ge 0, 0 \le x_2 \le t_2$ to $\mathbb{R}^{+2}$ or to $\mathbb{R}^2$ (that would allow the use of Theorems 1 and 2). Thus, we change the problem to end up with a SOS program that either solves or approximates problem (45). Specifically, consider the following problem related to (45):

$$
\begin{aligned}
\underline{d}' = \quad & \sup \quad && y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{such that} && p(x_1, x_2) \le 1, \forall\, x_1 \ge t_1, x_2 \ge t_2 \\
& && p(x_1, x_2) \le 0, \forall\, x_1 \ge 0, x_2 \le t_2, \\
& && p(x_1, x_2) \le 0, \forall\, x_1 \le t_1, x_2 \ge 0.
\end{aligned} \tag{46}
$$

Notice that (46) is less constrained than (45) (the last two constraints of (46) include more values of x_1 and x_2). Thus, $\underline{d}'$ is a lower bound on $\underline{d}$; that is $\underline{d}' \le \underline{d}$ (in fact, intuition suggests that $\underline{d}' = \underline{d}$). Therefore, $\underline{d}'$ and the corresponding upper bound of Section 3.2.1 still give a "100% confidence interval" on the value of the cumulative probability of interest.

Now, problem (46) is equivalent to (applying variable substitutions similar to the ones used in Section 3.2.1, and multiplying by -1 to get $\geq$ constraints):

$$
\begin{array}{lll}
\underline{d}' = & \sup & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{such that} & 1 - p(x_1 + t_1, x_2 + t_2) \geq 0, \forall\, x_1, x_2 \geq 0 \\
& & -p(x_1, t_2 - x_2) \geq 0, \qquad \forall\, x_1, x_2 \geq 0, \\
& & -p(t_1 - x_1, x_2) \geq 0, \qquad \forall\, x_1, x_2 \geq 0.
\end{array} \tag{47}
$$

Similar to the upper bound problem, we now let:

$$
\begin{aligned}
q_1(x_1, x_2) &= 1 - p(x_1 + t_1, x_2 + t_2) \\
q_2(x_1, x_2) &= -p(x_1, t_2 - x_2) \\
q_3(x_1, x_2) &= -p(t_1 - x_1, x_2);
\end{aligned}
$$

that is,

$$
\begin{aligned}
q_1(x_1, x_2) = \;& -(y_{00} + y_{10}t_1 + y_{01}t_2 + y_{20}t_1^2 + y_{02}t_2^2 + y_{11}t_1t_2 - 1) \\
& -(y_{10} + 2t_1y_{20} + y_{11}t_2)x_1 - (y_{01} + 2t_2y_{02} + y_{11}t_1)x_2 \\
& -y_{20}x_1^2 - y_{02}x_2^2 - y_{11}x_1x_2 \\
q_2(x_1, x_2) = \;& -(y_{00} + y_{01}t_2 + y_{02}t_2^2) \\
& -(y_{10} + y_{11}t_2)x_1 - (-y_{01} - 2y_{02}t_2)x_2 \\
& -y_{20}x_1^2 - y_{02}x_2^2 + y_{11}x_1x_2 \\
q_3(x_1, x_2) = \;& -(y_{00} + y_{10}t_1 + y_{20}t_1^2) \\
& -(-y_{10} - 2y_{20}t_1)x_1 - (y_{01} + y_{11}t_1)x_2 \\
& -y_{20}x_1^2 - y_{02}x_2^2 + y_{11}x_1x_2.
\end{aligned}
$$

To finish, from Theorem 2, it follows that (47) (with the first constraint written in terms of $q_i(x_1, x_2)$, $i = 1, \dots, 3$) is equivalent to the following SOS program:

$$
\begin{array}{lll}
\underline{d}' = & \sup & y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{such that} & q_1(x_1^2, x_2^2) \text{ is a SOS polynomial} \\
& & q_2(x_1^2, x_2^2) \text{ is a SOS polynomial} \\
& & q_3(x_1^2, x_2^2) \text{ is a SOS polynomial.}
\end{array} \tag{48}
$$

The SOS program (48) can be readily solved with a SOS programming solver. Thus, if problem (39) is feasible (cf. Observation 9), it follows from Observation 11 that we can numerically approximate the ruin probability semiparametric lower bound $\underline{p}$ by solving problem (48) with a SOS solver. Furthermore, notice that by solving (43) and (48) we obtain a "100% confidence interval" $\underline{d}' \leq \mathbb{P}\mathrm{r}(X_1 \geq t_1 \text{ and } X_2 \geq t_2) \leq \overline{p}$ on the value of the probability when given only up to the second order moment information on the non-negative random variables X_1, X_2.

