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Value-at-risk given a linguistic perspective
-- Some preliminary observations

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Abstract

This article provides an overview of three diverse fuzzy logic methodologies that can be employed to compute the value-at-risk of a portfolio, given a linguistic perspective. The methodologies are: fuzzy histograms, credibility measures, and a fuzzy Chebyshev inequality. Some preliminary observations about how they can be implemented are also provided. The article concludes with a comment on the material covered and suggestions for further studies.

1. Introduction

The value-at-risk (VaR), at a confidence level $\alpha \in (0, 1)$ and a fixed horizon, is given by the smallest number such that the probability that a loss exceeds that number is no larger than $(1 - \alpha)$. When the distribution of losses is known (or can be assumed) and adequate relevant data is available, estimates of the core parameters can be procured, and the VaR can be computed. However, often there is distributional ambiguity, in the sense that there is uncertainty regarding underlying distributions, and, frequently, this uncertainty is associated with parameters that are described in linguistic form owing to vagueness in the historical data and/or imprecision in the opinions of experts. Fuzzy logic (FL) methodologies can be employed to help resolve such situations.

This article provides an overview of some of these FL methodologies and some preliminary observations about how they can be implemented to compute the VaR, given a linguistic perspective. In particular, we present a synopsis of three diverse FL methodologies that can be employed to establish the VaR: fuzzy histograms, credibility measures, and a fuzzy Chebyshev inequality. Following this, the article concludes with a comment on the material covered and suggestions for further studies.

2. Estimating the VaR using a histogram

This section discusses the use of histograms to estimate a pdf, with which a VaR can be computed. First a crisp histogram model is presented. This is followed by a discussion of a fuzzy histogram model. The section ends with a comparison of a fuzzy histogram-based VaR with a GARCH-based VaR, using the VaR model validation of Kupiec (1995).

2.1 Estimating a pdf using crisp histograms[1]

Let

X be a sample space
C_k, k = 1, …, K, be disjoint classes of X

Then, using crisp histograms, a pdf can be estimated as:

$$f_k(x) = \begin{cases} \dfrac{P(C_k)}{c_k} & \text{if } x \in C_k \\ 0 & \text{if } x \notin C_k \end{cases}$$

where:

$P(C_k)$ is estimated using the relative frequency of samples $x_j \in C_k$ and
c_k is the scaling scalar, and equals the size of class C_k (which in the one-dimensional case, equals the length of the interval C_k)

The pdf f(x) is approximated by a summation of the $f_k(x)$ functions:

$$f(x) \approx \sum_{k=1}^{K} f_k(x)$$

Figure 1 shows a representation between a pdf and its associated crisp histogram[2] based on the foregoing methodology.

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[1] This section is based on Almeida and Kaymak (2008).

[2] Adapted from Almeida and Kaymak (2008) Figure 1(a).

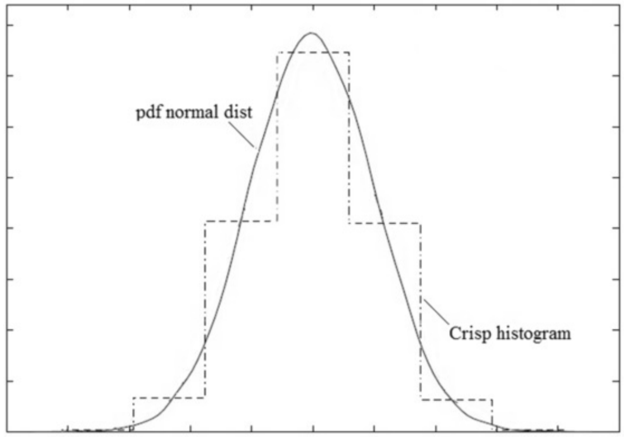

Figure 1: pdf v. crisp histogram

2.2 Estimating a pdf using fuzzy histograms[3]

In a similar fashion to the previous section, a pdf can be defined on a fuzzily partitioned sample space and approximated by a fuzzy histogram.[4]

