Optimal investment strategies and intergenerational risk sharing for target benefit pension plans

Barbara Sanders
with Suxin Wang and Yi Lu

Department of Statistics and Actuarial Science
Simon Fraser University

July 29, 2017

Outline

Target Benefit Plans

Key features:

- Predefined contribution level
- Sponsor liability limited to contributions
- Target benefit level
- Actual benefits vary

- Collective asset pool
- Members bear risk collectively

Target Benefit Plans

Key features:

- Predefined contribution level
- Sponsor liability limited to contributions
- Target benefit level
- Actual benefits vary

- Collective asset pool
- Members bear risk collectively

Target Benefit Plans

Practical objectives:

- Provide adequate benefits
- Maintain stability
- Respect intergenerational equity

Key question:

Given some starting asset value and contribution commitment, how should assets be invested and benefits be paid out to achieve these goals?

Target Benefit Plans

Practical objectives:

- Provide adequate benefits
- Maintain stability
- Respect intergenerational equity

Key question:

Given some starting asset value and contribution commitment, how should assets be invested and benefits be paid out to achieve these goals?

Stochastic optimization in pension literature

- DB optimization: asset mix and contribution rate

  Cairns (1996, 2000), Haberman and Sung (2004), Josa-Fombellida and Rincon-Zapatero (2001, 2004, 2008), Ngwira and Gerard (2007), etc.

- DC optimization: asset mix and payout pattern

  Gerrard et al. (2004), He and Liang (2013, 2015), etc.

- Gollier (2008): asset mix, benefit payout, dividend policy
- Cui *et al.* (2011): asset mix, contribution rate, benefit payout

Dynamics of financial market

- Risk-free asset $S_0(t)$

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \quad t \geq 0,$$

where r_0 represents the risk-free interest rate.

- Risky asset $S_1(t)$

$$\mathrm{d}S_1(t) = S_1(t)[\mu \mathrm{d}t + \sigma \mathrm{d}W(t)], \quad t \geq 0,$$

where μ is the appreciation rate of the stock, σ is the volatility rate, and $W(t)$ is a standard Brownian motion.

Plan membership

- The fundamental elements of demographic model:

  $n(t)$: density of new entrants aged a at time t,
  $s(x)$: survival function with $s(a) = 1$ and $a \leq x \leq \omega$.

- The density of those who attain age x at time t is

$$n(t - (x - a))s(x), \qquad x > a.$$

Salary process

- Dynamics of salary rate for a member who retires at time t:

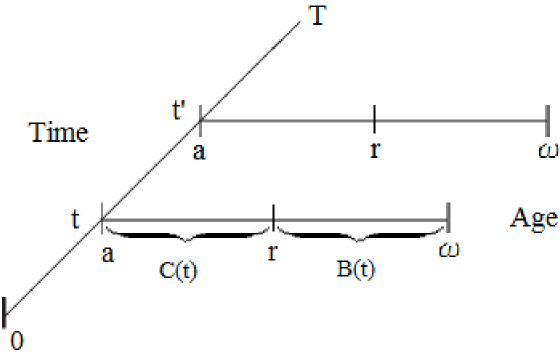
$$\mathrm{d}L(t) = L(t)\left(\alpha \mathrm{d}t + \eta \mathrm{d}\overline{W}(t)\right), \quad t \geq 0,$$

  where $\alpha \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. $\overline{W}$ is a standard Brownian motion correlated with W, such that $E[W(t)\overline{W}(t)] = \rho t$.

