An Integro-differential Equation for a Sparre Andersen Model with Investments

Corina Constantinescu & Enrique Thomann
Department of Mathematics, Oregon State University

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• Classical actuarial problem - the collective risk Sparre Andersen model.

• Additional non-traditional feature, investments in a risky asset with returns modeled by a stochastic process.

• Focus of the analysis: the probability of ruin.

• Decay of the probability of ruin in the case of Erlang(n) distribution for inter-claims returns modeled by a geometric Brownian motion.
Sparre Andersen model

\[ U_t = u + ct - \sum_{k=1}^{N(t)} X_k \]

- \( u \) - initial surplus
- \( c \) - premium rate
- \( X_k \) iid - "light" claims - \( F_X \sim \) exponentially bounded tail
- \( N(t) \) - renewal process
- \( T_1, T_2, \cdots \) times when claims occur
- \( \tau_0 = 0, \tau_n = T_n - T_{n-1} \) inter-arrival times, independent, identical distributed r.v.
Sparre Andersen model with investments

- Consider that the company invests all its money, continuously, in a risky asset modeled by a non-negative stochastic process.
- NOTE: The ruin may happen only at the time of a claim, $T_k$.
- The model
  \[ U_k = Z_{\tau_k}^{U_{k-1}} - X_k \]
  is a discrete Markov process, where $U_k = U_{T_k}$. 
Definitions
The time of ruin:

\[ T_u = \inf_{t \geq 0} \{ U(t) < 0 \mid U(0) = u \} \]

The probability of ruin with infinite horizon:

\[ \Psi(u) = P(T_u < \infty) \]
Objectives

1. An equation for the probability of ruin in the Sparre Andersen model with investments
2. Particular case: Investigate the decay of the probability of ruin if the interarrival times are Erlang $(n, \beta)$, with returns from investments modeled by GBM $(a, \sigma^2)$.

Main tools

1. Integro-differential equation: generators arguments
2. Decay: Karamata-Tauberian arguments
Assumptions

- $(X_k)_k$ - claim sizes - "light" or well-behaved distributions $F_X$ with exponentially bounded tail

\[ 1 - F_X(x) \leq ce^{-ax} \]

for some $a$ and $c$ and for all $x \geq 0$.

- $(\tau_k)_k$ - inter-arrival times - $f_\tau$ satisfies an ODE with constant coefficients

\[ \mathcal{L} \left( \frac{d}{dt} \right) f_\tau(t) = 0 \]

Example: $f_\tau(t) = \beta e^{-\beta t}$ then $\mathcal{L} \left( \frac{d}{dt} \right) f_\tau(t) = (\frac{d}{dt} + \beta) f_\tau(t) = 0$

- $Z^u_t$ - returns from investments up to time $t$, starting with an initial capital $u$ - the company invests all its money, continuously into a risky asset modeled by a non-negative stochastic process with infinitesimal generator $A$
Transition operator of the discrete Markov process

For our discrete Markov process $U_0, U_1, U_2, \cdots$ (where $U_k = U_{T_k}$), on the set of all real-valued, bounded, Borel measurable functions $g$, define the transition operator

$$T_k g(u) = E(g(U_k) \mid U_0 = u) = E_{U_k} g(U_k).$$

Then $M_n = f(U_n) - \sum_{k=0}^{n-1} (T_1 - I) f(U_k)$ is a martingale.

**Proof:** $E(M_{n+1} \mid \sigma(U_0, U_1, \cdots U_n)) = E(g(U_{n+1}) \mid U_0, U_1, \cdots U_n) - \sum_{k=0}^{n} (T_1 - I) g(U_k) = T_1 g(U_n) - T_1 g(U_n) + g(U_n) - \sum_{k=0}^{n-1} (T_1 - I) g(U_k) = M_n.$
Theorem