Let

X be a sample space

A_k , k = 1, …, K, be fuzzy partitions of X

$\mu_{A_k}(x)$ be a membership function (MF) that describes A_k

Then, an estimate of the (fuzzy) column $\overline{f_k(x)}$ for fuzzy class A_k is:

$$\overline{f_k(x)} = \frac{\Pr(A_k)\,\mu_{A_k}(x)}{\int_{-\infty}^{\infty} \mu_{A_k}(x)\,dx}$$

where:

The numerator describes a probability weighted with a MF

.

The denominator is a scaling factor representing the fuzzified size of class A_k .

As with the crisp case, the complete pdf f(x) is approximated by summing the functions $\overline{f_k(x)}$:

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[3] This section is based on Almeida et al (2009).

[4] In addition to the method described here, a fuzzy histogram can be developed based on a probabilistic fuzzy system. See, for example, van den Berg et al (2011) and Shapiro (2013).

$$f(x) \approx \sum_{k=1}^{K} \overline{f_k(x)}$$

Figure 2 shows a representation between a pdf and its associated fuzzy histogram[5], given the underlying MFs. Notice, in Figure 2(b), that the number of MFs is equal to the number of crisp partitions and that the first and last MF are reverse-S and S-shaped, respectively. As suggested by Figure 2(a), the fuzzy histogram provides a better approximation of the pdf than the crisp histogram, because of the overlap of the MFs in the former.

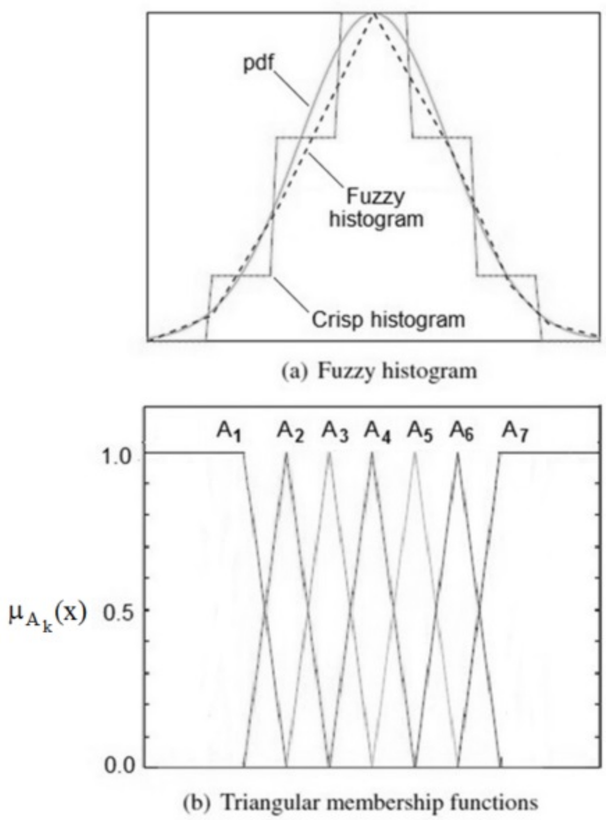

Figure 2: pdf v. fuzzy histogram

2.3 Example: Fuzzy Histogram v. GARCH model

Following Almeida et al (2009), the VaR based on a fuzzy histogram along the lines of that shown in Figure 2(a) might be compared with the VaR based on a GARCH (1,1) model[6], using

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[5] Adapted from Almeida and Kaymak (2008) Figure 1.

