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THE GAMMA-EXPONENTIAL MIX MODEL CDF

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Agenda

- Background
- Prior results
- Derivation of closed form for CDF of Gamma-Exponential Mix Distribution

What's the Problem?

- Life insurance experience studies focus on A/E ratios
  - use pivot tables or similar tools for flexible views
- Dispersion information rarely included
  - Approximations (if any) usually based only on policy counts using Poisson / Normal
  - Accurate methods using amounts difficult to pivot
  - Closed form of aggregate claims distribution not available

What is needed?

- Need algorithm to calculate CI's and p-values by amount that
  - Accurately captures skewness of distribution
  - Can be applied in pivot table
- Ideally use closed form distribution
  - If approximation, must be good

Prior Results

Edwalds 2003

Edwalds & Hilton 2013

The Gamma-Exponential Mix Model

- Random variable X has Gamma-Exponential Mix distribution (GE Mix) if

$$X = X_0 \pm X_1$$

$$X_0 \sim Gamma(\beta, \gamma)$$

$$X_1 \sim Exponential(\lambda)$$

- Convolution of Gamma and Exponential

First Three Central Moments

- GE Mix Model

$$\mu(X) = \beta \cdot \gamma \pm {}^{1}\!/\!_{\lambda}$$

$$\mu_2(X) = \beta^2 \cdot \gamma + {}^{1}\!/\!_{\lambda^2}$$

$$\mu_3(X) = 2 \cdot \beta^3 \cdot \gamma \pm {}^{2}\!/\!_{\lambda^3}$$

- Expected claims

$$\mu(L) = \sum_j A_j \cdot q_j$$

$$\mu_2(L) = \sum_j A_j^2 \cdot q_j \cdot (1 - q_j)$$

$$\mu_3(L) = \sum_j A_j^3 \cdot q_j \cdot (1 - q_j) \cdot (1 - 2 \cdot q_j)$$

Fitting GE Mix to Expected Claims Distribution

- Solve for Λ:

$$\Lambda^3 - 2\cdot\frac{\mu_2(L)}{\mu(L)}\cdot\Lambda^2 + \frac{\mu_3(L)}{2\cdot\mu(L)}\cdot\Lambda + \frac{[\mu_2(L)]^2}{\mu(L)} - \frac{\mu_3(L)}{2} = 0$$

$$\lambda = {}^{1}\!/\!_{|\Lambda|}$$

$$\beta = \frac{\mu_2(L) - \Lambda^2}{\mu(L) - \Lambda}$$

$$\gamma = \frac{\mu(L) - \Lambda}{\beta}$$

- Bounds: $\Lambda < \mu(L)$

  $0 < \Lambda^2 < \mu_2(L)$

- The sign of Λ determines whether convolution is + or –

Derivation of GE Mix CDF

Derivation of GE Mix CDF

- $f_G(x_1) = \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta \cdot \Gamma(\gamma)}$ (Gamma Distribution)
- $f_E(x_2) = \lambda e^{-\lambda x_2}$ (Exponential Distribution)
- Two cases:
  - $\Lambda > 0$
  - $\Lambda < 0$

Case 1: $\Lambda > 0, z = x_1 + x_2$

- $F_{GE}(z, \gamma, \beta, \lambda) =$

$$\int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta\Gamma(\gamma)} \int_0^{z-x_1} \lambda e^{-\lambda x_2} dx_2 dx_1 =$$

$$\int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta\Gamma(\gamma)} \left[1 - e^{-\lambda(z-x_1)}\right] dx_1 =$$

$$\int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta\Gamma(\gamma)} dx_1 - e^{-\lambda z} \int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{\lambda x_1 - x_1/\beta}}{\beta\Gamma(\gamma)} dx_1$$

Focus on 2nd part

- $$e^{-\lambda z} \int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{\lambda x_1 - {x_1}/{\beta}}}{\beta\Gamma(\gamma)} dx_1 =$$

$$e^{-\lambda z} \int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{(1-\beta\lambda)^\gamma}{(1-\beta\lambda)^\gamma} \frac{e^{{(\beta\lambda-1)x_1}/{\beta}}}{\beta\Gamma(\gamma)} dx_1 =$$

$$\frac{e^{-\lambda z}}{(1-\beta\lambda)^\gamma} \int_0^z \left(\frac{x_1}{b}\right)^{\gamma-1} \frac{e^{-{x_1}/{b}}}{b\Gamma(\gamma)} dx_1, \text{ where } b = \frac{\beta}{1-\beta\lambda}$$

Case 1: $\Lambda > 0, z = x_1 + x_2$

- $F_{GE}(z, \gamma, \beta, \lambda) =$
$$\int_0^z \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta\Gamma(\gamma)} dx_1 - \frac{e^{-\lambda z}}{(1-\beta\lambda)^\gamma} \int_0^z \left(\frac{x_1}{b}\right)^{\gamma-1} \frac{e^{-x_1/b}}{b\Gamma(\gamma)} dx_1$$
- $= F_G(z, \gamma, \beta) - \frac{e^{-\lambda z}}{(1-\beta\lambda)^\gamma} F_G(z, \gamma, b)$ where $b = \frac{\beta}{1-\beta\lambda}$