If $f_\tau$ satisfies the ODE with constant coefficients

$$\mathcal{L}\left(\frac{d}{dt}\right)f_\tau(t) = 0$$

and

1. $f_\tau^{(k)}(0) = 0$, the $k$–th derivatives of $f_\tau$, for $k = 0, \cdots, n - 2$
2. $\lim_{x \to \infty} f^{(k)}(x) = 0$, for $k = 0, \cdots, n - 1$

then for any $g \in \mathcal{D}_{A(n)}$

$$\mathcal{L}^*(A) T_1 g(u) = f_\tau^{(n-1)}(0) \int_0^\infty g(u - x) f_X(x) dx$$

where $A$ denotes the infinitesimal generator of the investment process $Z_t$, $n$ represents the order of the ODE with constant coefficients satisfied by $f_\tau$. 
Relation to the ruin probability

Theorem. Assume that on the event $\{T_u = \infty\}$, $U_t \to \infty$ as $t \to \infty$. If $g \in \mathcal{D}_{A_U}$ satisfies

$$\mathcal{L}^*(A)g(u) = f^{(n-1)}(0) \int_0^\infty g(u - x)f_X(x)dx$$

...together with the boundary conditions

$$g(u) = 1 \quad \text{if} \quad u < 0$$

$$\lim_{u \to \infty} g(u) = 0$$

then

$$g(u) = P(T_u < \infty)$$
Sketch of proof:

- $g(U_k)$ is a martingale, $T_u$ stopping time
- $g(u) = E_u g(U_{T_u\wedge T_k}) = E_u g(U_{T_u\wedge T_k} 1_{\{T_u < T_k\}}) + E_u g(U_{T_u\wedge T_k} 1_{\{T_u > T_k\}}) = g(U_{T_u})P(T_u < T_k) + g(U_{T_k})P(T_u > T_k)$ (let $t \to \infty$)
- $g(u) = 1 \times P(T_u < \infty) + 0 \times P(T_u > \infty) = P(T_u < \infty)$
Examples

Integro-differential equation for Cramer Lundberg model - $\exp(\beta)$

$$
\mathcal{L}(\frac{d}{dt}) f_{\tau}(t) = (\frac{d}{dt} + \beta)f_{\tau}(t) = 0 \implies \mathcal{L}^* (\frac{d}{dt}) = (-\frac{d}{dt} + \beta)
$$

therefore

$$
\mathcal{L}^*(A) \Psi(u) = (-A + \beta) \Psi(u) = \beta \int_{0}^{\infty} \Psi(u - x)f_X(x)dx
$$
Examples
If no investments, \( A = c \frac{d}{du}, \)

\[
(-c \frac{d}{du} + \beta)\Psi(u) = \beta \int_0^\infty \Psi(u - x)f_X(x)dx
\]

\[
\Psi'(u) = \frac{\beta}{c} \Psi(u) - \frac{\beta}{c} \int_0^\infty \Psi(u - x)f_X(x)dx
\]
Particular case

- \( f_X \sim \text{finite moments in the neighborhood of the origin} \)
- \( f_\tau \sim \text{Erlang } (n, \beta) - \mathcal{L} \left( \frac{d}{dt} f_\tau(t) = \left( \frac{d}{dt} + \beta \right)^n f_\tau(t) \right) = 0 \)
- \( Z \sim \text{GBM}(a, \sigma^2) \) returns,

\[
dZ = (c + aZ)\,dt + \sigma Z\,dW_t
\]

\[
A = (c + au) \frac{d}{du} + \frac{\sigma^2}{2} u^2 \frac{d^2}{du^2}
\]

Then the surplus model is:

\[
U(t) = u + ct + a \int_0^t U(s)\,ds + \sigma \int_0^t U(s)\,dW_s - \sum_{0}^{N(t)} X_k.
\]
Integro-differential equation for Erlang(n) with investments

The integro-differential equation for a Sparre Andersen model when the time in between claims is Erlang($n$, $\beta$)

$$(-A + \beta)^n \Psi(u) = \beta^n \int_0^\infty \Psi(u - x)f_X(x)dx$$

together with the boundary conditions for $\Psi$.