[6] In a GARCH (1,1) model, the variance at period t + 1 depends on the variance and the realized returns at period t, that is, $\sigma_{t+1}^2 = \gamma \overline{\overline{\sigma}}^2 + \alpha r_t^2 + \beta \sigma_t^2$, where r_t, the return at period t, is normally distributed with constant mean and

the Kupiec (1995) statistical test (see Appendix A). A representation of the results is shown in Figure 3[7], which shows the number of exceptions that have occurred in the validation data at a 95% confidence interval, for the 5 assets, S1, ..., S5. As depicted, the non-rejection region is between 16 and 36 rejections, which only the fuzzy histogram model satisfies for all assets.

Figure 3: Back-testing failure rates

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variable variance (local volatility), σ_t = local volatility, $\overline{\overline{\sigma}}$ = global volatility, and α, β and γ are positive constants, with $\alpha + \beta + \gamma = 1$. According to Hansen and Lunde (2005), a GARCH(1,1) model is very hard to beat in practice.

[7] Adapted from Almeida et al (2009) Table 6.

3. Estimating the VaR using a credibility measure

Given perceived limitations of the possibility measure, Liu and Liu (2002) suggested replacing it with what they termed a credibility measure, which they defined as the average of its possibility and necessity measures. This section is based on this credibility measure. It begins with a definition of the credibility measure, and then discusses how a credibilistic VaR can be developed.

3.1 The credibility measure[8]

Let

$(\Theta, \mathcal{P}(\Theta), \text{Pos})$ be a possibility space and the set $A \in \mathcal{P}(\Theta)$

Pos(A) be a possibility measure on Θ, where:

Pos(A) is the possibility that A will occur
$\text{Pos}(\Theta) = 1$ (Normality)
$\text{Pos}(\varnothing) = 0$ (Zero null set)
$\text{Pos}\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sup_i \text{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ (Maximality)

Nec(A) be the necessity measure of A (the impossibility of the occurrence of A^c), where:

$$\text{Nec}(A) = 1 - \text{Pos}(A^C) = 1 - \sup_{x \in A^C} \mu(x)$$

Then, the credibility measure of A is defined by

$$\text{Cr} = \tfrac{1}{2}[\text{Pos}(A) + \text{Nec}(A)]$$

That is, the credibility of a fuzzy event is the average of its possibility and necessity.

This characterization is not new. The right hand side of the foregoing equation was previously discussed in Gaines (1976, p. 639) and Dubois and Prade (1988, p. 125).

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[8] This section is based on Liu and Liu (2002) and Liu (2004, Ch. 3). See, also, Tanaka and Guo (1999, p. 134) and Bandemer (2006, p. 100).

3.2 Credibilistic VaR[9]

Let

ξ be a fuzzy variable.

$f(x, \xi)$ is the loss associated with a decision vector x

Then, the credibility of $f(x,\xi)$ not exceeding a threshold r is given by

$$\Psi(x, r) = \mathrm{Cr}\{f(x, \xi) \le r\}$$

and the credibilistic VaR (CrVaR) is defined by:

$$\mathrm{CrVaR}_\beta = \inf\{r \in \mathbb{R} \mid \psi(x, r) \ge \beta\}$$

where $\beta \in (0,1)$ is a prescribed confidence level.

3.3 Credibilistic VaR based on a triangular fuzzy variable[10]

As a specific application, consider a CrVaR based on a triangular fuzzy variable.

Let

$\xi = (a, b, c)$ be a triangular fuzzy variable.

Then the possibility, necessity and credibility measures are as shown in Table 1.

Table 1: Possibility, necessity and credibility measures

Measure	Possibility	Necessity	Credibility
	$\mathrm{Pos}\{\xi \le x\}$	$\mathrm{Nec}\{\xi \le x\}$	$\mathrm{Cr}\{\xi \le x\}$
$x \le a$	0	0	0
$a \le x \le b$	$\frac{x-a}{b-a}$	0	$\frac{x-a}{2(b-a)}$
$b \le x \le c$	1	$\frac{x-b}{c-b}$	$\frac{c-2b+x}{2(c-b)}$
$x \ge c$	1	1	1

[9] This section is based on Ma et al (2009).

[10] This section is based on Peng and Li (2009).