- For a retiree age x at time t ($x \geq r$), define assumed salary at retirement ($x - r$ years ago) as

$$\widetilde{L}(x,t) = L(t)e^{-\alpha(x-r)}, \quad t \geq 0,\ x \geq r.$$

The time-age structure of the pension plan



Contribution process

- Individual contribution rate for active member aged x at time $t \geq 0$:

$$C(x,t) = c_0(x)e^{\alpha t}, \quad a \leq x < r.$$

- Aggregate contribution rate in respect of all active members at time t:

$$C(t) = \int_a^r n(t-x+a)s(x)C(x,t)\mathrm{d}x, \quad t \geq 0.$$

Benefit payment process

Individual pension payment rate at time t:

- for a new retiree aged r:

$$B(r,t) = f(t)L(t)$$

- for an existing retiree aged $x > r$:

$$\begin{aligned} B(x,t) &= f(t)\widetilde{L}(x,t)e^{\varsigma(x-r)} \\ &= f(t)L(t)e^{-(\alpha-\varsigma)(x-r)} \end{aligned}$$

Benefit payment process

Individual pension payment rate at time t:

- for a new retiree aged r:

$$B(r,t) = f(t)L(t)$$

- for an existing retiree aged $x > r$:

$$\begin{aligned} B(x,t) &= f(t)\widetilde{L}(x,t)e^{\zeta(x-r)} \\ &= f(t)L(t)e^{-(\alpha-\zeta)(x-r)} \end{aligned}$$

Benefit payment process

- Aggregate pension benefit rate for all retirees at time t:

$$B(t) = \int_r^\omega n(t-x+a)s(x)B(x,t)\mathrm{d}x = I(t)f(t)L(t), \quad t \geq 0.$$

- The updated aggregate target benefit is $B^* e^{\beta t}$.

Fund dynamics

The pension fund dynamic can be described as

$$\begin{cases} \mathrm{d}X(t) = \pi(t)\frac{\mathrm{d}S_1(t)}{S_1(t)} + (X(t) - \pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + (C(t) - B(t))\mathrm{d}t, \\ X(0) = x_0. \end{cases}$$

The objective function

- Let $J(t,x,l)$ be the objective function at time t with the fund value and the salary level being x and l. It is defined as

$$\begin{cases} J(t,x,l) = E_{\pi,f}\Big\{ \int_t^T \Big[\big(B(s) - B^* e^{\beta s}\big)^2 - \lambda_1 \big(B(s) - B^* e^{\beta s}\big)\Big] e^{-r_0 s}\mathrm{d}s \\ \qquad\qquad + \lambda_2 \big(X(T) - x_0 e^{r_0 T}\big)^2 e^{-r_0 T}\Big\}, \\ J(T,x,l) = \lambda_2 \big(X(T) - x_0 e^{r_0 T}\big)^2 e^{-r_0 T}. \end{cases}$$

- The value function is defined as

$$\phi(t,x,l) := \min_{(\pi,f)\in\Pi} J(t,x,l), \qquad t,x,l > 0.$$

See Ngwira and Gerrard (2007), He and Liang (2015).

Using variational methods and Itô's formula, we get the following HJB equation satisfied by the value function $\phi(t, x, l)$:

$$\min_{\pi,f} \Big\{ \phi_t + \left[r_0 x + (\mu - r_0)\pi + C_1(t)e^{\alpha t} - fl \cdot I(t) \right] \phi_x + \alpha l \phi_l + \frac{1}{2}\pi^2\sigma^2\phi_{xx} + \frac{1}{2}\eta^2 l^2 \phi_{ll} + \rho\sigma\eta l \pi \phi_{xl} + \left[\left(fl \cdot I(t) - B^* e^{\beta t} \right)^2 - \lambda_1 \left(fl \cdot I(t) - B^* e^{\beta t} \right) \right] e^{-r_0 t} \Big\} = 0.$$

Solution to the optimization problem

- Optimal strategies are

$$\pi^*(t,x,l) = -\frac{\delta}{2\sigma}\left[2x + Q(t)\right],$$
$$f^*(t,x,l) = \frac{1}{l \cdot l(t)}\left[\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\left(2x + Q(t)\right)P(t) + B^* e^{\beta t}\right],$$

where $\delta = (\mu - r_0)/\sigma$ is the Sharp Ratio.