If the investments are made in a stock modeled by a geometric brownian motion

$$(-(c + au)\frac{d}{du} - \frac{\sigma^2 u^2}{2} \frac{d^2}{du^2} + \beta)^n \Psi(u) = \beta^n \int_0^\infty \Psi(u - x)f_X(x)dx$$

Then the decay of the probability of ruin is algebraic

$$\lim_{u \to \infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = K_n$$

for (small volatility) $1 < \frac{2a}{\sigma^2} < 2$. 
Steps in establishing the algebraic decay rate

1. Take Laplace transform

2. Regularity at zero of the homogeneous ODE obtained in the Laplace side implies that \( \hat{\Psi}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k \).

3. Karamata -Tauberian arguments
Laplace transform

- **Erlang** \((n, \beta)\)

\[
(-\hat{A} + \beta)^n \hat{\Psi}(s) = \beta^n \hat{f}_X \hat{\Psi}(s)
\]

\[
(-1)^n \hat{A}^n \hat{\Psi}(s) + \cdots + \beta^n = \beta^n \hat{\Psi}(s) \hat{f}_X(s) + \beta^n \left( \frac{1}{s} - \frac{\hat{f}_X(s)}{s} \right)
\]

- \(2n\)-th order ODE:

\[
y^{(2n)} + p_1(s)y^{(2n-1)} + p_2(s)y^{(2n-2)} + \cdots + p_{2n}(s)y = p_{2n+1}(s)
\]

- regularity at zero

\[
\Rightarrow \hat{\Psi}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k
\]
Regularity at zero

Determine $\rho$:

- The coefficient of the $s^\rho$ term should be zero, i.e.
  \[
  (-\delta + \beta)^n - \beta^n = 0
  \]
  where
  \[
  \delta = [\sigma^2(\rho + 2) - a](\rho + 1)
  \]
- For $k = 0, 1, 2, \cdots, n - 1$
  \[
  \delta = \beta(1 - e^{\frac{2\pi ik}{n}})
  \]
- distinguish two cases, $n$ odd or even
Case 1. $n = \text{odd}$

\[
\rho_1 = 0
\]
\[
\rho_2 = -2 + \frac{2a}{\sigma^2}
\]

\[
\rho_1 \leq \rho_2
\]

- $\rho_1$ doesn’t produce decay of the probability of ruin
- $\rho_2$ is the leading term,

\[
\lim_{s \to 0} \hat{\Psi}(s)s^{2-\frac{2a}{\sigma^2}} = K_n
\]

\[
\lim_{u \to \infty} \Psi(u)u^{-1+\frac{2a}{\sigma^2}} = K_n
\]

for $1 < \frac{2a}{\sigma^2} < 2$. 
Case 2. $n = \text{even}$

\[
\begin{align*}
\rho_1 &= 0 \\
\rho_2 &= -2 + \frac{2a}{\sigma^2} \\
\rho_{3,4} &= \frac{\rho_2 - 1}{2} \pm \sqrt{\left(\frac{\rho_2 + 1}{2}\right)^2 + \frac{4\beta}{\sigma^2}}
\end{align*}
\]

\[
\rho_4 \leq \rho_1 \leq \rho_2 \leq \rho_3
\]

- $\rho_4, \rho_1$ do not produce decay of the probability of ruin
- By Karamata arguments and ordering of the ruin probabilities for Erlang of different $n$ it can be shown that for any $n$,

\[
\lim_{u \to \infty} \Psi(u)u^{-1 + \frac{2a}{\sigma^2}} = K_n
\]

for $1 < \frac{2a}{\sigma^2} < 2$. 
Conclusions

1. For a Sparre Andersen model, perturbed by a stochastic process, a very general integro-differential equation for the ruin probability can be written, if the inter-claim arrivals are mixture of Erlangs.

2. For any $n$, the Sparre Andersen model with inter-arrival times distributed Erlang ($n$) and investments in a stock modeled by a GBM with small volatility, has an algebraic decay rate, depending on the parameter of the investments only.

3. Conjecture: in the case of high volatility, $\sigma^2 > 2a$, the ruin is certain.
Future questions

1. $f_\tau$ satisfies an ODE with polynomial coefficients
2. $f_\tau \sim Gamma(\alpha, \beta)$
3. Gerber-Shiu functions
4. Optimal investment strategy
References


References