APPENDIX B: OBTAIN BOUNDS ON STOP-LOSS PAYMENTS FROM A TRANSFORMED PROBLEM

Let $\psi(Z) = Z - \phi(Z)$ where $\phi(Z)$ is the stop-loss payment function defined in problem (25) and $\psi(Z)$ is the transform function (34).

If the moment matrix Σ satisfies the feasibility requirement (cf. Observation 6), we can numerically obtain the semiparametric upper bound $\overline{p}(\psi)$. The upper bound of $\psi(Z)$, $\overline{p}(\psi) = \sup\{\mathrm{E}_\pi[\psi(Z)]\}$ given the same moment information, equals μ_z minus the lower bound of $\phi(Z)$. That is, $\overline{p}(\psi) = \mu_z - \underline{p}(\phi)$.

Proof. On one side,

$$\begin{aligned} \overline{p}(\psi) &= \sup\{\mathrm{E}_\pi[\psi(Z)]\} \\ E_\pi[\psi(Z)] &= \mu_z - E_\pi[\phi(Z)] \\ \overline{p}(\psi) &\geq \mu_z - E_\pi[\phi(Z)] \quad \forall\,\pi \text{ with given moments} \\ \overline{p}(\psi) &\geq \mu_z - \underline{p}(\phi). \end{aligned} \tag{49}$$

On the other side,

$$\begin{aligned} \underline{p}(\phi) &= \inf\{\mathrm{E}_\pi[\phi(Z)]\} \\ \text{since } \phi(Z) &= Z - \psi(Z) \\ \mathrm{E}_\pi[\phi(Z)] &= \mu_z - E_\pi[\psi(Z)] \\ \underline{p}(\phi) &\leq \mu_z - E_\pi[\psi(Z)] \quad \forall\,\pi \text{ with given moments} \\ \underline{p}(\phi) &\leq \mu_z - \overline{p}(\psi) \\ \overline{p}(\psi) &\leq \mu_z - \underline{p}(\phi). \end{aligned} \tag{50}$$

In order to satisfy both (49) and (50) simultaneously, $\overline{p}(\psi)$ must equal $\mu_z - \underline{p}(\phi)$. □

The lower bound of $\psi(Z)$, $\underline{p}(\psi) = \inf\{\mathrm{E}_\pi[\psi(Z)]\}$ given the same moment information, equals μ_z minus the upper bound of $\phi(Z)$. That is, $\underline{p}(\psi) = \mu_z - \overline{p}(\phi)$.

Proof. On one side,

$$\begin{aligned} \underline{p}(\psi) &= \inf\{\mathrm{E}_\pi[\psi(Z)]\} \\ E_\pi[\psi(Z)] &= \mu_z - E_\pi[\phi(Z)] \\ \underline{p}(\psi) &\leq \mu_z - E_\pi[\phi(Z)] \quad \forall\,\pi \text{ with given moments} \\ \underline{p}(\psi) &\leq \mu_z - \overline{p}(\phi). \end{aligned} \tag{51}$$

On the other side,

$$
\begin{aligned}
\overline{p}(\phi) &= \sup\{\mathrm{E}_\pi[\phi(Z)]\} \\
\mathrm{E}_\pi[\phi(Z)] &= \mu_z - E_\pi[\psi(Z)] \\
\overline{p}(\phi) &\geq \mu_z - E_\pi[\psi(Z)] \quad \forall\,\pi \text{ with given moments} \\
\overline{p}(\phi) &\geq \mu_z - \underline{p}(\psi) \\
\underline{p}(\psi) &\geq \mu_z - \overline{p}(\phi).
\end{aligned} \tag{52}
$$

In order to satisfy both (51) and (52) simultaneously, $\underline{p}(\psi)$ must equal $\mu_z - \overline{p}(\phi)$. $\square$