It follows that its CrVaR is:

$$\xi_{VaR}(\alpha) = \begin{cases} a + 2(b-a)\alpha, & \alpha \le 0.5 \\ 2b + 2(c-b)\alpha - c, & \alpha > 0.5 \end{cases}$$

This result can be validated by referring to Figure 4, where the red (dark) line represents the credibility distribution.

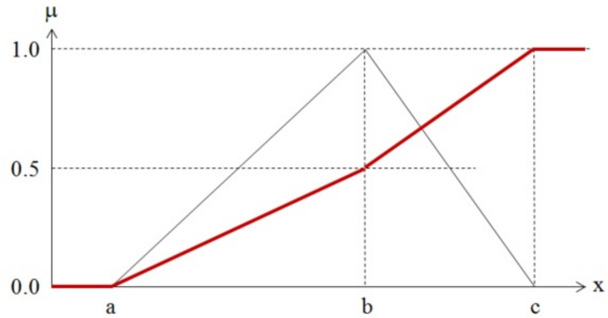

Figure 4: Credibilistic VaR, given a TFN

As indicated in the figure, an inflection point is at 0.5. Below that, (x-a) / (b-a) = α / (1/2), where α represents the value of μ that coincides with x. Since $\xi_{VaR}(\alpha)$ = x, the result follows. An analogous argument holds above 0.5.

4. Estimating the VaR using Chebyshev's inequality[11]

Chebyshev’s inequality provides a methodology for forecasting the VaR for a wide class of distributions. This section begins with the presentation of a crisp VaR under Chebyshev's inequality, and then proceeds to the presentation of its counterpart, a fuzzy VaR.

4.1 A crisp VaR under Chebyshev's inequality

For any random variable with mean[12], m, and finite variance, σ^2, it follows from Chebyshev's inequality that:[13]

$$P(|X - m| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

where $|X - m| \leq t$ denotes $-t + m \leq X \leq t + m$.

For the basic case of a symmetric distribution:

$$P(X \leq m - k\sigma) \leq \frac{1}{2}\left(1 - \left(1 - \frac{1}{k^2}\right)\right) = \frac{1}{2k^2}$$

where, for the tail probability (1-ε),

$$\frac{1}{2k^2} = (1 - \varepsilon) \Rightarrow k = \sqrt{\frac{1}{2(1-\varepsilon)}}\ .$$

Since the VaR is the negative of the quantile, it follows that the 100 ε % VaR is:

$$VaR_{\varepsilon} = \sigma\sqrt{\frac{1}{2(1-\varepsilon)}} - m$$

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[11] This section is based on Appadoo et al (2015).

[12] Here, m, rather than μ, is used to represent the mean, in order to eliminate potential confusion with the MF symbol μ.

[13] Soong (2004, p. 86)

4.2 A fuzzy VaR under Chebyshev's inequality

Extending the previous section to a fuzzy context, consider the forecast of a FVaR for a normally distributed portfolio.

If

The portfolio process follows a Gaussian AR(1) process of the form:

$$y_t - m = \phi(y_{t-1} - m) + a_t$$

The conditional distribution of y_t is symmetric

Then

The one step ahead fuzzy forecast of 100 ε % VaR based on $y_1, \ldots, y_n$ is given by

$$\overline{\mathrm{VaR}}_n(1) = \sigma\sqrt{\frac{1}{2(1-\varepsilon)}} - \overline{m} - \overline{\phi}(y_n - \overline{m}).$$

where, following Thavaneswarana et al (2009, p. 361):[14]

$$\overline{m(\alpha)} = [m_L(\alpha), m_U(\alpha)] = [\hat{m} - Z^*_{\alpha/2}(SE(\hat{m}),\ \hat{m} + Z^*_{\alpha/2}(SE(\hat{m}))]$$

$$\overline{\phi(\alpha)} = [\phi_L(\alpha), \phi_U(\alpha)] = [\hat{\phi} - Z^*_{\alpha/2}(SE(\hat{\phi}),\ \hat{\phi} + Z^*_{\alpha/2}(SE(\hat{\phi}))]$$