Case 2: $\Lambda < 0,\ z = x_1 - x_2$

- $f_{GE}(z,\gamma,\beta,\lambda) = \int_z^{\infty} \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{e^{-x_1/\beta}}{\beta\Gamma(\gamma)} \lambda e^{-\lambda(x_1-z)}\, dx_1 =$

  $\lambda e^{\lambda z} \int_z^{\infty} \left(\frac{x_1}{\beta}\right)^{\gamma-1} \frac{(1+\beta\lambda)^{\gamma}}{(1+\beta\lambda)^{\gamma}} \frac{e^{-\lambda x_1 - x_1/\beta}}{\beta\Gamma(\gamma)} dx_1$

  $= \frac{\lambda e^{\lambda z}}{(1+\beta\lambda)^{\gamma}} \int_z^{\infty} \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)} dx_1,$ where $b' = \frac{\beta}{1+\beta\lambda}$

- $f_{GE}(z,\gamma,\beta,\lambda) = \frac{\lambda e^{\lambda z}}{(1+\beta\lambda)^{\gamma}} \left(1 - F_G(z,\gamma,b')\right)$

Case 2: $\Lambda < 0,\, z = x_1 - x_2$

- $F_{GE}(z,\gamma,\beta,\lambda) - F_{GE}(0,\gamma,\beta,\lambda) = \int_0^z f_{GE}(y,\gamma,\beta,\lambda)\, dy =$

$$\int_0^z \frac{\lambda e^{\lambda y}}{(1+\beta\lambda)^\gamma} \int_y^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)}\, dx_1\, dy =$$

$$\frac{\lambda}{(1+\beta\lambda)^\gamma} \int_0^z e^{\lambda y} \int_y^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)}\, dx_1 dy$$

Use integration by parts

- $u = \int_y^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)} dx_1$
- $du = -\left(\frac{y}{b'}\right)^{\gamma-1} \frac{e^{-y/b'}}{b'\Gamma(\gamma)}$
- $dv = e^{\lambda y} dy$
- $v = \frac{1}{\lambda} e^{\lambda y}$

Use integration by parts

- $\int_0^z e^{\lambda y} \int_y^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)} dx_1 dy =$
$\frac{1}{\lambda} e^{\lambda z} \int_z^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)} dx_1 -$
$\frac{1}{\lambda} \int_0^\infty \left(\frac{x_1}{b'}\right)^{\gamma-1} \frac{e^{-x_1/b'}}{b'\Gamma(\gamma)} dx_1 + \int_0^z \frac{1}{\lambda} e^{\lambda y} \left(\frac{y}{b'}\right)^{\gamma-1} \frac{e^{-y/b'}}{b'\Gamma(\gamma)} dy$

- $F_{GE}(z,\gamma,\beta,\lambda) - F_{GE}(0,\gamma,\beta,\lambda) = \frac{\lambda}{(1+\beta\lambda)^{\gamma}}\frac{1}{\lambda}\Big[e^{\lambda z}(1 -$

Case 2: $\Lambda < 0, z = x_1 - x_2$

- $F_{GE}(z,\gamma,\beta,\lambda) - F_{GE}(0,\gamma,\beta,\lambda) =$
$$\int_0^z \left(\frac{y}{\beta}\right)^{\gamma-1} \frac{e^{-y/\beta}}{\beta\Gamma(\gamma)} dy - \frac{1}{(1+\beta\lambda)^\gamma}$$
$$+ \frac{e^{\lambda z}}{(1+\beta\lambda)^\gamma}\left(1 - F_G(z,\gamma,b')\right)$$
$$= F_G(z,\gamma,\beta) - \frac{1}{(1+\beta\lambda)^\gamma} + \frac{e^{\lambda z}}{(1+\beta\lambda)^\gamma}\left(1 - F_G(z,\gamma,b')\right)$$

Case 2: $\Lambda < 0, z = x_1 - x_2$

- $F_{GE}(z, \gamma, \beta, \lambda) = F_G(z, \gamma, \beta) + \frac{e^{\lambda z}}{(1+\beta\lambda)^{\gamma}}(1 -$

Putting it Together

- Case 1: $\Lambda > 0$
  - $F_{GE}(z,\gamma,\beta,\lambda) = F_G(z,\gamma,\beta) - \frac{e^{-\lambda z}}{(1-\beta\lambda)^\gamma} F_G(z,\gamma,b)$
    - where $b = \frac{\beta}{1-\beta\lambda}$
- Case 2: $\Lambda < 0$
  - $F_{GE}(z,\gamma,\beta,\lambda) = F_G(z,\gamma,\beta) + \frac{e^{\lambda z}}{(1+\beta\lambda)^\gamma}\left(1 - F_G(z,\gamma,b')\right)$
    - where $b' = \frac{\beta}{1+\beta\lambda}$

Questions?

Thank You!