- The corresponding value function is given by

$$\phi(t,x,l) = \lambda_2 e^{-r_0 t} P(t)[x^2 + xQ(t)] + K(t).$$

$$P(t) = \begin{cases} \frac{1}{\lambda_2(T-t)+1}, & r_0 = \delta^2, \\ \frac{r_0-\delta^2}{\lambda_2+(r_0-\delta^2-\lambda)e^{-(r_0-\delta^2)(T-t)}}, & r_0 \neq \delta^2, \end{cases}$$

$$Q(t) = \begin{cases} 2e^{r_0 t}\left[\int_t^T C_1(s)e^{(\alpha-r_0)s}\mathrm{d}s - B^*(T-t) - x_0\right], & \beta = r_0, \\ 2e^{r_0 t}\left[\int_t^T C_1(s)e^{(\alpha-r_0)s}\mathrm{d}s - B^*\frac{\left(e^{(\beta-r_0)T}-e^{(\beta-r_0)t}\right)}{\beta-r_0} - x_0\right], & \beta \neq r_0, \end{cases}$$

$$K(t) = \lambda_2 \int_t^T e^{-r_0 t}\Big\{P(s)Q(s)\Big[C_1(s)e^{\alpha s} - B^* e^{\beta s} - \frac{1}{4}\left(\delta^2 + \lambda_2 P(s)\right)Q(s)\Big] - \frac{\lambda_1^2}{4}\Big\}\mathrm{d}s.$$

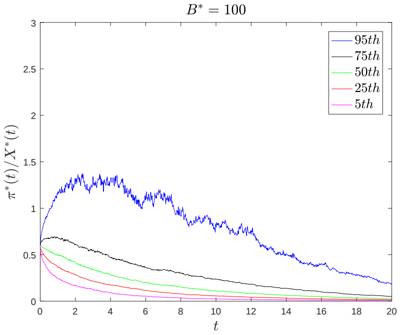
Assumptions for numerical illustrations

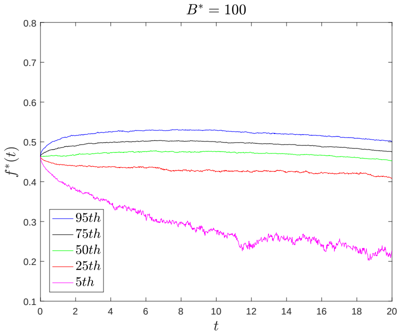
- $a = 30$, $r = 65$, $\omega = 100$.
- Force of mortality follows Makeham's Law.
- $n(t) = 10$ for all $t \geq 0$, implying a stationary population.
- $B^* = 100$, $\beta = 0.025$.
- Cost-of-living adjustment rate $\zeta = 0.02$.
- $r_0 = 0.01$, $\mu = 0.1$, $\sigma = 0.3$, $\Rightarrow \delta = 0.3$.
- $\alpha = 0.03$, $\eta = 0.01$; initial salary rate $L(0) = 1$.
- Correlation coefficient $\rho = 0.1$; $\lambda_1 = 15$, $\lambda_2 = 0.2$.
- $X(0) = 2500$, $c_0 = 0.1$.

See Dickson et al. (2013)

Numerical analysis

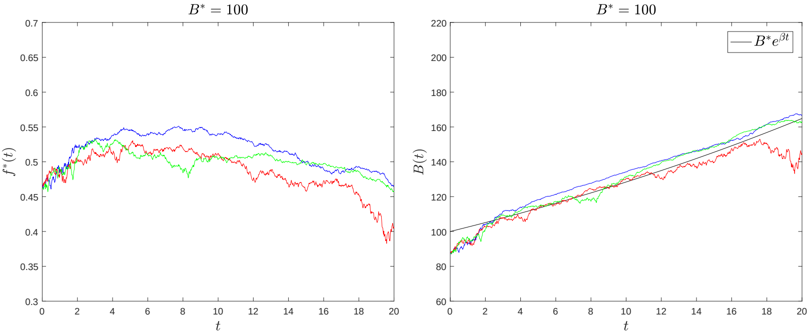
Percentiles of $\pi^*(t)/X^*(t)$ and $f^*(t)$