Finally,

$\overline{\mathrm{VaR}}_n(1) = [\mathrm{VaR}_{n+1,L}, \mathrm{VaR}_{n+1,U}]$, which are given by:

$$\mathrm{VaR}_{n+1,L} = \sigma\sqrt{\frac{1}{2(1-\varepsilon)}} - m_L(\alpha) - \mathrm{Min}[\phi_j(\alpha)(y_n - m_k(\alpha))],\ j,k \in \{L,U\}$$
$$\mathrm{VaR}_{n+1,U} = \sigma\sqrt{\frac{1}{2(1-\varepsilon)}} - m_U(\alpha) - \mathrm{Max}[\phi_j(\alpha)(y_n - m_k(\alpha))],\ j,k \in \{L,U\}$$

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[14] Here, $\overline{m}$ signifies a triangular fuzzy number whose α-cuts are denoted by $\overline{m(\alpha)} = [m_L(\alpha), m_U(\alpha)]$, where $m_L(\alpha)$ and $m_U(\alpha)$ represents the infima and the suprema of the α-cuts.

5. Closing comments

This article presented some preliminary observations about implementing a VaR model when parameters are linguistic. The topics covered included a fuzzy histogram VaR model, a credibilistic VaR model, and a VaR model based on Chebyshev's inequality. The discussion of these topics was far from exhaustive. But if this overview provides a sense of how linguistic-based VaR models can be addressed, it will have served its purpose.

As far as future research is concerned, there are many interesting avenues that can be explored. In addition to extending the cross-section of FL methodologies that can be used to model a FVaR, a comprehensive analysis of each of them can be done, including broadening the universe of discourse to include fat-tailed distributions.[15] Another dimension to be explored is the application of these FL methodologies to the conditional value-at-risk (CVaR) [16], that is, the expected loss, given that losses exceed the VaR. While the VaR often appears to be the measure of choice among risk managers,[17] there is considerable discussion regarding its limitations by the advocates of the CVaR.[18] Finally, in this article the VaR was addressed in the context of a portfolio model. As an alternative, the foregoing analysis could be applied from a loss model perspective, which focuses on a VaR in the right-hand tail of a loss distribution.[19]

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[15] See, for example, Longin (2000) and Thavaneswaran (2009).

[16] CVaR is also referred to as expected shortfall (ES) and conditional tail expectation (CTE)

[17] See Hull (2009, p. 445).

[18] See, for example, Artzner et al (1999), Bodie et al (2014, p. 140) and Rachev et al. (2008, p. 194).

[19] See, for example, Klugman et al (2008, p. 44).

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Appendix A: Kupiec's VaR Model validation

Model verification is universally recognized as a key component of quantitative models that measure market risk. Greenspan (1996, p. 502), in commenting on the issue, wrote that disclosure associated with the VaR "is enlightening only when accompanied by a thorough discussion of how the risk measures were calculated and how they related to actual performance." The VaR model validation test of Kupiec (1995) was developed in response to these types of concerns.

Kupiec's VaR Model validation test can be summarized as follows.

Let:

N = the number of exceptions
T = the total number of observations
c = the confidence level
UC = unconditional coverage
LR_{uc}, the likelihood ratio statistic

Then:

$$LR_{uc} = -2\ln\left[(1-c)^{T-N}c^{N}\right] + 2\ln\left\{\left[1-\left(\frac{N}{T}\right)\right]^{T-N}\left(\frac{N}{T}\right)^{N}\right\}$$

The N/T ratio is asymptotically χ^2-distributed, with 1 degree of freedom, under the null hypothesis of a valid VaR model, and the confidence regions are defined by the tail point of the log-likelihood ratio LR_{uc}.