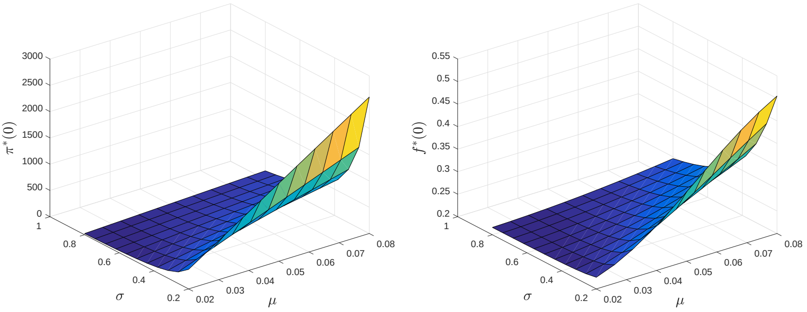
Numerical analysis

Sample paths of $f^*(t)$ and $B(t)$



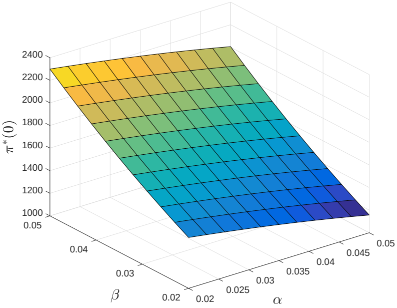
Numerical analysis

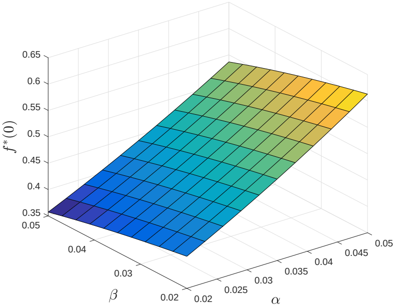
Effects of asset returns



Numerical analysis

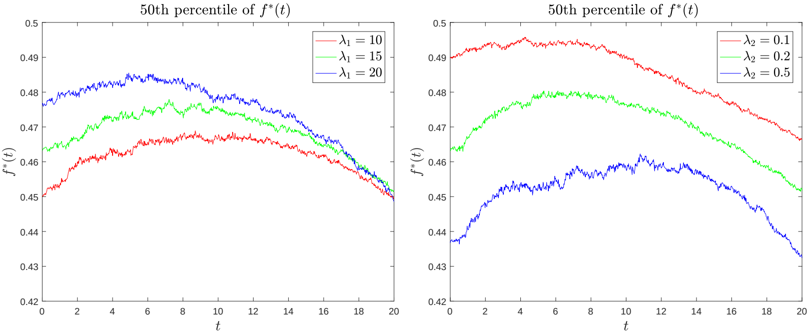
Effects of salary and target benefit growth rates





Numerical analysis

Medians of $f^*(t)$ for different values of λ_1 and λ_2



Conclusion

- We apply the Black-Scholes framework for plan assets, and consider a correlated salary process.
- We consider three key objectives of plan trustees (benefit adequacy, stability and intergenerational equity).
- We derive closed form expressions for optimal investments and payouts.
- The model is useful for identifying combinations of inputs that can meet stakeholders' stated objectives.

References

- Cui, J., De Jong, F., and Ponds, E. (2011). Intergenerational risk sharing within funded pension schemes. *Journal of Pension Economics and Finance*, 10(01):1-29.
- Dickson, D. C., Hardy, M. R., and Waters, H. R. (2013). *Actuarial mathematics for life contingent risks*. Cambridge University Press.
- Gollier, C. (2008). Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics*, 92(5):1463-1485.
- He, L. and Liang, Z. (2015). Optimal assets allocation and benefit outgo policies of DC pension plan with compulsory conversion claims. *Insurance: Mathematics and Economics*, 61:227-234.
- Ngwira, B. and Gerrard, R. (2007). Stochastic pension fund control in the presence of Poisson jumps. *Insurance: Mathematics and Economics*, 40(2):283-292.

Questions?